# The small $k_{\mathrm{T}}$ region in the parton branching approach and relation to CSS

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**Abstract**

The Parton Branching (PB) method is applied to identify the different terms in the Collins-Soper-Sterman (CSS) formalism by the introduction of a soft resolution scale $z_{dyn}$ motivated by angular ordering and the requirement of resolvable branchings. The soft contributions, which are an essential part of the DGLAP evolution and the PB approach provide a natural explanation of the "non-perturbative" Sudakov form factor in CSS language. The PB approach allows a direct calculation of this non-perturbative Sudakov form factor.

# 1 Introduction

The transverse momentum spectrum of Drell-Yan (DY) lepton pairs in hadron collisions is sensitive at very small $p_{\rm T}$ to the intrinsic motion of the partons inside the hadrons. At larger transverse momentum soft gluon emissions have to be resummed. Then, at large transverse momentum fixed higher order calculations become important. The DY $p_{\rm T}$- spectrum has been investigated in [1–6] and later formulated as the Collins Soper Sterman (CSS) approach in [7]. In Ref. [8] it is argued, that the CSS equation follows from a more general TMD factorization approach.

The DY spectrum is also rather well described by parton shower approaches as implemented in PYTHIA [9,10], HERWIG [11,12] and SHERPA [13,14]. However, the description of the lowest DY $p_{\rm T}$ spectrum is not consistent for different centre of mass energies [28,29]

The Parton Branching (PB) method [15,16], which is based on a branching solution of the DGLAP [17–20] evolution equation allows a very good description of the DY $p_{\rm T}$ spectrum from very low to large center-of-mass energies without the need for adjusting additional parameters [21,22].

In the CSS approach, parameters for the intrinsic transverse momentum distribution as well as for the so-called non-perturbative Sudakov form factor need to be extracted from experimental measurements. On the contrary, the PB approach needs no extra introduction of the non-perturbative Sudakov form factor.

In this note we show how the CSS formalism directly emerges from the PB approach, and how the non-perturbative Sudakov form factor is related to soft gluon emissions, which are essential and treated already by the PB solution of the DGLAP equation.

# 2 PB method

The PB method [15,16] has been shown to provide an exact solution of the DGLAP evolution equations. The PB method is based on the concept of resolvable and non-resolvable branchings [16, 23] via Sudakov form factors. The evolution equation for the parton density* of

*Please note, it is the parton density, not the momentum weighted density $\tilde{f} = xf$

parton $a$ with momentum fraction $x$ at the scale $\mu^2$ reads in PB language:

$$f_a(x,\mu^2) = \Delta_a^S(\mu^2)\, f_a(x,\mu_0^2) + \sum_b \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \frac{\Delta_a^S(\mu^2)}{\Delta_a^S(\mu'^2)} \int_x^{z_M} \frac{dz}{z}\, P_{ab}(\alpha_s(\mu'^2), z)\, f_b\left(\frac{x}{z}, \mu'^2\right) \quad (1)$$

with $\mu_0$ being the starting scale. The DGLAP splitting function for $b \to a + c$ is given by $P_{ab}(z) = P_{ab}^R(z) + \delta(1-z) D_{ab}$ with $P_{ab}^R$ being the real emission probability and $D_{ab} = \delta_{ab} d_a$ (in the notation of Ref. [15]). The parameter $z_M = 1 - \epsilon$ is needed to allow numerical integration over $z$, where $\epsilon \to 0$ is required to reproduce DGLAP. The Sudakov form factor $\Delta_a^S(\mu^2)$ is defined as:

$$\Delta_a^S(\mu^2, \mu_0^2) = \exp\left( - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_0^{z_M} dz P_{ab}(\alpha_s(\mu'^2), z) \right) \quad . \quad (2)$$

The expression of this Sudakov form factor is different from the one used in Ref. [15], since the full splitting function, including virtual contributions is applied.

Equation eq.(1) can be easily extended for transverse momentum dependent parton densities, following the same arguments given in Ref. [15]:

$$\begin{aligned} \mathcal{A}_a(x,\mathbf{k},\mu^2) &= \Delta_a^S(\mu^2)\, \mathcal{A}_a(x,\mathbf{k},\mu_0^2) + \int \frac{d^2\mathbf{q}'}{\pi \mathbf{q}'^2} \frac{\Delta_a^S(\mu^2)}{\Delta_a^S(\mathbf{q}'^2)} \Theta(\mu^2 - \mathbf{q}'^2)\, \Theta(\mathbf{q}'^2 - \mu_0^2) \\ &\times \left[ \int_x^{z_M} \frac{dz}{z}\, P_{ab}(\alpha_s(\mathbf{q}'^2), z)\, \mathcal{A}_a\left(\frac{x}{z}, \mathbf{k} + (1-z)\mathbf{q}', \mathbf{q}'^2\right) \right] \quad , \end{aligned} \quad (3)$$

In order to arrive at an evolution equation for momentum weighted parton distributions ($\tilde{\mathcal{A}} = a\mathcal{A}$ as given in Ref. [15]), the momentum sum rule for the splitting functions to eq.(1) is applied. The splitting functions in eq.(1) are the full DGLAP splitting functions.

## 3 The PB Sudakov form factor and CSS

In the following we concentrate on the Sudakov form factor given in eq.(2) and reformulate it to provide a similar structure as obtained in CSS. The splitting function $P_{ab}(\alpha_s, z)$ is given in lowest order by (in the notation of [15]):

$$P_{ab}(\alpha_s, z) = d_a(\alpha_s)\delta(1-z) + k_a(\alpha_s)\frac{1}{1-z} + R_{ab}(\alpha_s, z) \quad (4)$$

with:

$$d_q^{(0)} = \frac{3}{2} C_F \frac{\alpha_s}{2\pi} \;, \quad k_q^{(0)} = 2 C_F \frac{\alpha_s}{2\pi}, \quad (5)$$

$$d_g^{(0)} = \frac{11}{6} C_A + \frac{2}{3} T_R N_f \frac{\alpha_s}{2\pi} \;, \quad k_g^{(0)} = 2 C_A \frac{\alpha_s}{2\pi} \quad (6)$$

and $R_{ab}$ containing analytic terms for $z \to 1$. Inserting this into eq.(1) (focusing on $z \to 1$, neglecting $R_{ab}$):

$$\Delta_a^S(\mu^2, \mu_0^2) = \exp\left( - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left[ \int_0^{z_M} dz \frac{k_q}{1-z} - d_q \right] \right) \tag{7}$$

As in the CSS approach, we divide different regions of $q_T$. For that, we insert a finite resolution scale $z_{dyn} = 1 - q_0/\mu'$, which is motivated from being able to resolve partons with a transverse momentum $q_t > q_0$ in an angular ordering environment [24–26].

$$\begin{aligned} \Delta_a^S(\mu^2, \mu_0^2) &= \exp\left( - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left[ \int_0^{z_{dyn}} dz \frac{k_q}{1-z} - d_q \right] \right) \\ &\times \exp\left( - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_{z_{dyn}}^{z_M} dz \frac{k_q}{1-z} \right) \end{aligned} \tag{8}$$

The $q_T > q_0$ region or resolvable region is treated with perturbative physics. While the $q_T < q_0$, the non-resolvable region, can become non-perturbative when $q_T^2 << Q^2$.

## The $q_T > q_0$ region

The term in eq. 8 that contributes to the $q_T > q_0$ region is,

$$\Delta_a^S(\mu^2, \mu_0^2) = \exp\left( - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left[ \int_0^{z_{dyn}} dz \frac{k_q}{1-z} - d_q \right] \right). \tag{9}$$

For the simple case, with $\alpha_s$ independent on $z$, we can perform the $z$ integral analytically:

$$\begin{aligned} \log \Delta_a^S(\mu^2, \mu_0^2) &= - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left[ \int_0^{z_{dyn}} dz \frac{k_a}{1-z} - d_a \right] \\ &= - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left[ \frac{1}{2} k_a \log\left( \frac{\mu'^2}{q_0^2} \right) - d_a \right] \\ &= -\frac{1}{2} \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left[ \log\left( \frac{q_0^2}{\mu'^2} \right) A + B \right] \end{aligned} \tag{10}$$

The factors $A$ and $B$ (the global $1/2$ arises since CSS has only one Sudakov form factor, while in PB each initial parton has its own Sudakov form factor) can be identified with the CSS terms, they are given in lowest order by (at higher order correspondingly):

$$A = -k_a \qquad B = -2d_a \tag{11}$$

One could mention studies on higher orders? Make a reference to future paper from Antwerp?

## The $q_T < q_0$ region

For the $q_T < q_0$ region, if $q_T^2 << Q^2$, we are in the non-perturbative region of the TMD. We refer to the term in the Sudakov form factor that contributes at $q_T < q_0$ as the "non-perturbative" Sudakov form factor:

$$\Delta_a^{non-pert.}(\mu, \mu_0) = \exp\left(-\int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_{z_{dyn}}^{z_M} dz \frac{k_q}{1-z}\right). \tag{12}$$

For the simple case, with $\alpha_s$ independent on $z$, we can perform the $z$ integral analytically:

$$\log \Delta_a^{non-pert.} = \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_{z_{dyn}}^{z_M} dz \frac{k_a}{1-z} = -\int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \frac{1}{2} k_a \log\left(\frac{\epsilon^2 \mu^2}{q_0^2}\right) \tag{13}$$

This term arising from the $z$ integral between $z_{dyn}$ and $z_M = 1-\epsilon$ covers the region of very soft gluon emissions, with transverse momenta $q_t < q_0$ extending to the very soft region. This region while covered by the DGLAP equation is important also for inclusive distributions, as we will show below. This very soft region is labelled as the " non-perturbative Sudakov form factor " in the CSS formalism [7]. The corresponding term in eq.(13) can be rewritten in the following form:

$$\begin{aligned} \log \Delta_a^{non-pert}(\mu^2, \mu_0^2) &= -\int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \frac{1}{2} k_q \log\left(\frac{q_0^2}{\mu'^2 \epsilon^2}\right) \\ &= -\log\left(\frac{\mu^2}{\mu_0^2}\right) g_1(\alpha_s, q_0) \ , \end{aligned} \tag{14}$$

with $g_1(\alpha_s, q_0)$ is calculable analytically in the simple case when $\alpha_s$ is independent on $z$ and $\mu'$, otherwise it has to be calculated numerically as in PB.

A remark on the integrand $\log\left(\frac{q_0^2}{\mu'^2 \epsilon^2}\right)$ is needed: since both $\mu_0$ and $\epsilon$ are non-zero, the value of the $\log$ stays finite, however for $\epsilon \to 0$ it can become large, which in the exponential of the Sudakov form factor contributes only very little, leaving the whole expression stable and finite.

In the PB-approach, the non-perturbative Sudakov form factor arises from the requirement, that the $z$ integration has to extend close to one, and is not cutoff by the angular ordering requirement $z_{dyn}$. Even for collinear distributions, limiting the $z$-integration by $z_{dyn}$ will lead to non-cancellation of important terms and will result in distributions which are no longer consistent with DGLAP.

# 4 Numerical results

In Fig. 1 we show parton distributions, obtained with the DGLAP evolution package QCDnum [27] and compare it with parton distributions obtained with the PB approach using the same starting distribution. We show distributions for the gluon and down quark parton densities for different values of $z_M$: $z_M \to 1$ and $z_M = z_{dyn} = 1 - q_0/q$ with $q_0 = 2$ GeV obtained with PB. While for $z_M \to 1$ the DGLAP distributions, as obtained by QCDnum are well reproduced, applying $z_M = z_{dyn}$ leads to significant deviations over the whole range.

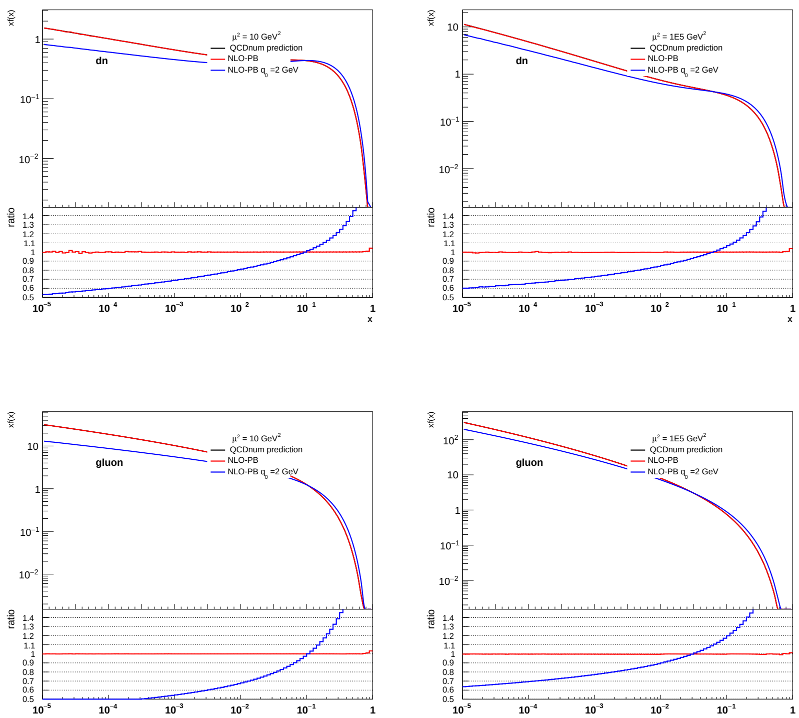


Figure 1: Integrated gluon and down-quark distributions at $\mu^2 = 10\,\text{GeV}^2$ (left column) and $\mu^2 = 10^5$ GeV$^2$ (right column) obtained from the PB approach for different values of $z_M$, compared with the result from QCDnum [27]. The ratio plots show the ratio of the results obtained with the PB approach to the result from QCDnum.

In the transverse momentum distributions, the effect of the $z_M$ cutoff is even more visible. In Fig. 2 the transverse momentum distributions obtained with the PB-approach are shown for the same configuration as in Fig. 1. In order to obtain clean configurations, no intrinsic $k_{\rm T}$ distribution is used ($q_s = 0$ GeV).

The transverse momentum distributions show very clearly the effect of applying the separation scale $z_M = z_{dyn} = 1 - q_0/q$, which was introduced in order to identify the logarithmic structure as in CSS. From the discussion on the Sudakov form factor as well as from the distributions in Fig. 2 it is obvious, that the soft, non-perturbative region plays an important role. The PB approach offers a very natural explanation of this non-perturbative region, and

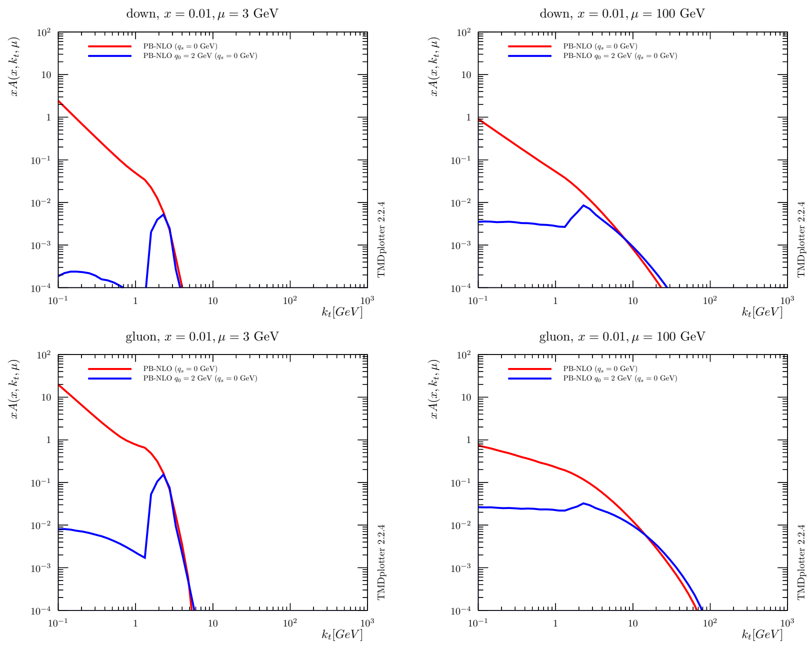


Figure 2: Transverse momentum distributions gluon and down-quarks at $\mu = 3$ GeV (left column) and $\mu = 100$ GeV (right column) obtained from the PB approach for $z_M \to 1$ as well as $z_M = z_{dyn} = 1 - q_0/q$ . Here, no intrinsic $k_{\mathrm{T}}$ distribution is included.

allows to calculate the corresponding non-perturbative Sudakov form factor explicitly.

# 5 Conclusion

Within the PB approach the main characteristics of the CSS formulation from DY production is automatically recovered when using the DGLAP splitting functions in the region of large $z$, neglecting finite and small $z$ terms. The identification of the $A$ and $B$ terms of the CSS formulation in the PB-approach requires the introduction of an angular ordering motivated soft resolution separation parameter $z_{dyn} < 1$. With this, the single and double logs of CSS are reproduced exactly up to next-to-leading log level.

The introduction of the artificial soft resolution parameter $z_{dyn}$ leaves a non-perturbative region which is automatically included in PB by the requirement to reproduce DGLAP ($z_M \to 1$) and can be calculated.

The PB approach provides a very natural explanation of the so-called non-perturbative Sudakov form factor, by the requirement, that on a collinear level, the DGLAP evolution equation is recovered exactly.

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