High-temperature effective field theories and the bubble wall speed

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How fast does the bubble grow?
DESY
17 May 2023

Overview

1. Motivation
   - The Holy Grail
   - Hierarchies of scale for phase transitions

2. High temperature dimensional reduction
   - $\phi^3 + \phi^4$
   - Real triplet scalar extended Standard Model

3. Real-time effective theories
   - Hard thermal loops
   - Langevin equations
   - ~~Effective kinetic theory~~

4. Conclusions

What is our quest?

 $\equiv$ bubble wall speed

What is our quest?

 $\equiv v_w^{(0)} + \varepsilon v_w^{(1)} + \varepsilon^2 v_w^{(2)} + O(\varepsilon^3)$

The first few orders of a controlled expansion in some small parameter,

$$\varepsilon \ll 1,$$

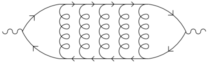
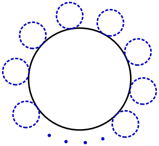
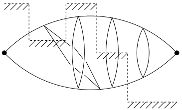
with unique and well-defined coefficients $v_w^{(n)}$, from first principles.

A direct attack is difficult

v_w is contained in the time dependence of n-point functions,

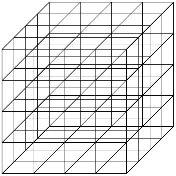
$$(\Box - V'')G(x,x') = -\delta(x-x') - \int_y \Pi(x,y)G(y,x'),$$

but these do not admit a simple perturbative expansion.

Arnold, Moore & Yaffe '03, Kapusta '79, Jeon '94

The lattice is limited



- Real-time sign problem thwarts importance sampling,

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \frac{1}{Z}\int_{\mathsf{SK}} \mathcal{D}\phi\ \mathcal{O}(t)\mathcal{O}(0)e^{iS[\phi]}.$$

- Chiral fermions, like e_{L}, e_{R}, can't be simulated on a lattice.

Nielsen & Ninomiya '81

Approach

Inspired by e.g. results using high-temperature dimensional reduction,

$$\frac{p_{\mathrm{QCD}}}{\frac{8\pi^2}{45}T^4} = p_0 + p_2 g^2 + p_3 g^3 + p_4 g^4 + p_5 g^5 + p_6 g^6 + O(g^7).$$

Zhai & Castening hep-ph/9507380
Kajantie et al. hep-ph/0211321
Hietanen et al. 0811.4664

We'll explore **effective field theory** approaches.

Hierarchies of scale for phase transitions

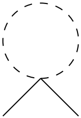
A hierarchy problem

Let's assume there is some very massive particle χ, $M_\chi \gg m_H$, coupled to the Standard Model Higgs Φ like

$$\mathscr{L} = \mathscr{L}_{\mathrm{SM}} + g^2 \Phi^\dagger \Phi \chi^\dagger \chi + \mathscr{L}_\chi .$$

If we integrate out χ, we find that the Higgs mass parameter gets a correction of the form

$$(\Delta m_H^2)\Phi^\dagger\Phi =$$



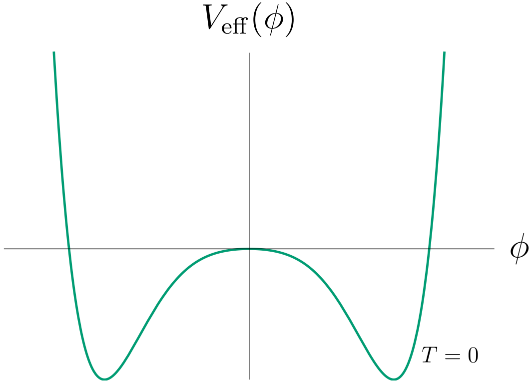
,

$$\sim g^2 M_\chi^2 \Phi^\dagger \Phi \, .$$

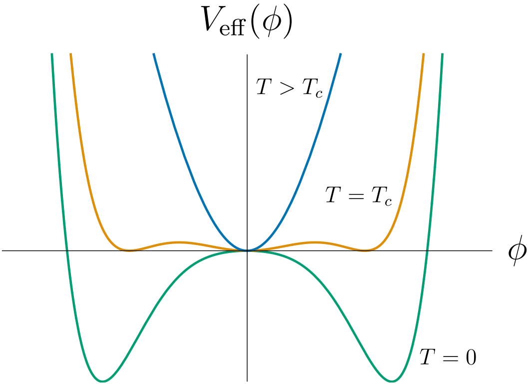
Relevant operators in the IR get large contributions from the UV,

$$\frac{\Delta m_H^2}{m_H^2} \sim g^2 \left(\frac{M_\chi}{m_H}\right)^2 .$$

Phase transitions



Phase transitions



For there to be a phase transition, thermal/quantum fluctuations should modify the potential at leading order,

$$V_{\text{eff}} = V_{\text{tree}} + \Delta V_{\text{fluct}}.$$

Hierarchies in phase transitions

So, for there to be a phase transition, we need

$$\frac{\Delta V_{\text{fluct}}}{V_{\text{tree}}} \sim g^2 N \left(\frac{\Lambda_{\text{fluct}}}{\Lambda_{\text{tree}}}\right)^{\sigma} \overset{!}{\sim} 1,$$

where $\sigma > 0$ for relevant operators.

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$\Rightarrow$ either:

(i) $g^2 N \gtrsim 1$, i.e. strong coupling

(ii) $\frac{\Lambda_{\text{fluct}}}{\Lambda_{\text{tree}}} \sim \frac{1}{(g^2 N)^{1/\sigma}} \gg 1$, i.e. scale hierarchy

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Perturbative phase transitions require scale hierarchies*

*There are some caveats.

Infrared strong coupling

Infrared bosons are highly occupied; the effective expansion parameter $\alpha_{\rm eff}$ grows

$$\alpha_{\rm eff} \sim g^2 \frac{1}{e^{E/T}-1} \approx g^2 \frac{T}{E}$$

Softer modes are classically occupied and more strongly coupled:

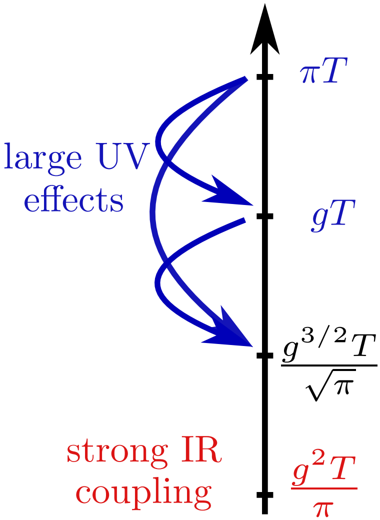
$$\begin{aligned}
\text{hard}: &\quad E \sim \pi T \Rightarrow \alpha_{\rm eff} \sim g^2 \sim 0.03, \\
\text{soft}: &\quad E \sim gT \Rightarrow \alpha_{\rm eff} \sim g \sim 0.18, \\
\text{supersoft}: &\quad E \sim g^{3/2} T \Rightarrow \alpha_{\rm eff} \sim g^{1/2} \sim 0.42, \\
\text{ultrasoft}: &\quad E \sim g^2 T \Rightarrow \alpha_{\rm eff} \sim g^0 \sim 1.
\end{aligned}$$

UV and IR problems

There are two main difficulties

- large UV effects break loop expansion
- IR becomes more strongly coupled

$$\frac{\Delta V_{\text{fluct}}}{V_{\text{tree}}} \sim \alpha_{\text{eff}} \left(\frac{\Lambda_{\text{fluct}}}{\Lambda_{\text{tree}}}\right)^{\sigma}$$



High-temperature dimensional reduction

Imaginary time formalism

- Thermodynamics $Z = \mathrm{Tr} e^{-\hat{H}/T}$ formulated in $\mathbb{R}^3 \times S^1$,

- Fields are expanded into Fourier modes:

$$\Phi(\mathbf{x}, \tau) = \sum_n \phi_n(\mathbf{x}) e^{i(n\pi T)\tau}$$

where n is even (odd) for bosons (fermions).

Matsubara modes

Substituting in the Fourier expansion (here for a scalar),

$$\int_0^{1/T} d\tau \int_x \left[\frac{1}{2}\overbrace{\Phi(x,\tau)}^{d+1}(-\nabla^2 - \partial_\tau^2 + m^2)\Phi(x,\tau)\right] =$$

$$\frac{1}{T}\sum_{n \text{ even}} \int_x \left[\frac{1}{2}\underbrace{\phi_n(x)}_{d}(-\nabla^2 + (n\pi T)^2 + m^2)\phi_n(x)\right].$$

The masses of the Fourier modes are

$$m_n^2 = (n\pi T)^2 + m^2.$$

One can therefore view a thermal field theory in $d+1$ dimensions as a Euclidean field theory in d dimensions with infinitely many fields.

Matsubara '55, Andersen '05

A simple example

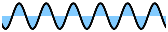
Real scalar model

A simple model,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \sigma\phi + \frac{m^2}{2}\phi^2 + \frac{\kappa}{3!}\phi^3 + \frac{g^2}{4!}\phi^4 + J_1\phi + J_2\phi^2,$$

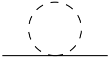
with only two relevant scales:

hard: $E \sim \pi T$ (nonzero Matsubara modes)



$$m_n^2 = m^2 + (n\pi T)^2 \text{ with } n \neq 0$$

soft: $E \sim gT$ (Debye screened)

$m_{\text{eff}}^2 \sim$  $\sim g^2 T^2$

Real scalar model

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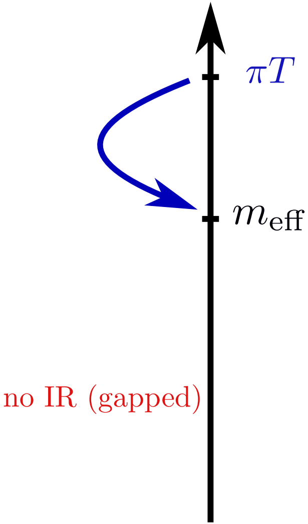
$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \sigma\phi + \frac{m^2}{2}\phi^2 + \frac{\kappa}{3!}\phi^3 + \frac{g^2}{4!}\phi^4 + J_1\phi + J_2\phi^2,$$

with only two scales: πT, $m_{\text{eff}} \sim gT$.

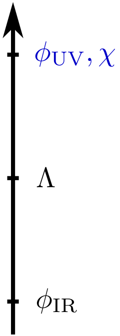
- large UV effects $(\pi T/m_{\text{eff}}) \sim \pi/g$
- IR coupling $\alpha_{\text{eff}} \sim g/\pi$



Wilsonian EFT

- Split degrees of freedom $\{\phi, \chi\}$ based on energy $\rightarrow$
- Integrate out the UV modes:

$$\int \mathcal{D}\phi \int \mathcal{D}\chi \; e^{-S[\phi,\chi]} = \int \mathcal{D}\phi_{\mathsf{IR}} \left(\int \mathcal{D}\phi_{\mathsf{UV}} \mathcal{D}\chi \; e^{-S[\phi,\chi]} \right)$$
$$= \int \mathcal{D}\phi_{\mathsf{IR}} \; e^{-S_{\mathsf{eff}}[\phi_{\mathsf{IR}}]}$$



Burgess '21, Hirvonen '22

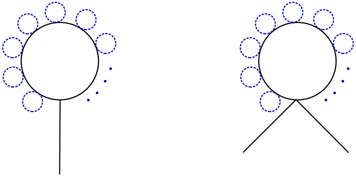
Resummations with EFT

By first integrating out the UV modes

$$S_{\text{eff}}[\phi_{\text{IR}}] = S_\phi[\phi_{\text{IR}}] - \log \int \mathcal{D}\phi_{\text{UV}}\mathcal{D}\chi \; e^{-S[\phi_{\text{IR}}+\phi_{\text{UV}},\chi]+S_\phi[\phi_{\text{IR}}]},$$

$$\approx S_\phi[\phi_{\text{IR}}] + \int_x \left[(\sigma_{\text{eff}} - \sigma)\phi_{\text{IR}} + \frac{1}{2}(m^2_{\text{eff}} - m^2)\phi^2_{\text{IR}}\right],$$

the daisy resummations arise naturally.



So do all other necessary resummations, order by order.

$\Rightarrow$ Solves UV problems

EFT factorisation

Contributions to physical quantities factorise

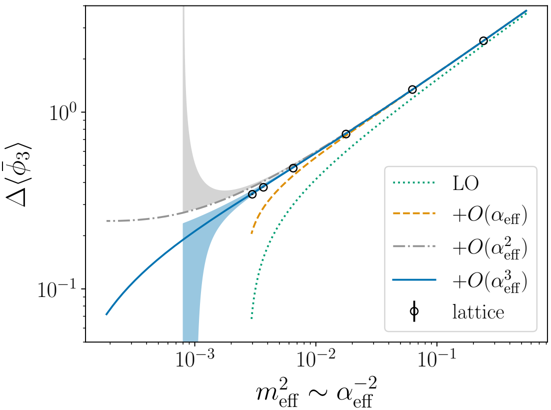
$$L = \underbrace{\frac{d\sigma_{\text{eff}}}{d\log T}}_{\text{hard modes}} \underbrace{\Delta\langle\bar{\phi}_{\text{IR}}\rangle}_{\text{soft modes}},$$

$$= \underbrace{(A + Bg^2 + O(g^4))}_{\text{hard modes}} \times \underbrace{(a + bg^1 + cg^2 + dg^3 + O(g^4))}_{\text{soft modes}}.$$

One must work harder for the soft modes,

$$\alpha_{\text{eff}} \approx \frac{g^2 T}{(4\pi)m_{\text{eff}}} \sim \frac{g}{(4\pi)}.$$

Lattice vs perturbation theory: real scalar model



$$\Delta\langle\bar{\phi}_{\rm IR}\rangle = \frac{1}{4\pi\alpha_{\rm eff}}\Big[\, 2 + \sqrt{3}\,\alpha_{\rm eff} + \frac{1}{2}\left(1 + 2\log\tilde{\mu}_3\right)\alpha^2_{\rm eff} + \sqrt{3}\left(-\frac{3}{8\sqrt{2}}\xi + \frac{21}{32}{\rm Li}_2\frac{1}{4} - \frac{7\pi^2}{128} - \frac{1}{2} + \frac{21}{64}\log^2\frac{4}{3} + \frac{5}{8}\log\frac{4}{3}\right)\alpha^3_{\rm eff} + O\left(\alpha^4_{\rm eff}\right)\Big]$$

OG 2101.05528

Some more complicated examples

Standard Model-like example

A more complicated model with more scale hierarchies,

$$\mathscr{L}_{\mathrm{SM}} \subset (D_\mu \Phi)^\dagger (D_\mu \Phi) - \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2.$$

At leading order, the high-T potential is

$$V_{\mathrm{eff}} \approx \frac{1}{2} m_{\mathrm{eff}}^2 \phi^2 + \frac{\lambda}{4} \phi^4.$$

This appears to have a 2$^{\mathrm{nd}}$ order transition as

$$m_{\mathrm{eff}} \sim -\mu^2 + g^2 T^2 = 0.$$

Before this happens, a 3$^{\mathrm{rd}}$ term comes to balance against these, as a new scale appears:

supersoft: $E \sim g^{3/2} T / \sqrt{\pi}$ (symmetry breaking)

$$V_{\mathrm{eff}} \approx \frac{1}{2} m_{\mathrm{eff}}^2 \phi^2 - \frac{T g^3}{16\pi} |\phi|^3 + \frac{\lambda}{4} \phi^4$$

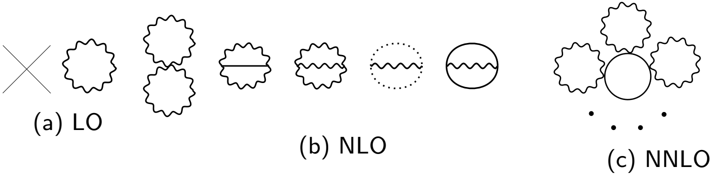
Arnold & Espinosa '92

Supersoft scale EFT

Integrating out the scales πT and gT gives

$$\mathscr{L}_{\text{eff}} = \frac{1}{2}\partial_i\varphi^\dagger\partial_i\varphi + \frac{m_3^2}{2}\varphi^\dagger\varphi - \frac{g_3^3}{4(4\pi)}(\varphi^\dagger\varphi)^{3/2} + \frac{\lambda_3}{4}(\varphi^\dagger\varphi)^2$$
$$- \frac{11g_3}{8(4\pi)}\frac{\partial_i\varphi^\dagger\partial_i\varphi}{(\varphi^\dagger\varphi)^{1/2}} - \frac{51}{64}\frac{g_3^4}{(4\pi)^2}\varphi^\dagger\varphi\log\frac{g_3^2\varphi^\dagger\varphi}{\tilde{\mu}_3^2}$$

After integrating out the scale πT, the relevant diagrams are



UV and IR in concert

For some observable $\mathcal{O}$ at $T = 0$

$$\mathcal{O}_0 = \underbrace{A}_{\text{0-loop}} + \underbrace{Bg^2}_{\text{1-loop}} + \underbrace{Cg^4}_{\text{2-loop}} + \underbrace{Dg^6}_{\text{3-loop}} + \underbrace{Eg^8}_{\text{4-loop}} + \dots$$

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At a Higgs-like first-order phase transition, instead

$$\mathcal{O}_T = \underbrace{a}_{\text{1-loop}^+} + \underbrace{bg^1}_{\text{2-loop}^+} + \underbrace{cg^{3/2}}_{\text{1-loop}^\dagger} + \underbrace{dg^2}_{\text{3-loop}^+} + \underbrace{eg^{5/2}}_{\text{3-loop}^\dagger} + \underbrace{fg^3}_{\infty\text{-loop}} + \dots$$

where $^+$ and † refer to different resummations of infinite classes of diagrams.

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where $^+$ and † refer to different resummations of infinite classes of diagrams.

$$\frac{\Delta\langle\Phi^\dagger\Phi\rangle}{g_3^2 T} = \frac{1}{2(8\pi x)^2}\left[1 + \frac{51}{2}x + 13\sqrt{2}x^{3/2} + O(x^2)\right],$$

where $x \equiv \lambda_3/g_3^2 \sim g$.

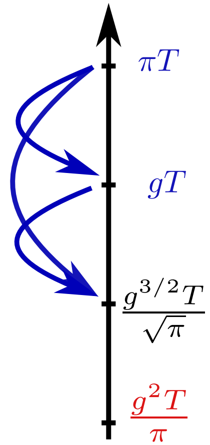
Ekstedt, OG & Löfgren 2205.07241

Scalar triplet extension of the Standard Model

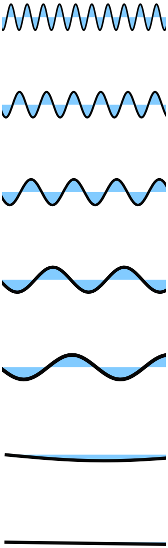
Let's also add a BSM SU(2)-triplet scalar,

$$\mathscr{L} = \mathscr{L}_{\mathsf{SM}} + \frac{a_2}{2}\Phi^\dagger\Phi\Sigma^a\Sigma^a$$
$$+ \frac{1}{2}D_\mu\Sigma^a D_\mu\Sigma^a + \frac{m_\Sigma^2}{2}\Sigma^a\Sigma^a + \frac{b_4}{4}(\Sigma^a\Sigma^a)^2.$$

- large UV effects
- strongly coupled IR



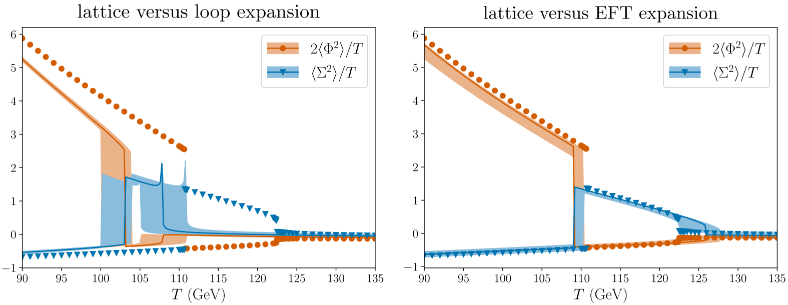
Scales in the free energy



Boltzmann: $E \gg T$	$\sim e^{-E/T}$
hard: $E \sim T$	$T^4(1 + g^2 + g^4 + \dots)$
semisoft: $E \sim \sqrt{g}\,T$	$T^4(g + g^{3/2} + g^{5/2} + \dots)$
soft: $E \sim gT$	$T^4(g^2 + g^3 + g^4 + \dots)$
supersoft: $E \sim g^{3/2}T$	$T^4(g^3 + g^4 + g^{9/2} + \dots)$
ultrasoft: $E \sim g^2 T$	$T^4(g^6 + g^6 + g^6 + \dots)$
background	

(typically considered)

Lattice versus perturbation theory



Scalar triplet extension of Standard Model,

$$\mathscr{L} = \mathscr{L}_{\mathsf{SM}} + \frac{a_2}{2}\Phi^\dagger\Phi\Sigma^a\Sigma^a + \frac{1}{2}D_\mu\Sigma^a D_\mu\Sigma^a + \frac{m_\Sigma^2}{2}\Sigma^a\Sigma^a + \frac{b_4}{4}(\Sigma^a\Sigma^a)^2$$

Niemi et al. 2005.11332, OG & Tenkanen forthcoming

Convergence and scales

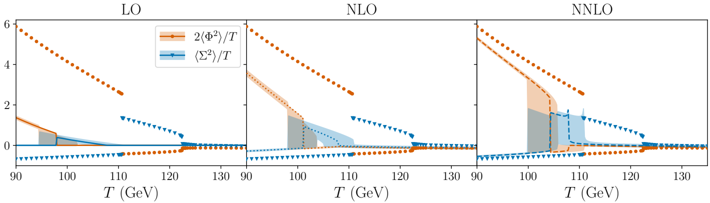


Figure: Assuming the scalars lie at the soft scale gT.

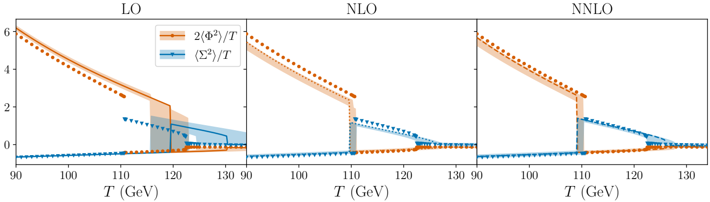
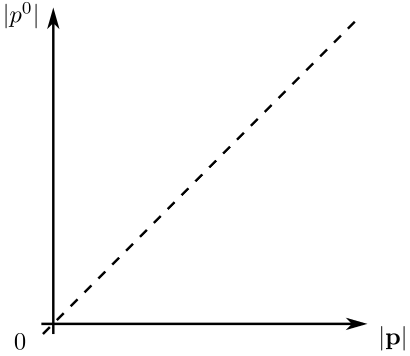


Figure: Assuming the scalars lie at the supersoft scale $g^{3/2}T/\sqrt{\pi}$.

Niemi et al. 2005.11332, OG & Tenkanen forthcoming

Real-time effective theories

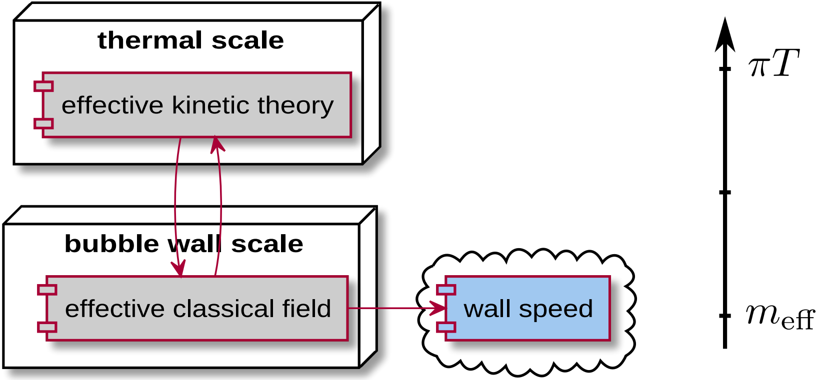
Scale hierarchies in real time



More possible scale hierarchies:

- $|p^0|, |\mathbf{p}| \ll \Lambda$
- $|\mathbf{p}| \ll \Lambda$, $|p^0| \sim \Lambda$
- $|p^0| \ll \Lambda$, $|\mathbf{p}| \sim \Lambda$
- $||p^0| - |\mathbf{p}|| \ll \Lambda$, $|p^0|, |\mathbf{p}| \sim \Lambda$

Outline bubble wall speed computation



Hard thermal loops

Real-time Wilsonian effective actions

Consider our Euclidean effective action from earlier,

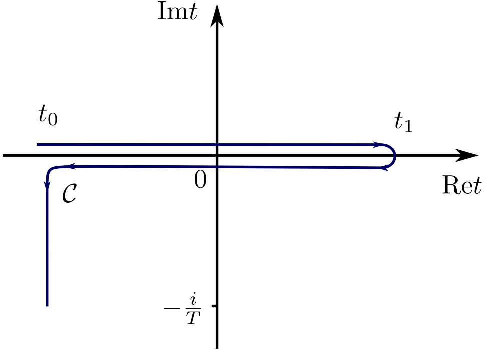
$$S_{\text{eff}}[\phi_{\text{IR}}] = S_\phi[\phi_{\text{IR}}] - \log \int \mathcal{D}\phi_{\text{UV}}\mathcal{D}\chi \; e^{-S[\phi_{\text{IR}}+\phi_{\text{UV}},\chi]+S_\phi[\phi_{\text{IR}}]}$$

How can we generalise this to real-time?

- consider soft external modes: $|p^0|, |\mathbf{p}| \sim gT$
- Integrate over hard internal loops $k_n, |\mathbf{k}| \sim \pi T$
- Taylor expand final result in soft quantities.

see e.g. Laine & Vuorinen '17

Quantum thermal evolution



$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle_{\mathsf{qm}} = \frac{1}{Z}\mathrm{Tr}\left[e^{-\hat{H}/T}\left(e^{i\hat{H}t}\mathcal{O}(0)e^{-i\hat{H}t}\right)\mathcal{O}(0)\right]$$
$$= \int_{\mathcal{C}} \mathcal{D}\phi\,\mathcal{O}(t)\mathcal{O}(0)e^{iS[\phi]}$$

Classicalisation

- Bose enhancement of IR modes

$$n_{\mathrm{B}}(E) = \frac{1}{e^{E/T} - 1}, \approx \frac{T}{E} \gg 1.$$

- Dynamics of QFT at nucleation scale ($\Lambda_{\mathrm{nucl}} \ll T$) expected to be quasi-classical.

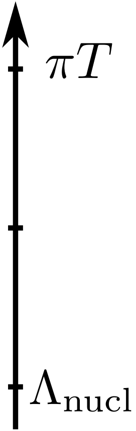
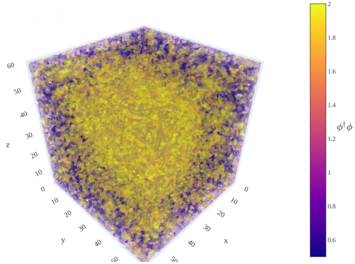


Figure: Nucleation scale much lower than thermal scale.

Classical thermal evolution

To determine real-time correlation functions,

$$\dot{\phi}(t,x) = \{\phi(t,x), H\},$$
$$\dot{\pi}(t,x) = \{\pi(t,x), H\},$$
$$\langle \phi(0,x_1)\phi(0,x_2)\rangle_{\mathrm{cl}} \equiv \frac{1}{Z_{\mathrm{cl}}}\int \mathcal{D}\phi\mathcal{D}\pi\,\phi(0,x_1)\phi(0,x_2)e^{-H[\phi,\pi]/T}.$$



UV catastrophe - the cut-off scale dominates everything!

Quantum versus classical

- Using counterterms from dimensional reduction, e.g. in $g_3^2\phi^4$

$$\delta m_3^2 = \frac{g_3^4}{24(4\pi)^2\epsilon}$$

  cancels classical UV catastrophe, giving finite result.

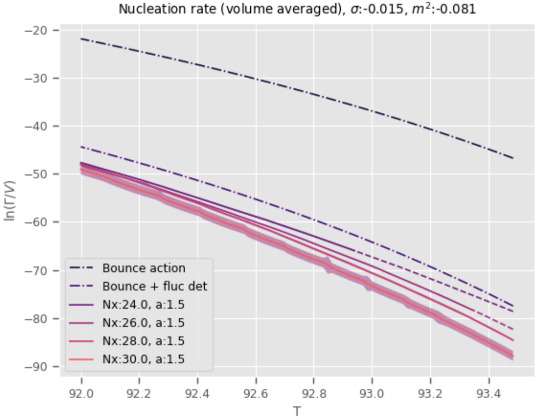
- Moreover, the finite classical and quantum remainders agree!

$$\langle \phi(t_1, x_1)\phi(t_2, x_2)\rangle_{\mathrm{cl}} \approx \langle \{\phi(t_1, x_1), \phi(t_2, x_2)\}\rangle_{\mathrm{qm}}$$

  N.B. in the classical evolution equations, the relevant potential is the **tree-level** potential for the EFT.

Aarts & Smit '97, Bödeker '97

Benchmarking against the lattice



$$\frac{\Gamma}{V} \sim A e^{-B}$$

$$H_{\text{eff}} = \int \mathrm{d}^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i\phi)^2 + \sigma_{\text{eff}}\phi + \frac{1}{2}(m_{\text{eff}}^2 + \delta m_{\text{eff}}^2)\phi^2 + \frac{g_{\text{eff}}^2}{4!}\phi^4\right]$$

OG, Kormu & Weir (forthcoming), Moore, Rummukainen & Tranberg '01

Soft gauge fields

The above strategy is more complicated for gauge fields:

- Thermal initial conditions are as before, based on the 3d EFT.
- But the evolution equations in the classical limit become nonlocal, here shown for the Abelian-Higgs model,

$$\partial_\mu F^{\mu\nu} = \frac{e^2 T^2}{3} \int \frac{d\Omega_v}{4\pi} \frac{v^\nu v^i}{v \cdot \partial} E^i + 2ie(\phi^* D^\nu \phi - \phi D^\nu \phi^*),$$
$$D_\mu D^\mu \phi = -m_T^2 \phi - 2\lambda(\phi^* \phi)\phi.$$

Recent developments in hard-thermal loops

The hard-thermal loop effective theory for gauge fields (here shown for QED)

$$\partial_\mu F^{\mu\nu} = \Pi^{\mu\nu} A_\mu,$$

has been recently extended to NLO,

$$\Pi^{\mu\nu}_{\mathrm{LO}}(K) = -\frac{e^2 T^2}{3} \int \frac{d\Omega_v}{4\pi} \left[n^\mu n^\nu + v^\mu v^\nu \frac{k_0}{v \cdot K} \right],$$

$$\Pi^{\mu\nu}_{\mathrm{NLO}}(K) = -\frac{e^4 T^2}{8\pi^2} \int \frac{d\Omega_v}{4\pi} \left\{ v^\mu v^\nu \left[\frac{(k^0)^2}{(v \cdot K)^2} - \frac{2k^0}{v \cdot K} \right] + \left[v^\mu n^\nu + n^\mu v^\nu \right] \frac{k^0}{v \cdot K} - n^\mu n^\nu \right\}.$$

Carignano et al. '20, Ekstedt '23, Gorda et al. '23

Langevin equations

Influence functional

Split the field based on spatial momentum

$$\Phi(t,\mathbf{p}) = \underbrace{\theta(\Lambda - |\mathbf{p}|)\Phi(t,\mathbf{p})}_{\Phi_{\rm IR}} + \underbrace{\theta(|\mathbf{p}| - \Lambda)\Phi(t,\mathbf{p})}_{\Phi_{\rm UV}},$$

Bödeker, McLerran & Smilga '95, Lombardo & Mazzitelli '95

and integrate over the UV modes,

$$\int \mathcal{D}\Phi\mathcal{D}\Phi' \rho_i e^{i(S[\Phi]-S[\Phi'])}$$

$$= \int \mathcal{D}\Phi_{\rm IR}\mathcal{D}\Phi'_{\rm IR}\rho_{{\rm IR},i} e^{i(S[\Phi_{\rm IR}]-S[\Phi'_{\rm IR}]+S_{\rm IF}[\Phi_{\rm IR},\Phi'_{\rm IR}])}.$$

The influence functional $S_{\rm IF}$ gives the effect of the UV modes, in the in-in formalism.

Feynman & Vernon '63

Complex influence functionals

In general the evolution of the IR modes is nonunitary

$$e^{i(S[\Phi_{\mathrm{IR}}]-S[\Phi'_{\mathrm{IR}}]+\mathrm{Re}S_{\mathrm{IF}}[\Phi_{\mathrm{IR}},\Phi'_{\mathrm{IR}}])} \times e^{-\mathrm{Im}S_{\mathrm{IF}}[\Phi_{\mathrm{IR}},\Phi'_{\mathrm{IR}}]}.$$

This complicates the naive semiclassical limit,

$$\frac{\delta}{\delta\Phi_{\mathrm{IR}}}(S[\Phi_{\mathrm{IR}}] + \mathrm{Re}S_{\mathrm{IF}}[\Phi_{\mathrm{IR}},\Phi'_{\mathrm{IR}}] + i\mathrm{Im}S_{\mathrm{IF}}[\Phi_{\mathrm{IR}},\Phi'_{\mathrm{IR}}]) = 0 \text{ ???}$$

Stochastic semiclassical limit

A possible solution is to introduce new stochastic variables, e.g.

$$e^{-\text{Im}S_{\text{IF}}[\Phi_{\text{IR}},\Phi'_{\text{IR}}]} \to e^{-\frac{1}{2}\Phi_{\text{IR}}\cdot\mathcal{I}_2\cdot\Phi_{\text{IR}}}, \\ = \frac{1}{\sqrt{\det\mathcal{I}}}\int \mathcal{D}\chi e^{-\frac{1}{2}\chi\cdot\mathcal{I}_2^{-1}\cdot\chi} e^{i\chi\cdot\Phi_{\text{IR}}},$$

where $a \cdot M \cdot b \equiv \int_x \int_y a(x)M(x,y)b(y)$. The effective action for Φ_{IR} is then real.

The semiclassical equations of motion become Langevin,

$$\frac{\delta}{\delta\Phi_{\text{IR}}}S[\Phi_{\text{IR}}] + \frac{\delta}{\delta\Phi_{\text{IR}}}\text{Re}S_{\text{IF}}[\Phi_{\text{IR}},\Phi'_{\text{IR}}] = \chi, \\ \langle\chi(x)\chi(y)\rangle = \mathcal{I}_2(x,y).$$

Effective stochastic $\lambda\phi^4$

Explicitly, for the $\lambda\phi^4$ theory,

$$-\Box\phi_{\rm IR}(x) + m^2\phi_{\rm IR}(x) + \lambda\phi_{\rm IR}(x)^3 + \int_{t_i}^{t} d^4y \,{\rm Re}\Gamma^{(2)}(x-y)\phi_{\rm IR}(y) = \chi(x) + \dots$$

where the stochastic variable satisfies

$$\langle\chi(x)\chi(y)\rangle = {\rm Im}\Gamma^{(2)}(x-y),$$

and where $\Gamma^{(2)}$ is the UV contribution to the IR self-energy. Here we have made an expansion in powers of $\phi_{\rm IR}$.

Greiner & Müller '·97

Time evolution of ultrasoft gauge bosons

Starting from HTLs for gauge fields, one can integrate out the soft scale, to arrive at an effective theory for the ultrasoft scale.

soft: $E \sim gT$

ultrasoft: $E \sim g^2 T/\pi$

To leading-log order the result is first-order Langevin,

$$(D_t A_i)^a = -\gamma \frac{\delta S_3}{\delta A_i^a} + \xi_i^a$$

where S_3 is the Euclidean action of the 3d EFT, $\gamma \sim \log(1/g)/T$ is the colour damping, and ξ_i is a Gaussian noise satisfying

$$\langle \xi_i^a(t, \mathbf{x}) \xi_j^b(u, \mathbf{y}) \rangle = 2\gamma \delta_{ij} \delta^{ab} \delta(\mathbf{x} - \mathbf{y}) \delta(t - u).$$

Bodeker '98

Time evolution of gauge-Higgs system

For gauge-Higgs theory, the coupled Langevin equations read

$$(D_t A_i)^a = -\gamma \frac{\delta S_3}{\delta A_i^a} + \xi_i^a,$$
$$D_t \phi = -\eta\gamma \frac{\delta S_3}{\delta \phi^\dagger} + \xi_\phi$$

where the Higgs noise terms satisfies

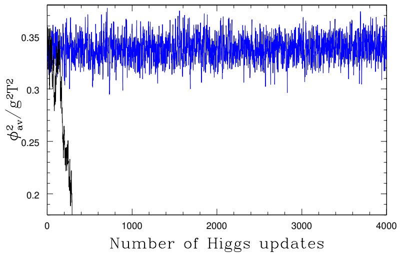
$$\langle \xi_\phi(t, \mathbf{x}) \xi_\phi^\dagger(u, \mathbf{y}) \rangle = 2\eta\gamma \mathbb{1} \delta(\mathbf{x} - \mathbf{y}) \delta(t - u),$$

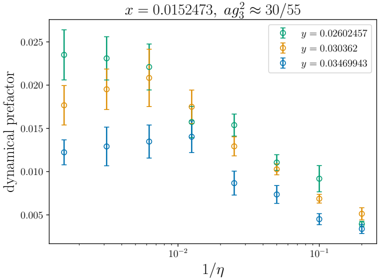
with $\eta \sim 1/g^2 \gg 1$, so that the Higgs evolves faster.

Bodeker '98, Moore '00

Gauge-Higgs coupled dynamics

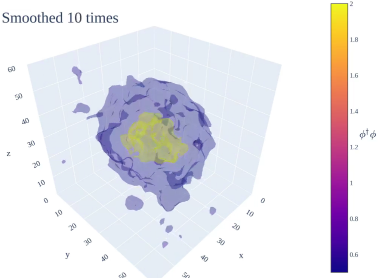
- Gauge-Higgs Langevin system tightly coupled.
- Even updating the Higgs infinitely fast, the system gets stuck.

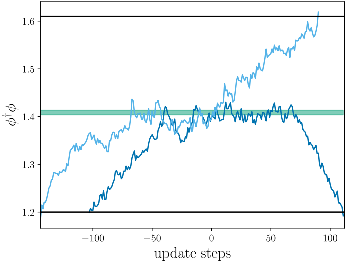




Moore & Rummukainen '00, OG, Güyer & Rummukainen '22

Early-time bubble dynamics





https://zenodo.org/record/6548608

Conclusions

- Goal is a controlled expansion,

$$= v_w^{(0)} + \varepsilon v_w^{(1)} + \varepsilon^2 v_w^{(2)} + O(\varepsilon^3)$$

- EFT can give accurate T_c, L/T^4
- Also for T_*, α_* and β/H_* (see Joonas's talk)
- Real-time physics has different EFTs in different regimes:
  - Hard-thermal loops
  - Langevin equations
  - Effective kinetic theory
- Recently developments to higher orders

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Thanks for listening!