Small x QCD and Multigluon States: N_c Dependence in a Toy Model

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Outline

Framework: Small x perturbative QCD in the LLA and in the weak field limit

Let us consider the linear evolution of composite states of reggeized gluons. It is in this system that integrability in QCD (in the large N_c limit) was first observed.

L.N. Lipatov
Faddeev, Korchemsky

Question: What about the N_c dependence in the spectrum? It is a difficult problem. Why not to play with a toy model?

- BFKL kernel and BKP kernels.
- The 4 gluon case: color structure
- A simple toy model
- Its spectrum dependence on N_c

Regge Kinematics in QCD and BFKL kernel in LLA

Elastic scattering : a+b$\longrightarrow$ a'+b'.
The amplitude $A(s,t)$ is related to the total "a+b"-cross section through unitarity of the S-matrix (optical theorem)

$$\boxed{\sigma_{tot} = \frac{1}{s} Im_s A(s, t=0)}$$

Multi Regge Kinematics (MRK)+Perturbative analysis with Feynman diagrams:

$$\boxed{A = \sum_n a_n^{LL} (\alpha_s \ln s)^n + \sum_n a_n^{NLL} \alpha_s (\alpha_s \ln s)^n + \cdots}$$

the leading contribution to the total cross section in the Regge limit comes essentially from ladder diagrams with gluons (spin 1) exchanged in the t-channel.

High energy factorization: impact factors Φ and resummed Green's functions G
Evolution in rapidity: dynamics in the transverse space
Complex notation for 2D vectors $\boldsymbol{p}_i$ and $\boldsymbol{\rho}_i \longrightarrow p = p_x + i\, p_y$

$$\boxed{H_{12} = \ln |p_1|^2 + \ln |p_2|^2 + \frac{1}{p_1 p_2^*} \ln |\rho_{12}|^2 \, p_1 p_2^* + \frac{1}{p_1^* p_2} \ln |\rho_{12}|^2 \, p_1^* p_2 - 4\Psi(1)}$$

LL BFKL kernel

In order to construct the Green's function G one has to study the Schrödinger like equation. The physical amplitudes are constructed from matrix elements between colorless impact factors

$$A = \langle \Phi_A | G | \Phi_B \rangle = \langle \Phi_A | e^{y H_{12}} | \Phi_B \rangle$$

BFKL kernel in LLA in Möbius representation

Convenient representation in the Möbius space.

- On such a space $H_{12} = h_{12} + h_{12}^*$, i.e. it is holomorphic separable, with
  $h_{12} = \sum_{r=1}^{2} \left(\ln p_r + \frac{1}{p_r} \ln(\rho_{12}) p_r - \Psi(1) \right)$
- this operator commutes with the generators of the global conformal group $SL(2, C)$
  $M_r^3 = \rho_r \partial_r \,, \;\; M_r^+ = \partial_r \,, \;\; M_r^- = -\rho_r^2 \partial_r \quad \rightarrow \quad [h_{12}, M_r^k] = 0$
- The eigenstates of the Casimir M^2 and h_{12} are

$$\psi_h(\rho_{10}, \rho_{20}) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^h = \langle 0 | \phi(\rho_1) \phi(\rho_2) O_h(\rho_0) | 0 \rangle$$

  where the composite operator O_h depend on the conformal weight $\boxed{h = \frac{1}{2} + i\nu + \frac{n}{2}}$

  which labels the principal series of unitary representations.
  The fields ϕ are associated to reggeized gluons with 0 weight.

  $M^2 \psi_h = h(h-1)\psi_h$ and $H_{12}\phi_h = \chi_h \phi_h \quad , \boxed{\chi_h = \psi(h) + \psi(1-h) - 2\psi(1)}$

- the full eigenstate of H_{12} is $E_{h\bar{h}}^{M} = \psi_h(\rho_{10}, \rho_{20}) \psi_{\bar{h}}(\rho_{10}^*, \rho_{20}^*)$ where $\bar{h} = 1/2 + i\nu - n/2$
- Clearly in this basis one can write a spectral representation for the kernel (Hamiltonian) and for the corresponding Green's function.

Colorless multigluon states in the LLA

- n **reggeized gluon states**: they evolve in rapidity according to the BKP Green's function constructed from the BKP Hamiltonian H_n, containing the informations about the full spectrum, before projection on the particular space of impact factors chosen:

$$\boxed{H_n = -\frac{1}{N_c}\sum_{i<j} T_i^a T_j^a H_{ij}}$$

- In the same way as for H_2 , H_n is conformal invariant.
- For $N_c \to \infty$ (one cylinder topology):

$$H_n = \frac{1}{2}\left(H_{12} + H_{23} + \cdots + H_{n1}\right)$$

- Holomorphic separability in Möbius space:
  $H_n = \frac{1}{2}(h_n + h_n^*)$ with $h_n = \sum_{i=1}^{n} h_{i,i+1}$.
- Integrability in the Möbius space in the large N_c limit due to $n-1$ integral of motions generated by the transfer matrix of an integrable non compact spin XXX model.

$$q_r = \sum_{1 \le i_1 \le i_2 \dots \le i_r \le n} \rho_{i_1 i_2} \rho_{i_2 i_3} \dots \rho_{i_r i_1}\ p_{i_1} p_{i_2} \dots p_{i_r}\ ,\ [q_r, q_s] = 0\,,\ \ [q_r, h] = 0$$

- duality symmetry D_n: $(D_n)^2 = R_n$ with R_n generating the rotation on the cylinder. D_n is a kind of supersymmetry!

Colorless 4 gluon states.

Color states of 4 gluons in a total colorless state can be classified looking at any 2 gluon subchannel.

- Let us consider the decomposition for a generic $SU(N_c)$ in terms of projectors $P[R_i]$ onto irreducible representations:
  $(N_c^2-1)\times(N_c^2-1): 1 = P_1 + P_{8A} + P_{8S} + P_{10+\bar{10}} + P_{27} + P_0 = \sum_i P[R_i], \quad TrP[R_i] = d_i$
- Choosing the projectors of the subchannel (12) as the basis of a vector space we have
  $v^{a_1a_2a_3a_4} = \sum_i v^i \left(P[R_i]^{a_3a_4}_{a_1a_2}\right) = \sum_i v^i P_{12}[R_i]$
  Let us write the color operators $T_iT_j = \sum_a T_i^a T_j^a$ on this basis.
- The color interaction in the "diagonal channel" is $T_iT_j = -\sum_k a_k P_{ij}[R_k]$ where
  $a_k = (N_c, \frac{N_c}{2}, \frac{N_c}{2}, 0, -1, 1)$

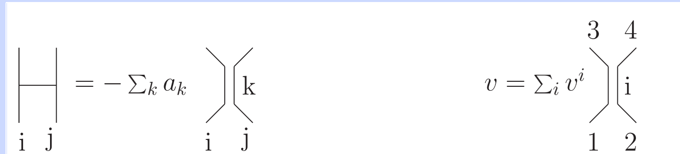
We can now write the action of these operators on a generic color state v in the basis $\{P_{12}[R_i]\}$.
The components transform according to

- $$\boxed{(T_1T_2v)^j = -a_j v^j = -\left(A\, v\right)^j}$$
- $$\boxed{(T_1T_3v)^j = -\sum_i \left(\sum_k C^j_k a_k C^k_i\right) v^i = -\left(C\,A\,C\, v\right)^j}$$
- $$\boxed{(T_1T_4v)^j = -\sum_i \left(\sum_k (-1)^{s_j} C^j_k a_k C^k_i (-1)^{s_i}\right) v^i = -\left(S\,C\,A\,C\,S\, v\right)^j}$$

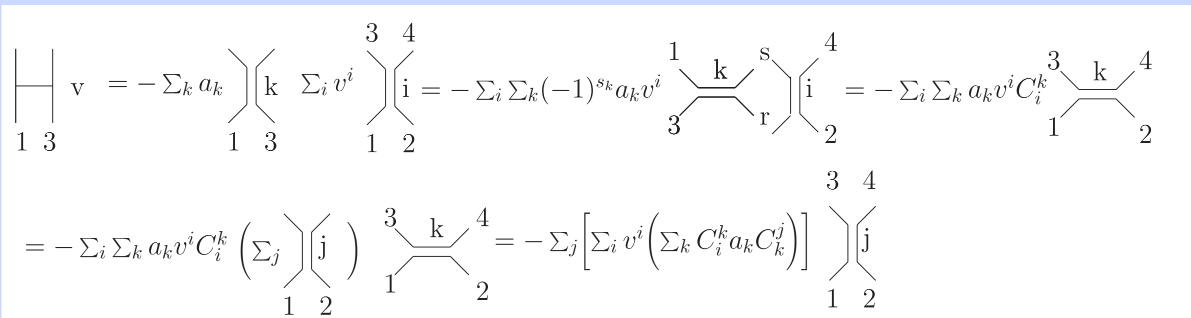
where the crossing matrices C are defined by the $6j$ symbols and S is the matrix associated to the parity of each representation.

More on the crossing matrix.

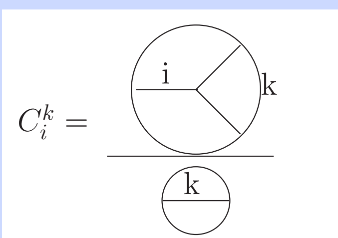
Using the definitions



let us compute the first non trivial crossing case



where the crossing matrix (essentially 6j symbols) can be written as



It is convenient to perform a similarity transformation to work with a symmetric crossing matrix. It is sufficient to introduce the matrix $\Delta = \mathrm{diag}(d_i)$ and define the new symmetric matrix

$C \rightarrow \Delta^{-\frac{1}{2}} C \Delta^{\frac{1}{2}}$ which acts on the vectors with components $v^i \rightarrow \left(\Delta^{-\frac{1}{2}} v\right)^i$

Cvitanovic- Nikolaev, Schäfer, Zakharov - Dokshitzer, Marchesini- Kovner, Lublinsky

4 Gluon Kernel

With the previous definitions the kernel becomes

$$\boxed{H_4 = \frac{1}{N_c}\left[A\left(H_{12}+H_{34}\right)+CAC\left(H_{13}+H_{24}\right)+SCACS\left(H_{14}+H_{23}\right)\right]}$$

Note that replacing H_{ij} with the unity operator one has $H_n = -\frac{1}{N_c}\left(-\frac{1}{2}\right)\sum_i T_i^2 = \frac{n}{2}1$

and indeed $A + CAC + SCACS = N_c 1$

Let us look also directly at the large N_c limit of the kernel. This limit depends on the impact factors.

- one cylinder topology: $T_i T_j \to -\frac{N_c}{2}\delta_{i+1,j}$ which correspond to
  $H_4 = H_{12} + H_{34} + H_{14} + H_{23}$
  The corresponding intercept has been computed usind the Baxter-Sklyanin methods for integrable models. De Vega, Lipatov - Derkachov et al.

- two cylinder topology: 3 possible singlet pairs corresponding to $H_4 = H_{12} + H_{34}$ and by their cyclic permutations.
  In this case the intercept is trivially twice the one of the BFKL pomeron.

Note that in the “real” world, corresponding to $N_c = 3$, the representation R_0 is absent (then we decompose the color space in 5-dimensional vectors) while this is not true when we consider the large N_c limit (and 6 dimensional vectors are used).

A Toy Model

To study some features of the color dependence we replace the interaction on the configuration space H_{ij} which depends on the Casimir L^2 of $SL(2,C)$ non compact spin. Here we consider the zero conformal spin ($n=0$) subsector. In this case such dependence is given by

$$\boxed{H_2 = 2\Re\psi\left(\frac{1}{2}+i\nu\right) - 2\psi(1) = 2\Re\psi\left(\frac{1}{2}+\sqrt{\frac{1}{4}+L^2}\right) - 2\psi(1)}$$

Let us consider a toy model based on the following assumptions

- degrees of freedom: non compact spin $SL(2,C) \to$ compact spin $SU(2)$ with the replacement $\boxed{\frac{1}{4}+L^2_{ij} \to -\alpha\, S^2_{ij}}$ where α is a free parameter.
- In order to use the same framework as the one seen for $SU(N_c)$ let us consider the discrete "configuration" degrees of freedom corresponding to spin one states and let us also consider a globally spin singlet state $S=\sum_i S_i = 0$
- From $S^2_{ij} = 4 + 2S_iS_j$ and the previous definitions we have the substitution $\boxed{H_{ij} \to f(S_iS_j)}$
- It is again convenient to use a basis of projectors $Q_{12}[R_i] = (Q_1, Q_3, Q_5)$ on the irreducible representations of the (12) channel. Defining $b_k = (2,1,-1)$ one has $S_iS_j = -\sum_k b_k Q_{ij}[R_k]$ so that we write the spectral representation
$$\boxed{f(S_1S_2) = \sum_k f(-b_k)Q_k}$$

A Toy Model: 2

Introducing suitable crossing matrix D and symmetry matrix S' we can write the action of the toy model operators on the spin part:

- $$\boxed{(f(S_1 S_2)u)^j = f(-b_j)v^j = (B\, u)^j}$$
- $$\boxed{(f(S_1 S_3)u)^j = \sum_i \left(\sum_k D^j_k f(-b_k) D^k_i \right) u^i = (D B\, D\, u)^j}$$
- $$\boxed{(f(S_1 S_4)u)^j = \sum_i \left(\sum_k (-1)^{s'_j} D^j_k f(-b_k) D^k_i (-1)^{s'_i} \right) u^i = (S' D B\, D S'\, u)^j}$$

We can now construct the toy model Hamiltonian:

$$\boxed{H_4 = \frac{2}{N_c} \left(A \otimes B + C A\, C \otimes D B\, D + S C A\, C S \otimes S' D B\, D S' \right)}$$

We shall analyze its spectrum dependence in N_c.

In the large N_c the eigenvalues flow to match the two following cases:

The one cylinder topology (1CT) corresponds to the simpler spin Hamiltonian

$$\boxed{H_4^{1cyl} = \left(B + S' D B\, D S' \right)}$$

The two cylinder topology (2CT) corresponds simply to

$$\boxed{H_4^{2cyl} = 2B}$$

A Toy Model: 3

Eigenvalues in the large N_c limit:

- 2CT: the max. eigenvalue $E = -2\chi_{BFKL} = 5.54518$
- 1CT: the max. eigenvalue depends on the parameter α.
  Let us fix it to match the leading eigenvalue corresponding to conformal spin $n = 0$ of the integrable XXX non compact spin chain. ($E = 0.67416$)

Eigenvalues:

$$\begin{pmatrix} N_c = 3 & \\ 7.04193 & (\times 1) \\ 5.51899 & (\times 2) \\ 1.12269 & (\times 2) \\ -3.89328 & (\times 2) \\ -4.04744 & (\times 1) \\ -4.27838 & (\times 1) \\ -7.81242 & (\times 1) \\ -9.18576 & (\times 2) \\ -12.6743 & (\times 2) \\ -14.1005 & (\times 1) \end{pmatrix} \longrightarrow \begin{pmatrix} N_c = \infty & \\ 5.54518 & (\times 3)\ 2CT \\ 0.67416 & (\times 3)\ 1CT \\ -4.27838 & (\times 3)\ 1CT \\ -7.81242 & (\times 3)\ 2CT \\ -8.67983 & (\times 3)\ 1CT \\ -10.0168 & (\times 3)\ 2CT \end{pmatrix}$$

To trace all the eigenstates at $N_c = \infty$ in the flow from finite N_c the $SU(N_c)$ basis containing the P_0 states should be used (with three more eigenstates).

The flow with N_c of the leading eigenvalue is given by $E_0(N_c) = E_0(\infty)\left(1 + \frac{2.465}{N_c^2}\right)$

Its large N_c approximation corresponds to an error of 27%.

A Toy Model: 4

It is interesting to look at the color content of the eigenstates.
Let us consider as an example the maximal eigenvalue $E_0(N_c)$.
At $N_c = 3$ the eigenvector is v_0 and at $N_c = \infty$ the eigenvectors are the w_i. In the basis $P_i Q_j$ they are

$$w_1 = \begin{pmatrix} 1 & P_1 Q_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.589824 & P_1 Q_1 \\ 0.0853017 & P_1 Q_5 \\ 0.343819 & P_{8A} Q_3 \\ 0.198919 & P_{8S} Q_1 \\ 0.198893 & P_{8S} Q_5 \\ 0.29322 & P_{10+\bar{10}} Q_3 \\ 0.178796 & P_{27} Q_1 \\ 0.574061 & P_{27} Q_5 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 1/3 & P_{10+\bar{10}} Q_1 \\ \frac{\sqrt{5}}{3} & P_{10+\bar{10}} Q_5 \\ \frac{1}{\sqrt{6}} & P_{27} Q_3 \\ \frac{1}{\sqrt{6}} & P_0 Q_3 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} & P_{10+\bar{10}} Q_3 \\ \frac{1}{3\sqrt{2}} & P_{27} Q_1 \\ \frac{\sqrt{5}}{3\sqrt{2}} & P_{27} Q_5 \\ \frac{1}{3\sqrt{2}} & P_0 Q_1 \\ \frac{\sqrt{5}}{3\sqrt{2}} & P_0 Q_5 \end{pmatrix}$$

For this specific model we note that in the large N_c limit the eigenvectors associated to the eigenvalues of the 2CT are actually independent on the form of f which defines the pomeron eigenvalues, but depends only on the spin structure.
The part associated to the 1CT is dependent on the function f.

Conclusions

- Since the spectrum of the BKP Hamiltonian is difficult to analyze at fixed N_c we have considered the case of 4 gluons and after rewriting in a specific basis the color part of the interaction we have considered a discrete toy model for the configuration part. The compact $SU(2)$ structure was chosen. Adjoint representation and null total spin were considered.
- A mapping in accord to pomeron spectrum and which fixes the large N_c eigenvalues of zero conformal spin was chosen.
- One can see in the leading eigenvalue large (30%) corrections comparing the finite to the infinite number of color cases.
- The dependence on N_c of the eigenvectors is also interesting.
- Question: how to analyze related quantum system with a different "spin" structure: spin of the single particle, different total spin? And what about another group or more complicated dynamics?
- It might be interesting to analyze similar toy models but with the property of being integrable in the large N_c limit.