

BFKL equation and anomalous dimensions in $N = 4$ SUSY

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1 BFKL equation (1975)

Impact parameter coordinates and momenta

$$\rho_k = x_k + iy_k, \quad \rho_k^* = x_k - iy_k, \quad p_k = i\frac{\partial}{\partial \rho_k}, \quad p_k^* = i\frac{\partial}{\partial \rho_k^*}$$

Balitsky-Fadin-Kuraev-Lipatov equation

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E$$

BFKL Hamiltonian

$$H_{12} = \ln |p_1 p_2|^2 + \frac{1}{p_1 p_2^*} (\ln |\rho_{12}|^2) p_1 p_2^* \\ + \frac{1}{p_1^* p_2} (\ln |\rho_{12}|^2) p_1^* p_2 - 4\psi(1), \quad \rho_{12} = \rho_1 - \rho_2$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \rightarrow \frac{a\rho_k + b}{c\rho_k + d},$$

$$m = 1/2 + i\nu + n/2, \quad \tilde{m} = 1/2 + i\nu - n/2$$

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n) = H \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n), \quad H = \sum_{k < l} \frac{\vec{T}_k \vec{T}_l}{-N_c} H_{kl}$$

2 Integrability

Holomorphic factorization for $N_c \rightarrow \infty$ (L. (1988))

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, \dots, \vec{\rho}_n) = \sum_{r,s} a_{r,s} \Psi_r(\rho_1, \dots, \rho_n) \Psi_s(\rho_1^*, \dots, \rho_n^*)$$

Duality symmetry (L. (1998))

$$\rho_{r,r+1} \rightarrow p_r \rightarrow \rho_{r-1,r}$$

Simplest integral of motion

$$A = q_n = \rho_{12}\rho_{23}\dots\rho_{n1} p_1 p_2 \dots p_n , \quad [h, A] = 0$$

Transfer and monodromy matrices (L. (1993))

$$T(u) = \text{tr } t(u) , \quad t(u) = L_1 L_2 \dots L_n = \sum_{r=0}^n u^{n-r} q_r ,$$

$$L_k = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k \end{pmatrix} , \quad \hat{l} = u \hat{1} + i \hat{P}$$

Yang-Baxter equation (L. (1993))

$$t_{r'_1}^{s_1}(u) t_{r'_2}^{s_2}(v) l_{r'_1 r'_2}^{r'_1 r'_2}(v-u) = l_{s'_1 s'_2}^{s_1 s_2}(v-u) t_{r'_2}^{s'_2}(v) t_{r'_1}^{s'_1}(u)$$

3 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4 \hat{a} \chi(n, \gamma) + 4 \hat{a}^2 \Delta(n, \gamma), \quad \hat{a} = g^2 N_c / (16\pi^2)$$

Non-analytic terms in QCD (K.,L. (2000))

$$\Delta_{QCD}(n, \gamma) = c_0 \delta_{n,0} + c_2 \delta_{n,2} + \dots$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}, \quad M = \gamma + \frac{|n|}{2},$$

$$\rho(M) = \beta'(M) + \frac{1}{2} \zeta(2), \quad \beta'(z) = \frac{1}{4} \left[\Psi' \left(\frac{z+1}{2} \right) - \Psi' \left(\frac{z}{2} \right) \right]$$

Maximal transcedentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M) \left(\Psi(1) - \Psi(M) \right)$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M} \left(\Psi'(k+1) - \frac{\Psi(k+1) - \Psi(1)}{k+M} \right)$$

4 $N = 4$ anomalous dimensions

Anomalous dimension matrix (L. (1997))

$$\begin{aligned}\gamma_{gg} &= -\frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} + 8S_1(j), \quad \gamma_{q\varphi} = -\frac{16}{j}, \\ \gamma_{gq} &= -\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1}, \quad \gamma_{g\varphi} = -\frac{8}{j-1} + \frac{8}{j}, \\ \gamma_{qg} &= -\frac{16}{j} + \frac{32}{j+1} - \frac{32}{j+2}, \quad \gamma_{qq} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j), \\ \gamma_{\varphi g} &= -\frac{24}{j+1} + \frac{24}{j+2}, \quad \gamma_{\varphi q} = -\frac{12}{j+1}, \quad \gamma_{\varphi\varphi} = 8S_1(j), \\ \tilde{\gamma}_{gg} &= -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j), \quad \tilde{\gamma}_{gq} = -\frac{8}{j} + \frac{4}{j+1}, \\ \tilde{\gamma}_{qg} &= \frac{16}{j} - \frac{32}{j+1}, \quad \tilde{\gamma}_{qq} = \frac{8}{j} - \frac{8}{j+1} + 8S_1(j)\end{aligned}$$

Diagonalization in the Born Approximation

$$\left| \begin{array}{ccc} S_1(j-2) & 0 & 0 \\ 0 & S_1(j) & 0 \\ 0 & 0 & S_1(j+2) \end{array} \right|, \left| \begin{array}{ccc} S_1(j-1) & 0 & 0 \\ 0 & 0 & S_1(j+1) \end{array} \right|$$

Integrable Heisenberg spin model (L. (1997))

5 Maximal transcendentality

Most transcendental functions (K.,L. (2002))

$$\gamma(j) = \hat{a}\gamma_1(j) + \hat{a}^2\gamma_2(j) + \hat{a}^3\gamma_3(j) + \dots, \quad \gamma_1(j+2) = -4S_1(j)$$

Two-loop dimension (K.,L.,V. (2003))

$$\frac{\gamma_2(j+2)}{8} = 2S_1(S_2 + S_{-2}) - 2S_{-2,1} + S_3 + S_{-3}$$

Three-loop dimension (K.,L.,O.,V. (2004))

$$\begin{aligned} \gamma_3(j+2)/32 &= -12(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}) \\ &+ 6(S_{-4,1} + S_{-3,2} + S_{-2,3}) - 3S_{-5} - 2S_3S_{-2} - S_5 \\ &- 2S_1^2(3S_{-3} + S_3 - 2S_{-2,1}) - S_2(S_{-3} + S_3 - 2S_{-2,1}) \\ &+ 24S_{-2,1,1,1} - S_1(8S_{-4} + S_{-2}^2 + 4S_2S_{-2} + 2S_2^2) \\ &- S_1(3S_4 - 12S_{-3,1} - 10S_{-2,2} + 16S_{-2,1,1}), \end{aligned}$$

$$S_a(j) = \sum_{m=1}^j \frac{1}{m^a}, \quad S_{a,b,c,\dots}(j) = \sum_{m=1}^j \frac{1}{m^a} S_{b,c,\dots}(m),$$

$$S_{-a}(j) = \sum_{m=1}^j \frac{(-1)^m}{m^a}, \quad S_{-a,b,\dots}(j) = \sum_{m=1}^j \frac{(-1)^m}{m^a} S_{b,\dots}(m),$$

$$\overline{S}_{-a,b,c,\dots}(j) = (-1)^j S_{-a,b,\dots}(j) + S_{-a,b,\dots}(\infty) \left(1 - (-1)^j \right)$$

6 Resummations of $\gamma(j)$

Singularities at $j = 1 + \omega \rightarrow 1$

$$\gamma(j) = 4\frac{\hat{a}}{\omega} + 0\frac{\hat{a}^2}{\omega} + 32\zeta_3\frac{\hat{a}^3}{\omega} - 16\left(32\zeta_3 + \frac{\pi^4}{9}\omega\right)\frac{\hat{a}^4}{\omega^4}$$

DL resummation at $j + 2r = \omega \rightarrow 0$

$$\begin{aligned} \gamma(j)(\omega - \gamma(j)) &= 4\hat{a}(1 - \omega S_1 - \omega^2(S_2 + \zeta_2)) \\ &+ 16\hat{a}^2(S_2 + \zeta_2 - S_1^2) + 4\hat{a}\gamma^2(j)(S_2 + S_{-2}) \end{aligned}$$

Anomalous dimensions at large j

$$\gamma_{uni} = a(z) \ln j, \quad z = \frac{\alpha N_c}{\pi} = 4\hat{a}$$

Perturbative results

$$a = -z + \frac{\pi^2}{12} z^2 - \frac{11}{720} \pi^4 z^3 + \dots$$

Gubser, Klebanov, Polyakov prediction

$$\lim_{z \rightarrow \infty} a = -z^{1/2} + \frac{3 \ln 2}{4\pi} + \dots$$

Resummation

$$\tilde{a} = -z + \frac{\pi^2}{12} \tilde{a}^2 = -z + \frac{\pi^2}{12} z^2 - \frac{1}{72} \pi^4 z^3 + \dots$$

7 BES equation (2006)

Algebraic equations and transcendentality (K.,L.)

$$a_{n,\epsilon} = \sum_{n'=1}^{\infty} K_{n,n'}(\epsilon) (\delta_{n',1} - a_{n',\epsilon}) , \quad \epsilon = \frac{1}{g\sqrt{2}},$$

$$K_{n,n'}(\epsilon) = 2n \sum_{R=0}^{\infty} i^{n+n'} (2\epsilon)^{-2R-n-n'} \zeta(2R+n+n') \frac{(2R+n+n'-1)! (2R+n+n')!}{R! (R+n)! (R+n')! (R+n+n')!}, \quad a(z) = \frac{2(1-a_{1,\epsilon})}{\epsilon^2}$$

Another representation of $K_{n,n'}(\epsilon)$

$$K_{n,n'}(\epsilon) \sim \sum_{k=1}^{\infty} k^{-n-n'} {}_4F_3 \left(\bar{n} + \frac{1}{2}, \bar{n}, \dots, 2\bar{n} + 1; \frac{4}{k^2 \epsilon^2} \right)$$

Essential singularity at $z = 0$ and $g \rightarrow \infty$

$$\chi_{sing}^{BES}(z) = \sum_{k=1}^{\infty} \frac{d_k}{z^k} = -\frac{i}{I_0(2\epsilon^{-1})} \int_L \frac{dz'}{2\pi i} \frac{\exp \frac{z'^2-1}{i\epsilon z'}}{z-z'}$$

Agreement with the AdS/CFT prediction (K.,L.)

$$\gamma_{sing} = \frac{2}{\epsilon} \frac{I_1(2\epsilon^{-1})}{I_0(2\epsilon^{-1})} \approx 2\sqrt{2}g - \frac{1}{2}$$

8 Pomeron and graviton

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D \nu^2$$

Anomalous dimension of twist-2 operators

$$\gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the conservation of $T_{\mu\nu}$

$$\gamma = (j-2) \left(\frac{1}{2} - \frac{1/\Delta}{1 + \sqrt{1 + (j-2)/\Delta}} \right)$$

AdS/CFT for the graviton Regge trajectory

$$j = 2 + \frac{\alpha'}{2} t, \quad t = E^2/R^2, \quad \alpha' = \frac{R^2}{2} \Delta$$

Gubser, Klebanov and Polyakov prediction

$$\gamma|_{z,j \rightarrow \infty} = -\sqrt{j-2} \Delta_{|j \rightarrow \infty}^{-1/2} = \sqrt{\pi j} z^{1/4}$$

Pomeron intercept at large α (K.,L.,O.,V.)

$$j = 2 - \Delta, \quad \Delta = \frac{1}{\pi} z^{-1/2} \approx \frac{\sqrt{3}}{2\pi} z^{-1/2},$$

$$\frac{\pi^2}{6} z = -\tilde{b} + \frac{1}{2} \tilde{b}^2, \quad b = \gamma'(2) = -\frac{\pi^2}{6} z + \frac{\pi^4}{72} z^2 - \frac{\pi^6}{540} z^3$$

9 Integrability approach

All-loop asymptotic Bethe Ansatz (BS (2005))

$$\left(\frac{x_k^+}{x_k^-} \right)^J = \prod_{r=1}^M \frac{x_k^- - x_r^+}{x_k^+ - x_r^-} \frac{1 - g^2/x_k^+ x_r^-}{1 - g^2/x_k^- x_r^+} \exp(2i\theta(u_k, u_r))$$

Bethe roots

$$x_k^\pm = x(u_k^\pm), \quad u^\pm = u \pm \frac{i}{2}, \quad x(u) = \frac{u}{2} \left(1 + \sqrt{1 - 4 \frac{g^2}{u^2}} \right)$$

Gauge invariance condition

$$\prod_{k=1}^M \frac{x_k^+}{x_k^-} = 1, \quad M = j - 2$$

Dressing phase (BES (2006))

$$\theta(u_k, u_j) = 4\zeta(3)g^6(q_2(u_k)q_3(u_j) - q_3(u_k)q_2(u_j)) + O(g^8)$$

Expression for anomalous dimensions

$$\gamma(g, M) = 2g^2 \sum_{k=1}^M \left(\frac{i}{x_k^+} - \frac{i}{x_k^-} \right)$$

10 Four-loop result (KLRSV)

$$\begin{aligned}
\frac{\gamma_4}{256} = & 4\mathbf{S}_{-7} + 6\mathbf{S}_7 + 2(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1}) \\
& + 3(-S_{-2,5} + S_{-2,3,-2}) + 4(S_{-2,1,4} - S_{-2,-2,-2,1} - S_{-2,1,2,-2} \\
& - S_{-2,2,1,-2} - S_{1,-2,1,3} - S_{1,-2,2,2} - S_{1,-2,3,1}) + 5(-S_{-3,4} + \\
& S_{-2,-2,-3}) + 6(-S_{5,-2} + S_{1,-2,4} - S_{-2,-2,1,-2} - S_{1,-2,-2,-2}) \\
& + 7(-S_{-2,-5} + S_{-3,-2,-2} + S_{-2,-3,-2} + S_{-2,-2,3}) + 8(S_{-4,1,2} \\
& + S_{-4,2,1} - S_{-5,-2} - S_{-4,3} - S_{-2,1,-2,-2} + S_{1,-2,1,1,-2}) + \\
& 9S_{3,-2,-2} - 10S_{1,-2,2,-2} + 11S_{-3,2,-2} + 12(S_{-2,2,-3} - S_{-6,1} \\
& + S_{1,4,-2} + S_{4,-2,1} + S_{4,1,-2} - S_{-3,1,1,-2} - S_{-2,2,-2,1} - \\
& S_{1,1,2,3} - S_{1,1,3,-2} - S_{1,1,3,2} - S_{1,2,1,3} - S_{1,2,2,-2} - \\
& S_{1,2,2,2} - S_{1,2,3,1} - S_{1,3,1,-2} - S_{1,3,1,2} - S_{1,3,2,1} - \\
& S_{2,-2,1,2} - S_{2,-2,2,1} - S_{2,1,1,3} - S_{2,1,2,-2} - S_{2,1,2,2} - \\
& S_{2,1,3,1} - S_{2,2,1,-2} - S_{2,2,1,2} - S_{2,2,2,1} - S_{2,3,1,1} - S_{3,1,1,-2} - \\
& S_{3,1,1,2} - S_{3,1,2,1} - S_{3,2,1,1}) + 13S_{2,-2,3} - 14S_{2,-2,1,-2} \\
& + \dots \\
& + \dots \\
& - 72S_{1,1,1,-4} - 80S_{1,1,-4,1} - \zeta(3)\mathbf{S}_1(\mathbf{S}_3 - \mathbf{S}_{-3} + 2\mathbf{S}_{-2,1})
\end{aligned}$$

11 Wrapping effect

Regge asymptotics

$$\gamma_4|_{j-1=\omega \rightarrow 0} = -\frac{512}{\omega^7}$$

BFKL prediction

$$\gamma_4|_{j-1=\omega \rightarrow 0} = -256 \left(4 \frac{\zeta(3)}{\omega^4} + \frac{5}{4} \frac{\zeta(4)}{\omega^3} \right)$$

Dressing phase modification (Wrapping effect)

$$\zeta(3) \rightarrow \frac{47}{24} \zeta(3) - \frac{1}{4} S_{-3} + \frac{3}{4} S_{-2} S_1 + \frac{3}{8} S_1 S_2$$

$$+ \frac{3}{8} S_3 + \frac{1}{6} S_{-2,1} - \frac{17}{24} S_{2,1}$$

Agreement with double-logarithmic predictions

$$\frac{1}{512} \gamma_4|_{j \rightarrow -2k+\omega} =$$

$$-\frac{5}{\omega^7} + 20 \frac{S_1}{\omega^6} + \frac{-24 S_1^2 + 14(S_2 + \zeta_2) - 4(S_2 + S_{-2})}{\omega^5} + \dots,$$

$$\frac{1}{512} \gamma_4|_{j \rightarrow -2k+1+\omega} = 0 \frac{1}{\omega^7}$$

12 Discussion

1. Pomeron as a composite state of reggeized gluons.
2. Integrability of the BFKL dynamics at $N_c \rightarrow \infty$.
3. Properties of the BFKL equation in $N = 4$ SUSY.
4. Integrability of DGLAP dynamics in $N = 4$ SUSY.
8. Maximal transcendentality.
9. Regge and Sudakov resummations.
10. BES equation for the casp anomalous dimension.
11. Pomeron-graviton interplay.
12. Integrability for γ in $N = 4$ SUSY.
13. Inconsistency of the four-loop result.
14. Wrapping effects from BFKL predictions.