INTEGRATED CORRELATORS FROM INTEGRABILITY

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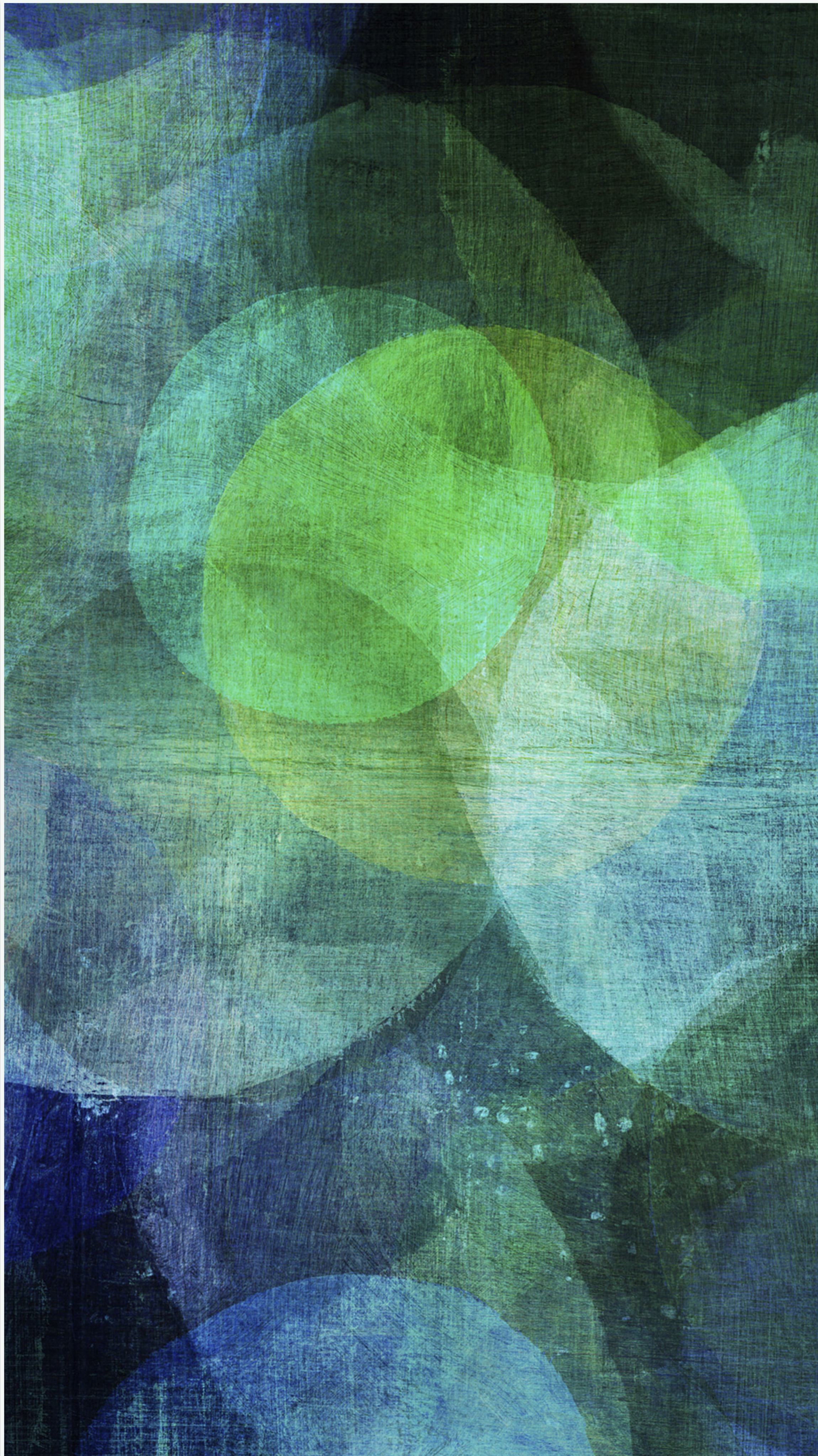
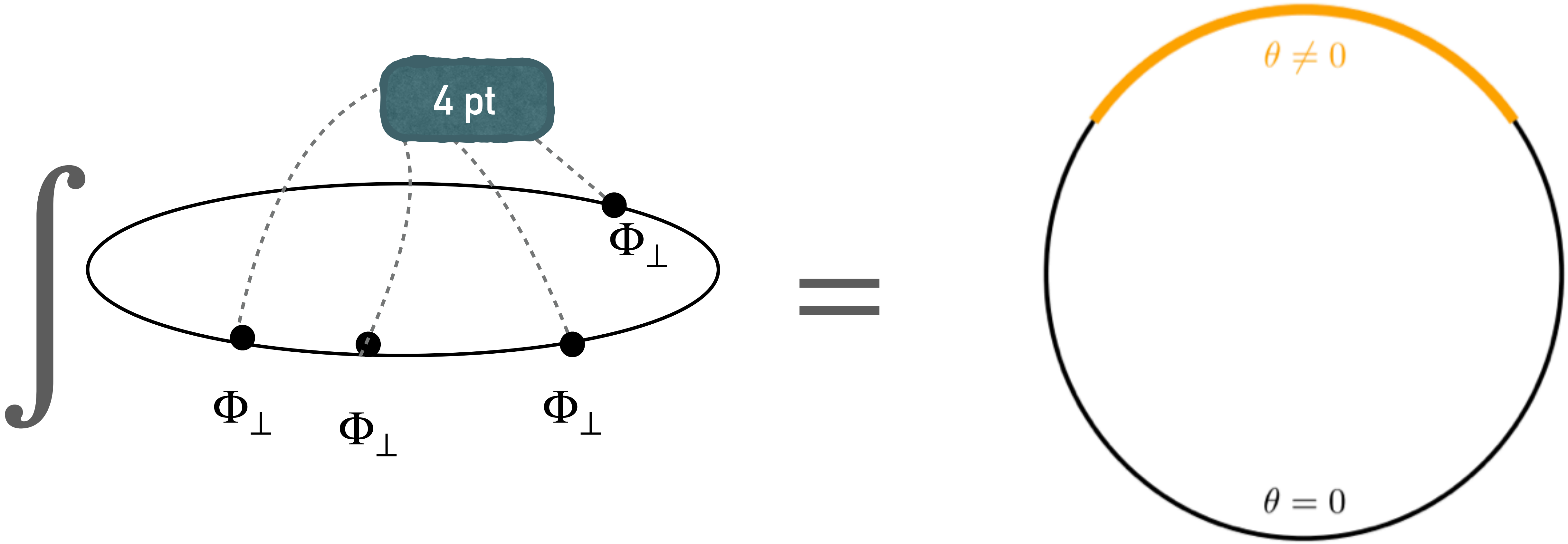


Based on 2211.03203 and other works with

Julius Julius, Nikolay Gromov, Michelangelo Preti

I will discuss a new type of **constraints on defect CFTs**,

coming from (**local**) **deformations** of the defect



I will discuss some powerful applications for integrable theories such as N=4 SYM, and a general derivation based on conformal pert. theory, which uses no susy, and no integrability except to fix constants.

Open question: can this be useful elsewhere (for non-integrable theories)?

Plan of the talk

- Motivation: new type of constraints for “Bootstrability”

  [cf. recent talk by Julius Julius in DESY!]

- Connecting the line CFT and the cusp, and simpler examples of the method

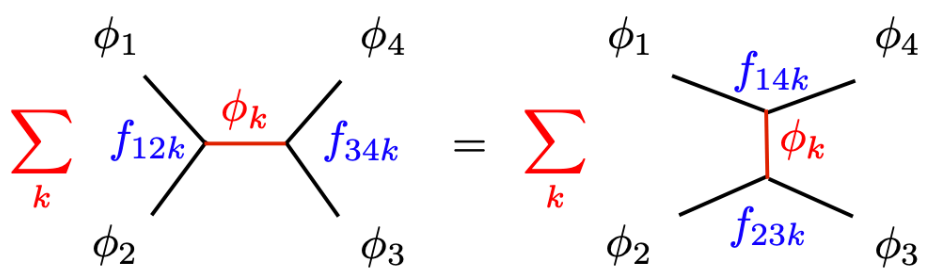
- Conformal perturbation theory derivation

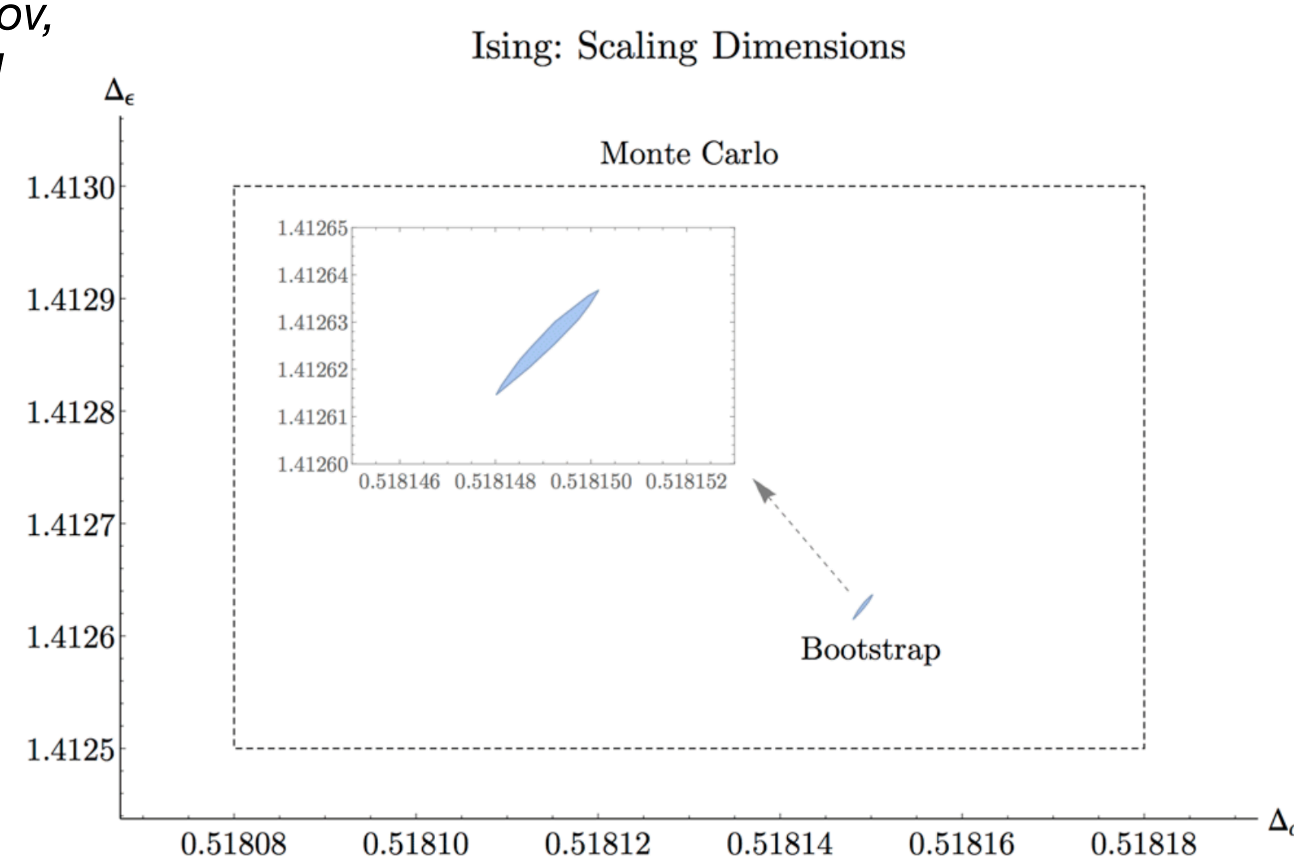
- Outlook on future applications

Motivation: bootstrability

Conformal bootstrap can “solve” CFTs, provided we can isolate them

[El-Showk,Paulos,Poland,Rychkov, Simmons-Duffin,Vichi ’12,’14]





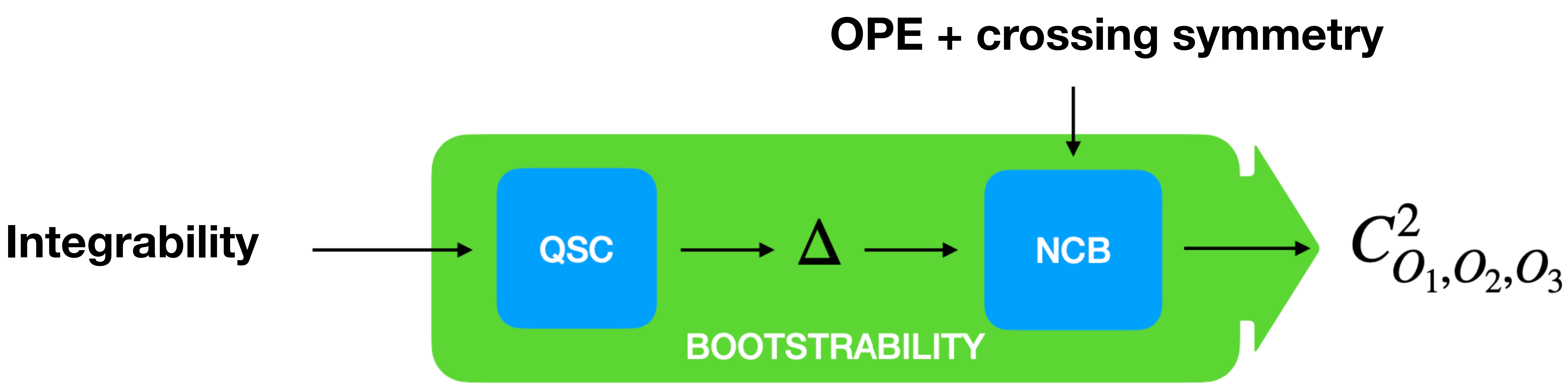
Nontrivial challenge for conformal gauge theories such as AdS/CFT models

(except in specific limits, e.g. strong ’t Hooft coupling)

In presence of integrability (and/or localization), we can provide further data to help isolate the theory!

Bootstrability 1.0:

Quantum **S**pectral **C**urve + **N**umerical **C**onformal **B**ootstrap



Aside: note that integrability should know much more than the spectrum (ask Till)! - but that is a story for another day.

Setup: line defect CFT

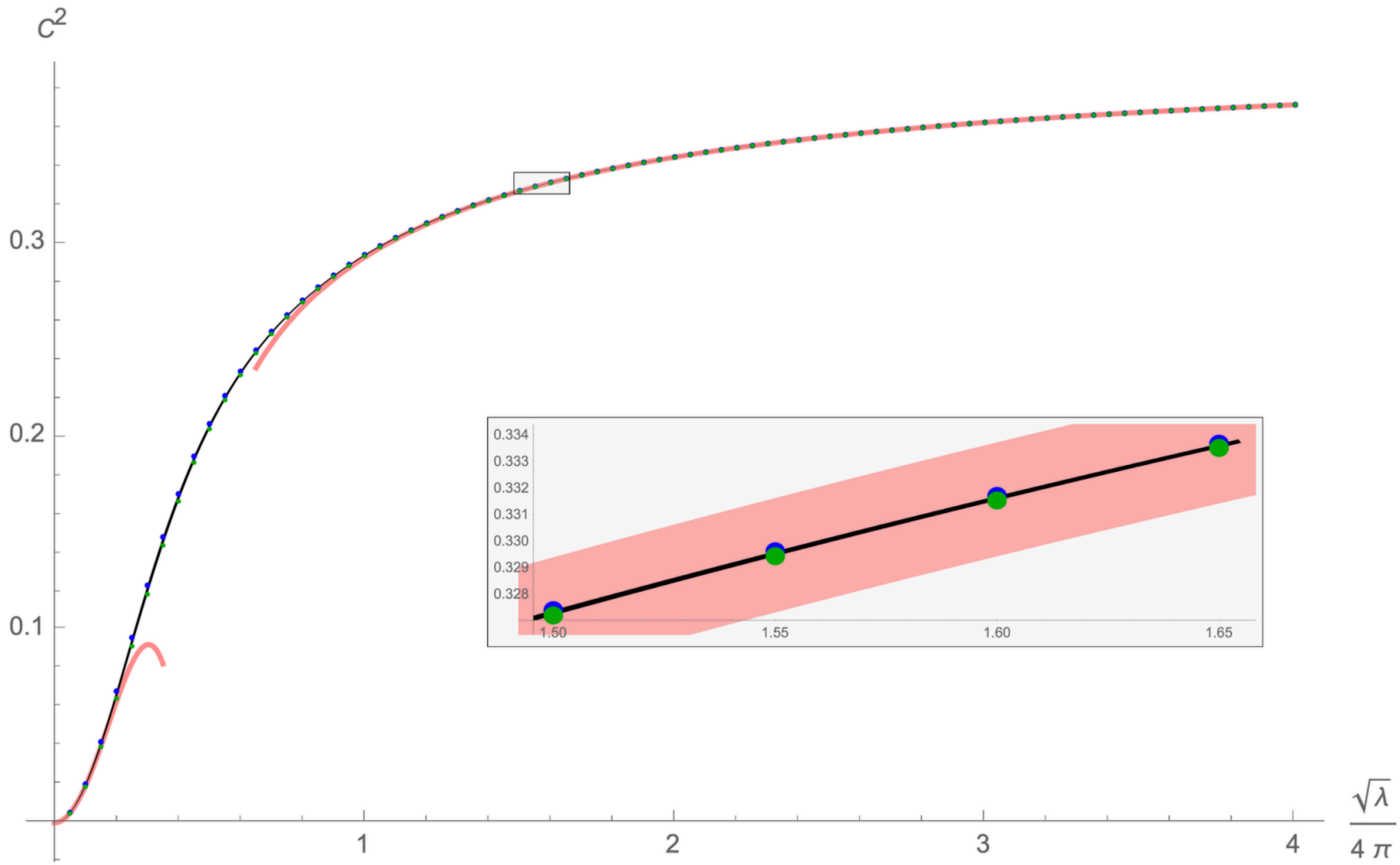
[AC, Gromov, Julius, Preti '21, '22]
[..+N.Sokolova, in progress]
bulk program also possible! [Caron-Huot, Coronado, Trinh, Zahraee '22]

Advantages: planar theory is consistent CFT on its own. Very nice results possible even with bootstrap alone [Liendo, Meneghelli, Mitev '18] [Meneghelli, Ferrero '21]

Simpler 1D CFT setup

Interesting eventually to study bulk-defect bootstrap

One can get very narrow bounds for OPE coefficients



Today: discuss a method which shrinks the bounds by 3-4 orders of magnitude, by using more information from integrability

A second type of constraints: spectrum of neighbouring theories

Similar to the way integrated correlators were obtained from localization

[Binder, Chester, Pufu, Wang ‘19]

$$Z(m, \lambda, \tau_p, \bar{\tau}_p) \longrightarrow \left. \frac{\partial_m^2 \partial_{\tau_p} \partial_{\bar{\tau}_p} \log Z}{\partial_{\tau_p} \partial_{\bar{\tau}_p} \log Z} \right|_{m=\tau=\bar{\tau}=0} \longrightarrow \text{integrated 4-pt in } \mathcal{N}\text{=4 SYM}$$

partition f. of theory with $\mathcal{N}=2$ deformed action

But we also know **many solvable deformations from integrability!**

e.g. we know full non-BPS spectrum on full conformal manifold
($\vec{\gamma}$ - deformations)

Today we look at **special deformations of 1D defect theory**, and use the integrability of **cusps on the defect in N=4 SYM**

While the localization constraints come finite, we need careful regularisation - calculation illustrates conformal perturbation theory at NLO.

Connecting line and cusp

1/2 BPS Susy Wilson line in N=4 SYM:

$$\mathcal{W}_{\mathcal{C}} = \frac{1}{N} \mathrm{Tr}\, \mathrm{P} \exp \int_{\mathcal{C}} dt \, \left(i\, A_\mu \dot{x}(t)^\mu + |\dot{x}(t)|\, n \cdot \Phi\right)$$

one of six scalar fields of N=4 SYM

$$\vec{n} \cdot \vec{\Phi} \equiv \Phi_{||} \equiv \Phi_6 \qquad \left\{\Phi^i\right\}_{i=1}^{5} \equiv \left\{\Phi^i_\perp\right\}_{i=1}^{5}$$

parallel — orthogonal

Symmetry breaking:

Spacetime $\Longrightarrow$ $\underbrace{SO(5,1)}_{4D\ conformal} \rightarrow \underbrace{SL(2)}_{1D\ conformal} \times SO(3)$

Global $\Longrightarrow$ $\underbrace{SO(6)}_{rotate\ \Phi^i} \rightarrow \underbrace{SO(5)}_{rotate\ \Phi^i_\perp}$

$$\langle\langle O_1(t_1) O_2(t_2) \ldots O_n(t_n) \rangle\rangle \equiv Tr\left[Pe^{\int_{-\infty}^{t_1} dt \left(iA_t + \Phi_{||}\right)} O_1(t_1)\, Pe^{\int_{t_1}^{t_2} dt \left(iA_t + \Phi_{||}\right)} O_2(t_2) \ldots O_n(t_n)\, Pe^{\int_{t_n}^{+\infty} dt \left(iA_t + \Phi_{||}\right)} \right]$$

1D CFT correlators

Defect CFT has distinguished operators with **protected dimensions**

$$\partial_\mu T^{\mu n} = \mathbb{D}^n(x_{||})\delta^3(x_\perp)$$

broken translations

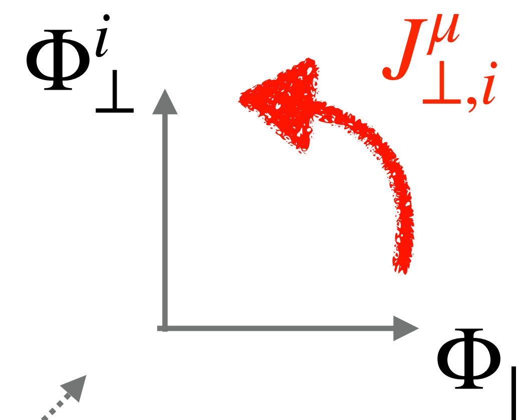
3 displacement operators

$\Delta = 2$

$$\partial_\mu J^\mu_{\perp,i} = \delta^{(3)}(x_\perp)\ \mathbb{T}_i(x_{||})$$

broken R-symmetry rotations

5 tilt operators $\Delta = 1$



(actually part of the same 1/2-BPS multiplet) [Liendo, Meneghelli, Mitev '18]

Spectrum from RG point of view:

- **one relevant operator ($\Delta = 0$)**
- **5 marginal operators ($\Phi^i_\perp$, $\Delta = 1$)**
- **Infinitely many (generally non-BPS) irrelevant ops with $\Delta > 1$**

The simplest 4-point function: four tilt operators

[Liendo, Meneghelli, Mitev '18]

$$\langle\langle \Phi_\perp^M(t_1)\Phi_\perp^N(t_2)\rangle\rangle = \frac{N_{\Phi_\perp}\delta^{MN}}{x_{12}^2}$$

Normalisation is physical.
We will review how it can be fixed shortly.

all polarisations written in terms of single function $f(x)$

$$\frac{\langle\langle \Phi_\perp^M(t_1)\ \Phi_\perp^N(t_2)\ \Phi_\perp^P(t_3)\ \Phi_\perp^Q(t_4)\rangle\rangle}{(N_\perp)^2 t_{12}^{-2}\ t_{34}^{-2}} = \left[\delta^{MN}\delta^{PQ}\mathscr{D}_1 + \delta^{MP}\delta^{NQ}\mathscr{D}_2 + \delta^{MQ}\delta^{NP}\mathscr{D}_3\right]\circ f(x)$$

known differential operators

Cross ratio $\quad x = \dfrac{t_{12}t_{34}}{t_{14}t_{23}}$

Operator product expansion

[Liendo, Meneghelli, Mitev '18]

$$f(\chi) = f_I(\chi) + C^2_{BPS}(\lambda) f_{\mathscr{B}_2}(\chi) + \sum_\Delta C^2_\Delta f_\Delta(\chi)$$

Identity

Protected operator

Non-protected infinite spectrum of neutral operators

Conformal blocks and $C_{BPS}(\lambda)$ are known

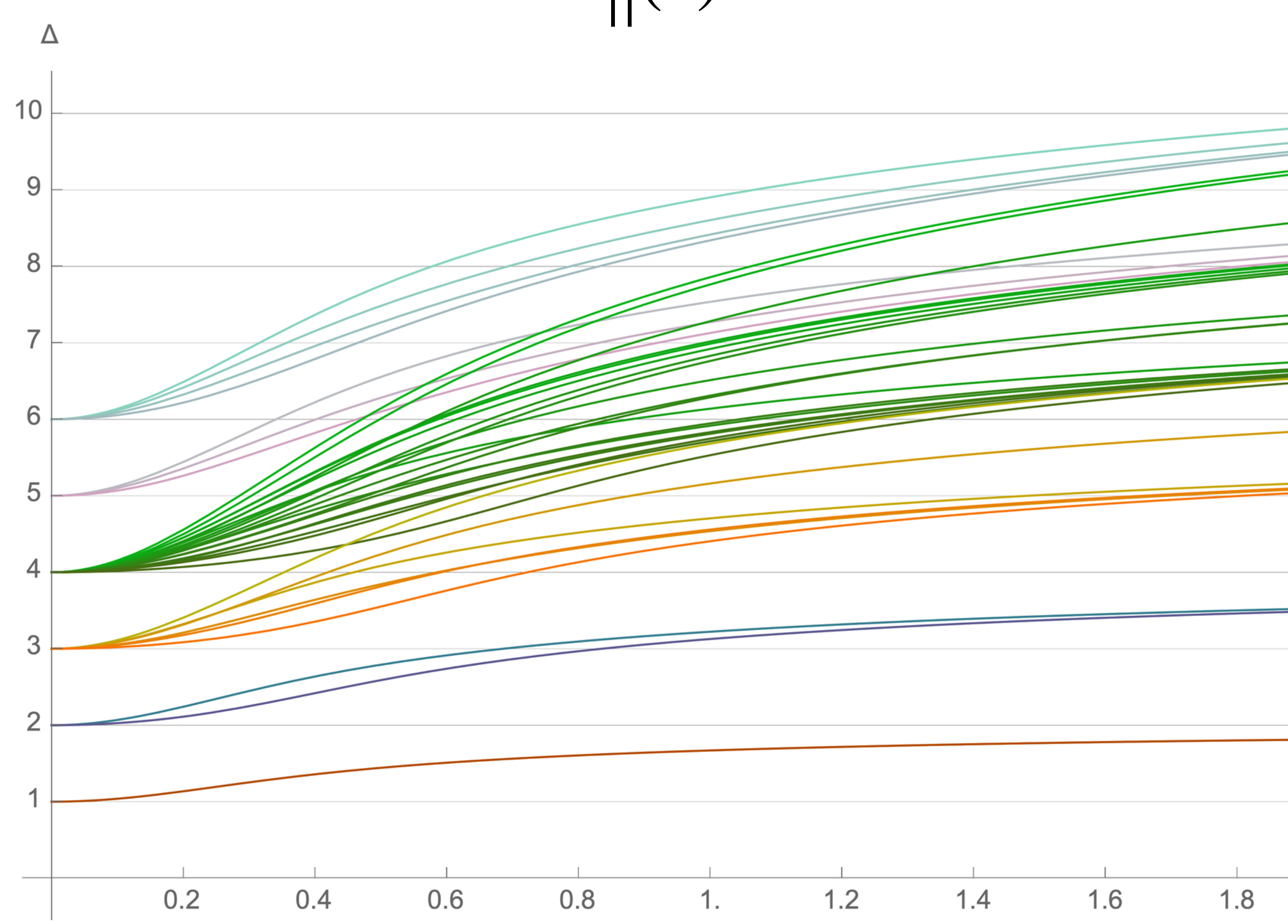
$$1 + C^2_{\mathrm{BPS}} = \frac{3WW''}{(W')^2}$$

$$W = \frac{2I_1(\sqrt{\lambda})}{\sqrt{\lambda}}$$

$\Phi_{||}$: leading non-protected irrelevant operator, $\Delta_{||}(\lambda) > 1$

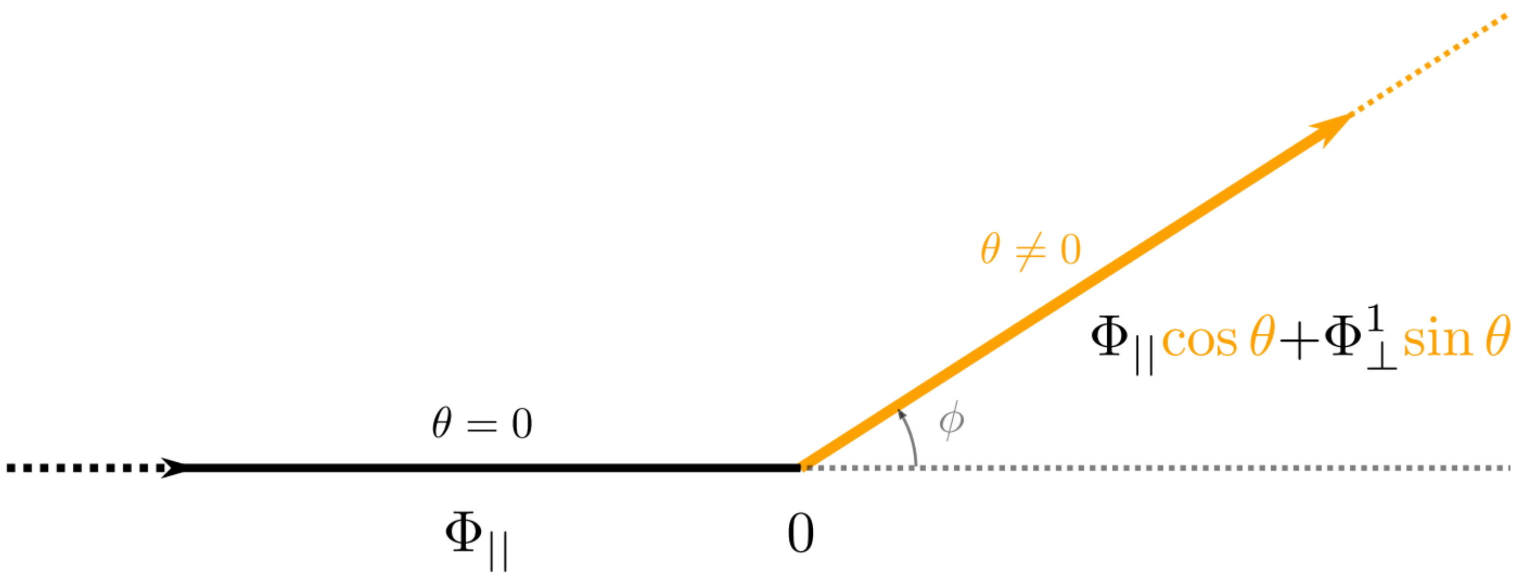
Spectrum is computable from integrability (QSC)

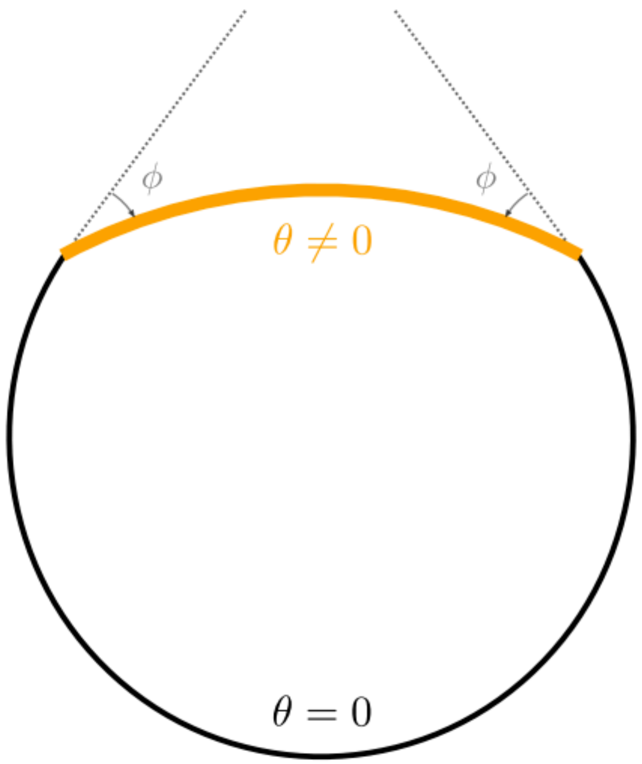
[Grabner, Gromov, Julius '20]
[Gromov, Julius, Sokolova to appear, all sectors]
[AC, Gromov, Julius, Preti '21]



The cusp anomalous dimension

$$\langle \mathcal{W}_{\text{cusp}} \rangle \sim (\epsilon_{\text{UV}})^{\Gamma_{\text{cusp}}(g,\phi,\theta)}$$



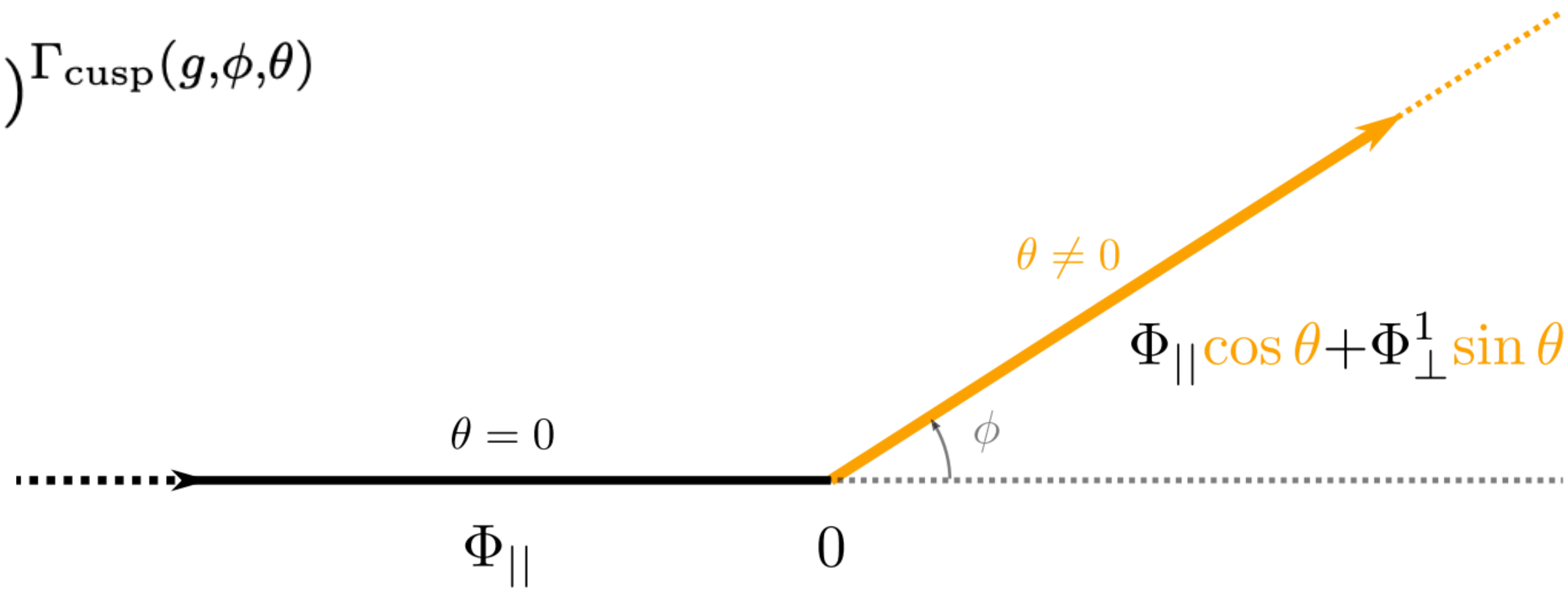


- ➤ Scaling dim. of non-local operators in 4D theory
- ➤ Can insert operators at the cusps
- ➤ Gives smooth deformation of spectrum of 1D CFT obtainable from integrability!

[Drukker '12] [Correa, Maldacena, Sever '12]
[Gromov, Levkovich-Maslyuk '16]

The vacuum cusp at small angles

$$\langle \mathcal{W}_{\text{cusp}} \rangle \sim (\epsilon_{\text{UV}})^{\Gamma_{\text{cusp}}(g,\phi,\theta)}$$



To $O(\theta^4)$ at $\phi = 0$, two key functions of the coupling appear

$$\Gamma_{\text{cusp}}(g, \phi = 0, \theta \to 0) = \mathbb{B}(g) \sin^2\theta + \frac{1}{4}\left(\mathbb{B}(g) + \mathbb{C}(g)\right) \sin^4\theta + \mathcal{O}(\sin^6\theta)$$

Bremsstrahlung function

“Curvature” function

$$\mathbb{B}(g) = \frac{g}{\pi}\frac{I_2(4\pi g)}{I_1(4\pi g)}\,,$$

$$\mathbb{C}(g) = -4\,\mathbb{B}^2(g) - \frac{1}{2}\oint \frac{du_x}{2\pi i}\oint \frac{du_y}{2\pi i} K_0(u_x - u_y) F[x,y]$$

Complicated integral expression (details omitted)

Known analytically also for general $\phi \neq 0, \quad \theta \sim \phi$

[Gromov, Levkovich-Maslyuk ’16]

We found two integrated 4-point identities involving those constants

$$\text{Constraint 1:} \quad \int_{\delta_x}^{1} \delta f(x) \left(\frac{1}{x} + \frac{1}{x^3} \right) dx - \frac{1}{2}(\mathbb{F} - 3) \log \delta_x - \mathbb{F} + 3 = \frac{3\mathbb{C} - \mathbb{B}}{8\,\mathbb{B}^2}$$

$$\text{Constraint 2:} \quad \int_{0}^{1} \frac{\delta f(x)}{x} dx = \frac{\mathbb{C}}{4\,\mathbb{B}^2} + \mathbb{F} - 3\,,$$

$$\mathbb{F} = 1 + C_{BPS}^2(\lambda)$$

$$\delta f(x) = f(x) - 2x + \frac{x}{1-x}$$

Note: two independent identities, with different origin

[Drukker Kong, Sakkas ’22]

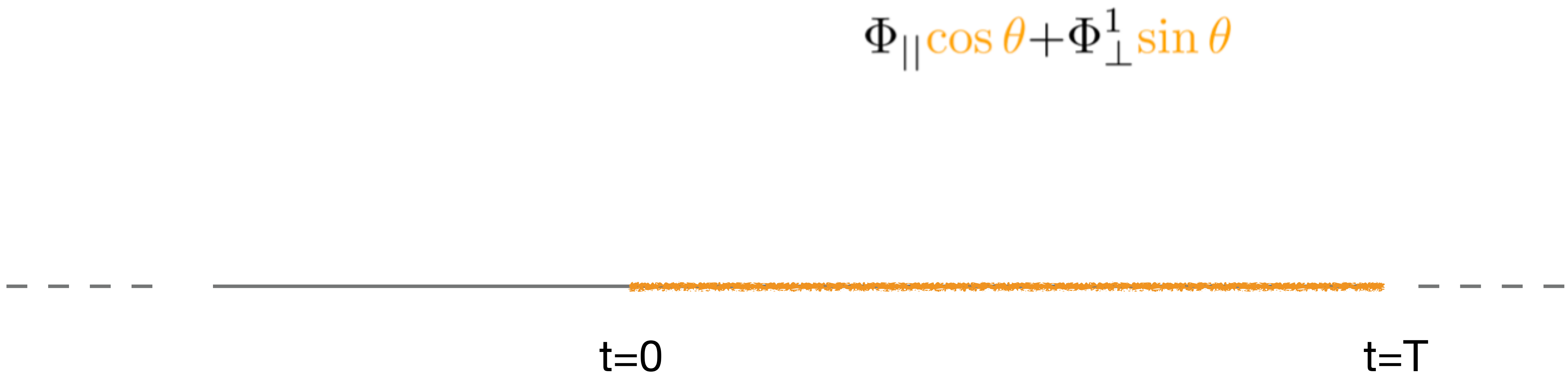
[AC, Gromov, Julius, Preti ’22]

To introduce the main idea of the method, let us look at two simple examples (postponing subtleties)

- Connecting $N_{\perp}$ and Bremsstrahlung [Correa, Maldacena, Sever '12]

- Deriving $C_{BPS}(\lambda)$ from integrability [AC, Gromov, Julius, Preti '22]

Warm up: fixing the tilt normalisation



$$\langle\, W_{-T}^{-\epsilon}(0)\; W_{+\epsilon}^{+T}(\theta)\,\rangle$$

Expand the Wilson line operator in θ …

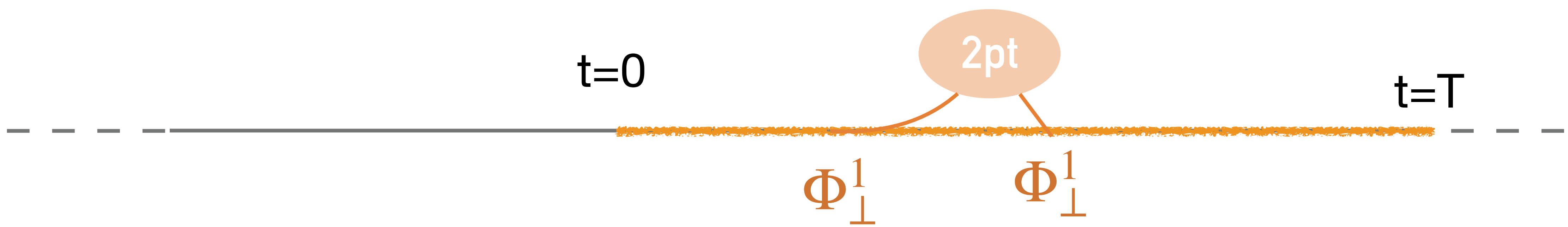
$$W_{t_1}^{t_2}(\theta) = Pexp\left(\int_{t_1}^{t_2} ds[A_0 + \Phi_{||}\cos\theta + \Phi_{\perp}^{1}\sin\theta]\right)$$

$$\sim \theta\; P\left[\int_{t_1}^{t_2} ds'\; \Phi_{\perp}^{1}(s')\; e^{\int_{t_1}^{t_2} ds[A_0+\Phi_{||}]}\right]$$

$$+\theta^2\; P\left[\left(\int_{t_1<s_1<s_2<t_2} ds_1 ds_2\; \Phi_{\perp}^{1}(s_1)\Phi_{\perp}^{1}(s_2) - \frac{1}{2}\int_{t_1}^{t_2} ds'\; \Phi_{||}(s')\right)\; e^{\int_{t_1}^{t_2} ds[A_0+\Phi_{||}]}\right]$$

$$+\ldots$$

Study at $O(\theta^2)$



$$\int_{0<s_1<s_2<T} ds_1 ds_2 \frac{N_\perp}{s_{12}^2} \simeq N_\perp \log\frac{\epsilon}{T}$$

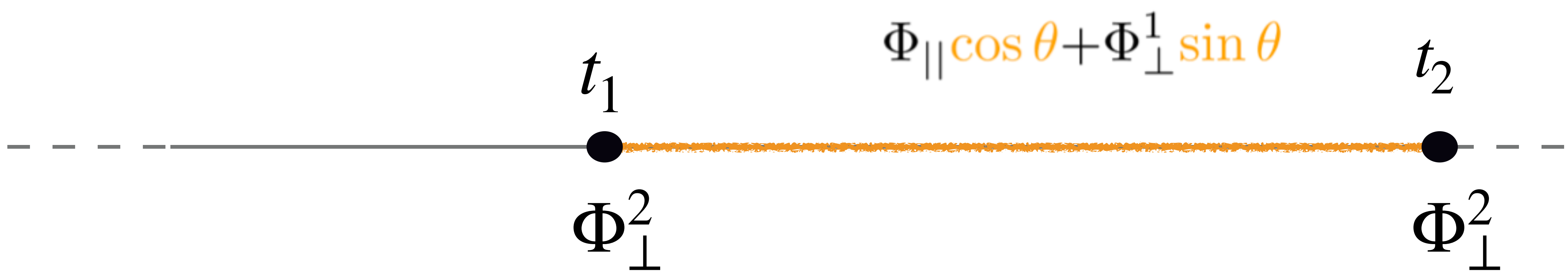
UV divergence should match $\quad (T/\epsilon)^{-2\Gamma_{\text{cusp}}} \simeq 1 - 2\theta^2 \mathbb{B} \log\frac{T}{\epsilon}$

Well known **relation between Bremsstrahlung and tilt normalisation**

$$N_\perp(\lambda) = 2\mathbb{B}(\lambda)$$

[Correa, Maldacena, Sever '12]

$C_{BPS}(\lambda)$ from integrability: add orthogonal charge



$$\langle \mathscr{W}_{\theta}(t_1, t_2)\rangle \propto \left(\frac{\epsilon}{t_{12}}\right)^{2(\Gamma_1(\lambda,\theta)-2)} \times t_{12}^{-2}$$

Cusp dimension in charged sector - also known!

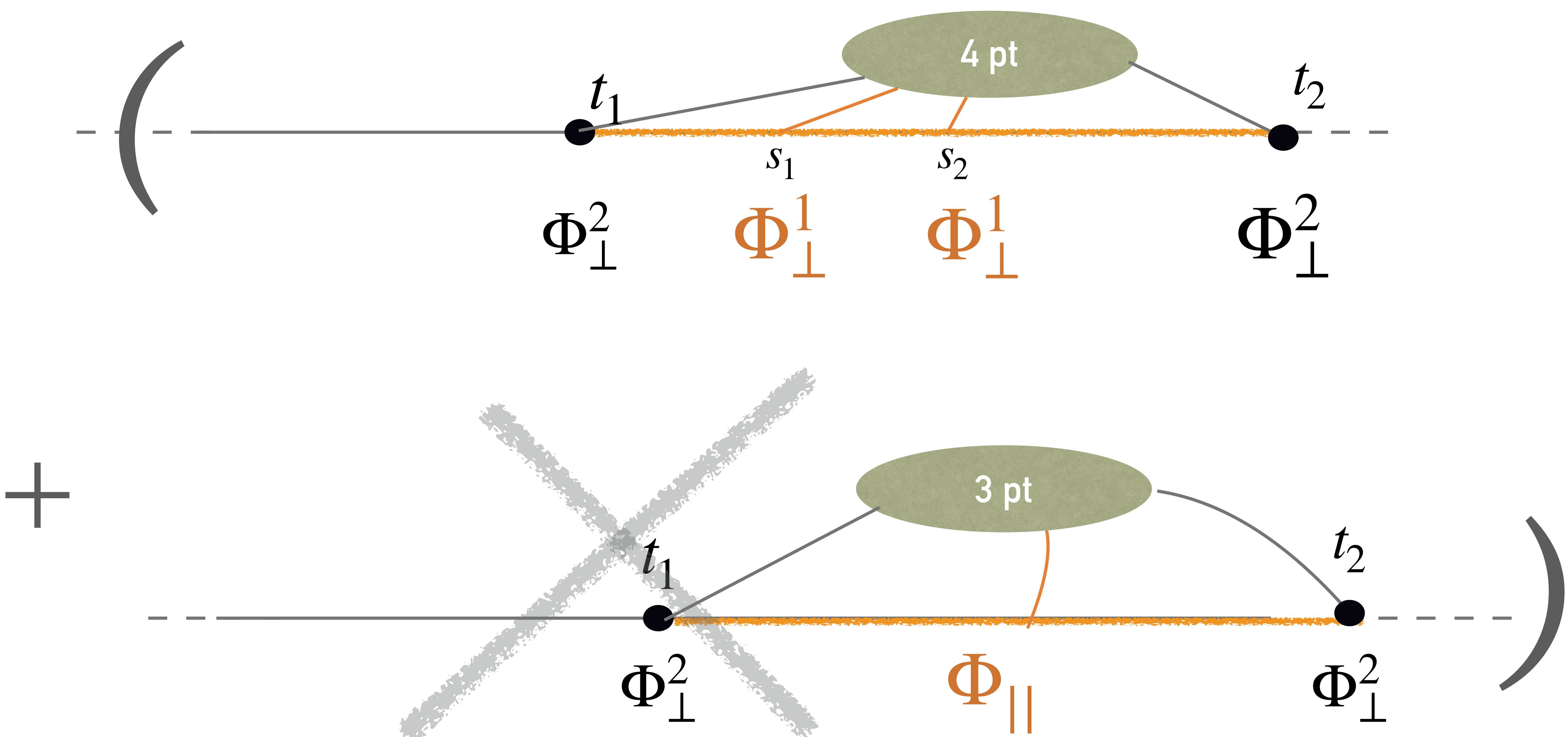
$$\Gamma_1(g,\theta) = 1 + \sin^2\theta\, \mathbb{B}_1(g) + \mathrm{O}(\sin^4\theta) \qquad \mathbb{B}_1 = \frac{g^2\left(-2I_3(4g\pi)^2 + I_1(4g\pi)I_3(4g\pi) + I_1(4g\pi)I_5(4g\pi)\right)}{I_1(4g\pi)^2 - I_1(4g\pi)I_3(4g\pi)}$$

[Gromov, Levkovich-Maslyuk ’16]

$$\langle \mathscr{W}_{\theta}(t_1, t_2)\rangle \sim \langle \mathscr{W}_{\theta=0}(t_1, t_2)\rangle \times \left(1 + 2\sin^2\theta\, \mathbb{B}_1 \frac{\log\frac{\epsilon}{t_{12}}}{t_{12}^2} + \dots\right)$$

Study at $O(\theta^2)$

$\Phi_{||}\cos\theta + \Phi_\perp^1 \sin\theta$



No log-divergence - can discard 3 pt

In this case, log-divergence of 4-point integral gives a solvable cross-ratio integral

$$\sim N_\perp^2 \left[-1 + \int_0^1 \partial_x \left(\frac{\delta f(x)}{x} \right) dx \right]$$

Boundary terms at $x = 0, x = 1$ reduce to explicit contribution from C_{BPS}^2.

Matching with cusp expansion $\mathbb{B}_1(\lambda)$:

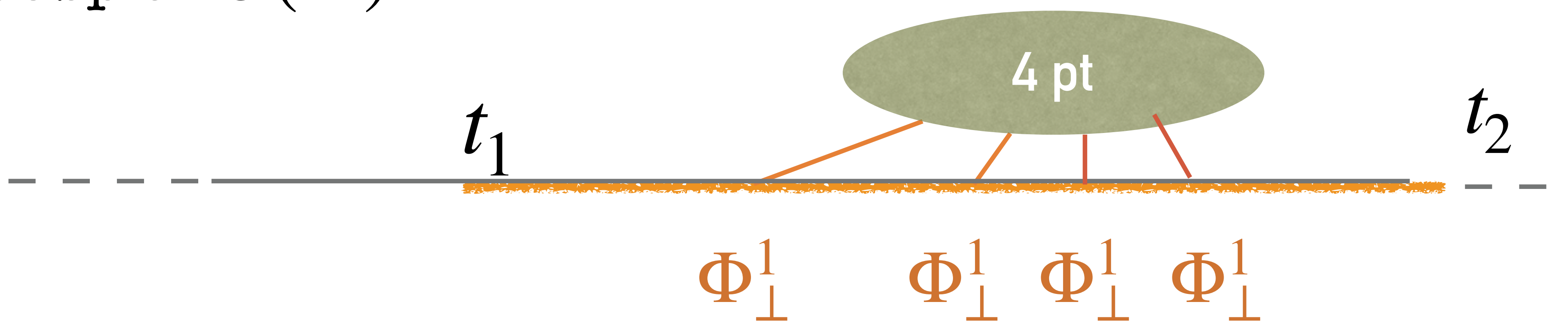
$$\Rightarrow \quad C_{BPS}^2(\lambda) = 1 + 2\mathbb{B}_1(\lambda)/N_\perp(\lambda) = 1 + \mathbb{B}_1(\lambda)/\mathbb{B}(\lambda)$$

Agreeing with localization calculation! [Liendo, Meneghelli, Mitev '18]

Can we just keep going...?

$\mathbb{C} \sim$ cusp at $O(\theta^4)$

$\sim$



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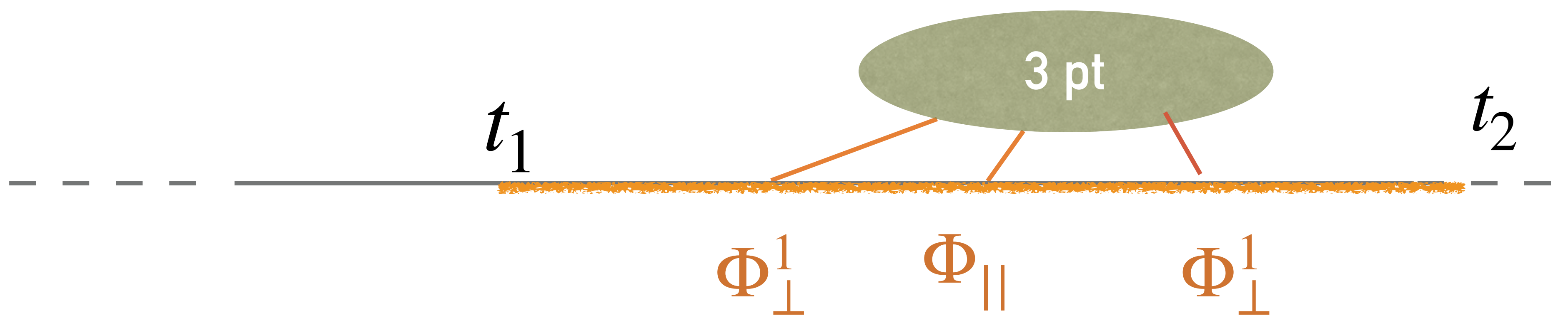


(+ cycl.) + 2 pt + 1 pt ... ?

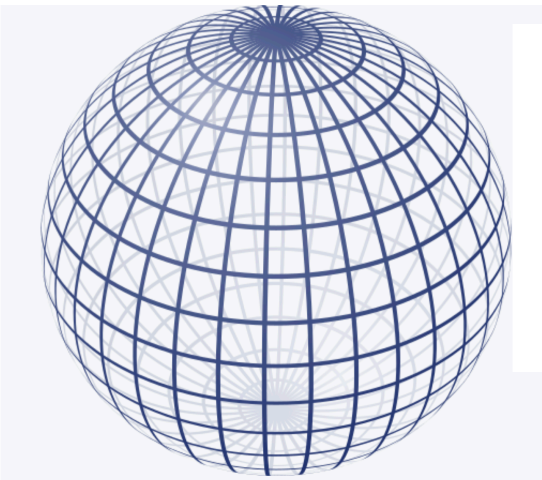
At such higher orders we need to be careful..!
e.g. $\log$-divergences hidden by $\log^2$ ones, so they become scheme-dependent…

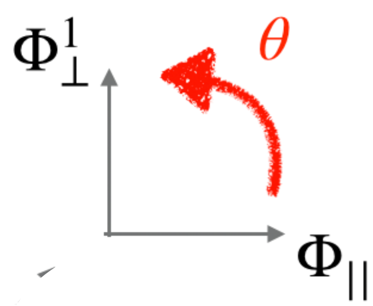
We need a better formalism to do it properly

Integrated 4-point functions from conformal perturbation theory

Abstract conformal perturbation theory

[Zamolodchikov '87] [Kutasov '89] [Cardy '96]....+...





θ-rotation of the defect in R-space

$\simeq$ motion on the conformal manifold of 1D CFT

We want to expand the action of CFT(θ) in terms of local ops in CFT(0)

$$\mathcal{A}_{\text{CFT}}(\theta) \sim \mathcal{A}_{\text{CFT}}(0) + \delta\mathcal{A}_{\text{CFT}}$$

$$\delta\mathcal{A}_{\text{CFT}} = \int dt\ \delta L(t) \sim \mathbf{s} \int dt\ O_{\Phi_\perp^1}(t) + \mathcal{O}(\mathbf{s}^2) \qquad \mathbf{s} = \sin\theta.$$

Leading order: tilt operator - the only marginal ops in the theory

Higher orders: can contain in principle all ops

$$\delta\mathcal{A}_{\text{CFT}} = \mathbf{s} \int dt\ O_{\Phi_\perp^1}(t) + \sum_{k=2}^{\infty} \mathbf{s}^k \sum_n b_{n,k}\, \epsilon^{\Delta_n - 1} \int dt\ O_n(t).$$

How do we use it? Like standard perturbation theory, but around interacting CFT

- Expand observables around CFT(0)

$$\langle \ldots \rangle_\theta = \langle \mathrm{P} \ldots e^{\int \delta L(t) dt} \rangle_{\mathrm{1D}}$$
$$= \int dt \, \langle \mathrm{P} \ldots \delta L(t) \rangle_{\mathrm{1D}} + \int_{s_1 < s_2} \langle \mathrm{P} \ldots \delta L(s_1) \delta L(s_2) \rangle_{\mathrm{1D}} + \ldots$$

- Choose a regularisation scheme. In our case:

Point-split all integrals with same ϵ cutoff, which also appears in the action

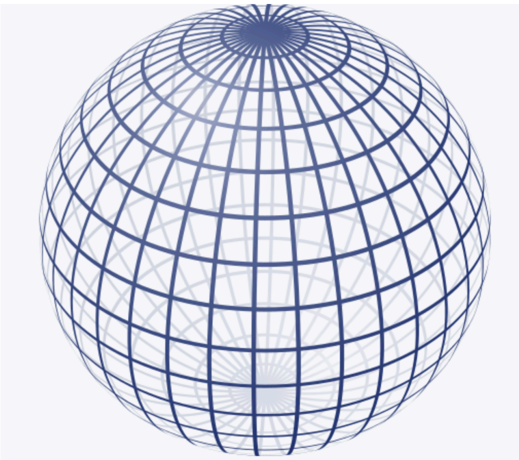
$$\delta \mathcal{A}_{\mathrm{CFT}} = \mathbf{s} \int dt \, O_{\Phi^1_\perp}(t) + \sum_{k=2}^{\infty} \mathbf{s}^k \sum_n b_{n,k} \, \epsilon^{\Delta_n - 1} \int dt \, O_n(t).$$

First do the integrals - then send $\epsilon \to 0$.

Irrelevant operators - which naively have “vanishing prefactor” - will contribute!

- **Coefficients in the action need to be fixed by physics!**

One combination of the constraints has a nice geometric proof



$\mathscr{A}_{CFT}(\overrightarrow{\theta}) = \mathscr{A}_{CFT}(\overrightarrow{0}) + \sum_{i=1}^{5} \theta^i \int dt\, O_\perp^i(t) + O(|\theta|^2)$

- Conformal manifold has natural Zamolodchikov metric

$$g_{ij} \propto \langle \Phi_\perp^i(0) \Phi_\perp^j(1) \rangle = \delta^{ij}\, N_\perp$$

- For a defect breaking global currents we can identify geometry

[Drukker, Kong, Sakkas ’22]

$$\mathscr{M} = G_{original}/G_{unbroken} = SO(6)/SO(5) \simeq S^5$$

5-sphere of radius $\sqrt{N_\perp}$

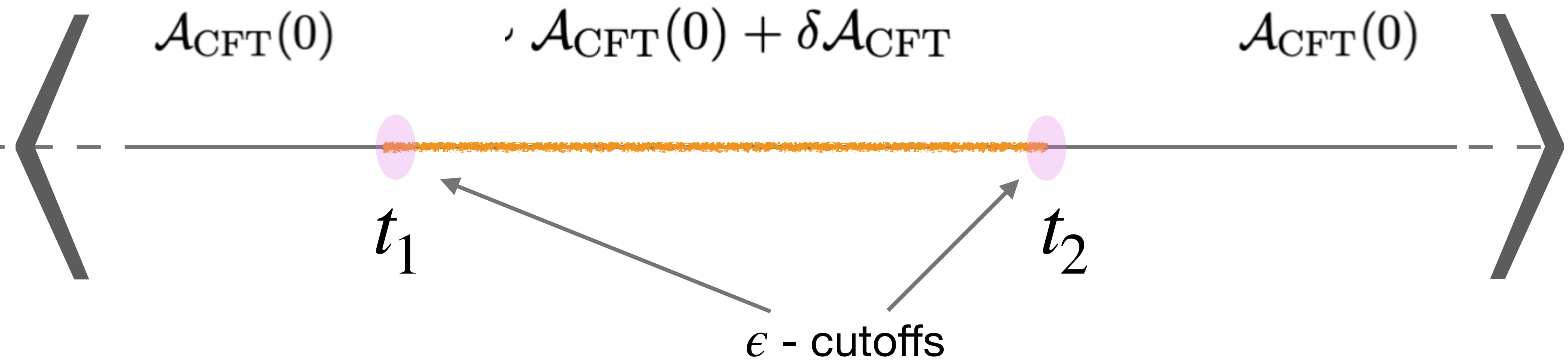
- Integrated 4-tilts give the Riemann tensor, which we now know!

[Kutasov ’12]

Linear combination of Constraint 1 and Constraint 2 where $\mathbb{C}$ is cancelled. [Drukker, Kong, Sakkas ’22]

To deduce the second constraint (with $\mathbb{C}$) we go back to cusps

/ Interfaces between two CFTs on the manifold



$$\propto t_{12}^{-2\Gamma_{cusp}(\theta)}$$

Easy to remove unphysical normalisation:

$$t_{12}^2\, \partial_{t_1}\partial_{t_2} \log\langle \dots \rangle = -\, 2\Gamma_{cusp}(\theta)$$

But for our derivation we need the action at NLO $O(\theta^2)$...

The expansion of the WL in N=4 SYM would suggest sth very simple:

$$\int dt\ \left[\mathbf{s}\ \Phi_{||}(t) + \sqrt{1-\mathbf{s}^2}\ \Phi^1_{\perp}(t)\right] \sim\ \mathbf{s}\int dt\ \Phi^1_{\perp}(t) - \frac{\mathbf{s}^2}{2}\int dt\ \Phi_{||}(t) + O(\mathbf{s}^4)$$

… but to avoid dangerous scheme dependence we will be totally agnostic:

$$\delta A_{CFT} = \mathbf{s}\int dt\ O^1_{\perp}(t) + \mathbf{s}^2 \sum_{\Delta_n} \epsilon^{\Delta_n - 1} b_{n,2} \int dt\ O_{\Delta_n}(t) + O(\mathbf{s}^4)$$

couplings (a.k.a. Wilson coefficients)

All operators!

How can we fix them?

- **We are on the conformal manifold.**

→ **All couplings should have vanishing beta functions.**

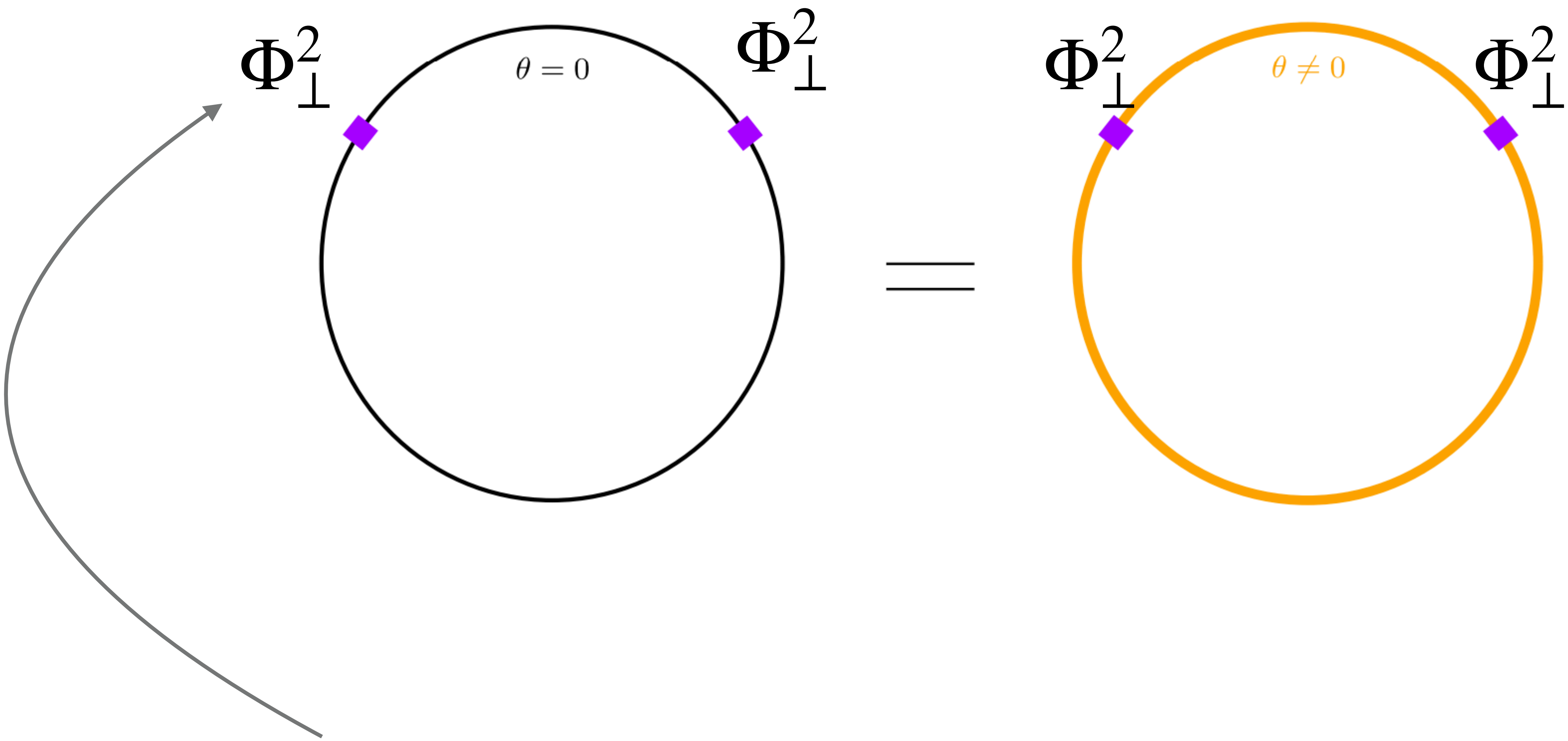
This should fix them up to reparametrisations of the manifold

- **The θ-rotation remains a symmetry on the space of theories.**

→ **When done on the whole line,**
it should actually do nothing to 1D CFT observables!

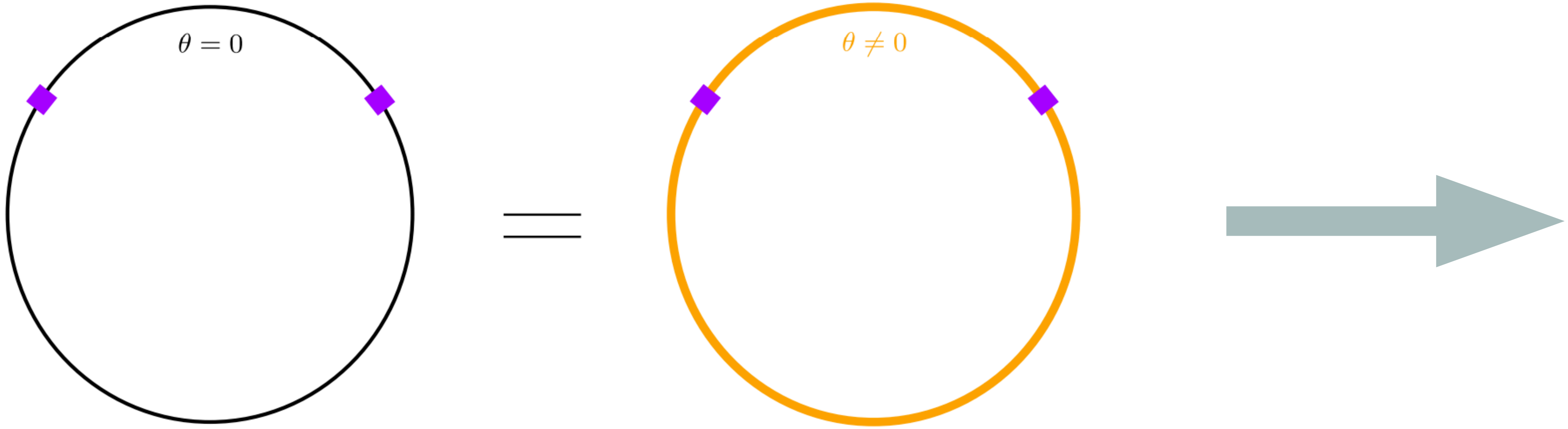
Very powerful - it is what we will actually use

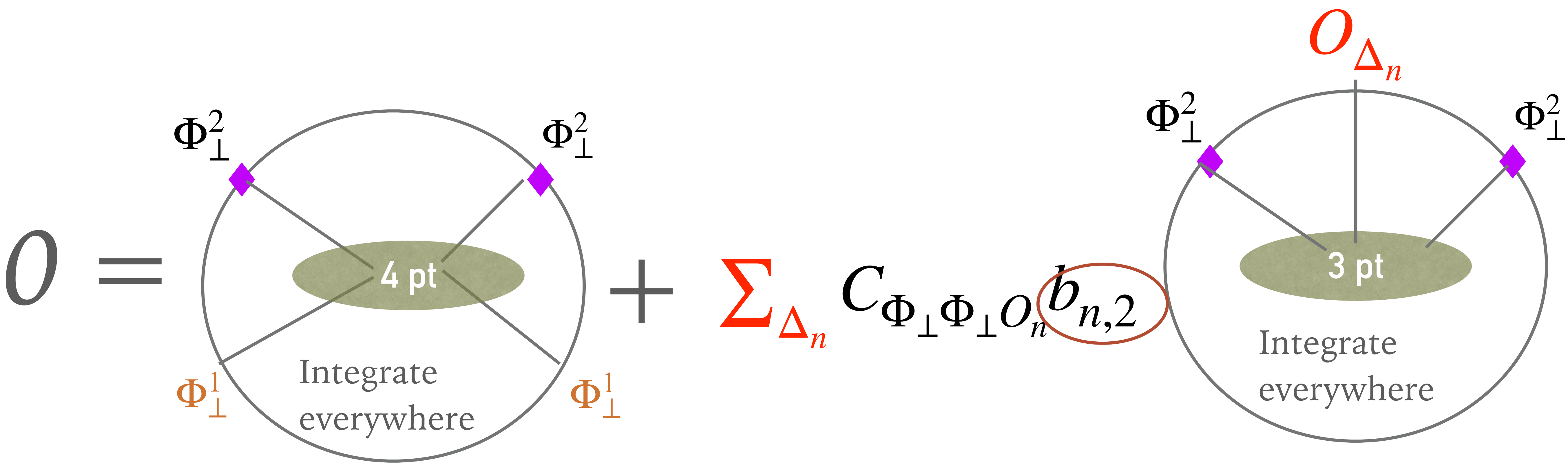
Key constraint in our derivation:



Tilt operator in direction orthogonal to both $\Phi_{||}$ **and** $\Phi_\perp^1$

(thus, the operator itself needs no redefinition with θ !)



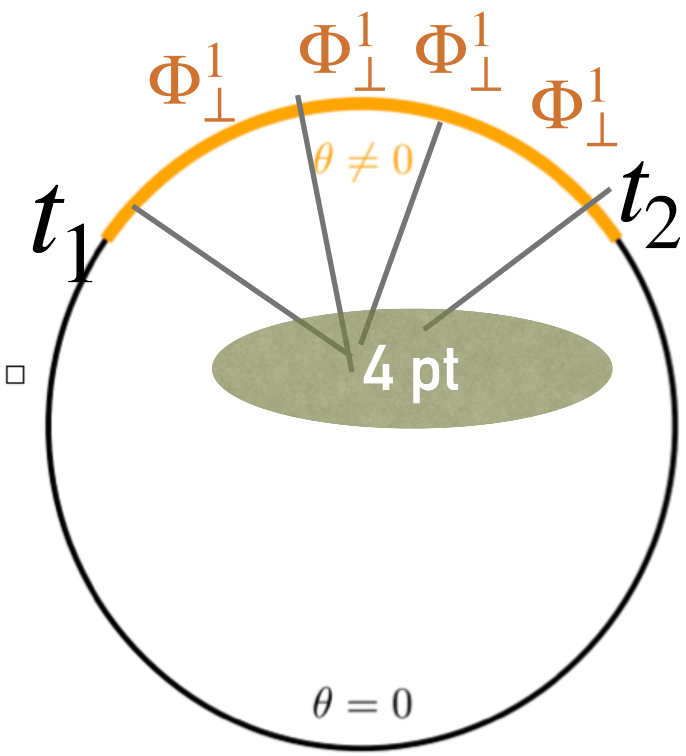


We get a sum rule constraining Wilson coefficients!

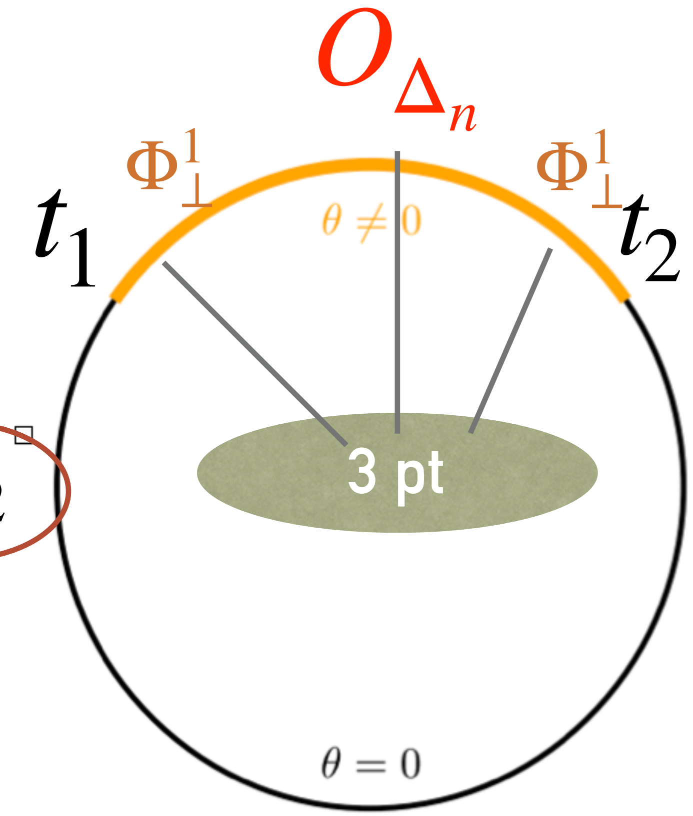
$$\frac{1}{\mathbb{B}}\sum_{\Delta_n>1} b_{n,2}\frac{C_n}{\Delta_n-1} = \int_{\delta_x}^{\frac{1}{2}} dx\,\frac{(x-2)\delta f(x)}{x^3} - (3-\mathbb{F})\log(\delta_x) + (\mathbb{F}-2)\ \log(2) + 1.$$

The “cusp” calculation gives

$t_{12}^2\, \partial_{t_1}\partial_{t_2} \log \Big\langle$



$+ \sum_{\Delta_n} C_{\Phi_\perp \Phi_\perp O_n} b_{n,2}$



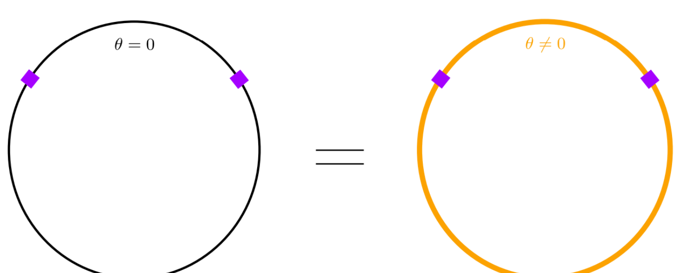
$+ \dots \Big\rangle \sim \mathbb{C}$

$$-\frac{1}{2}(\mathbb{C}+\mathbb{B}) = -(2\mathbb{B}) \sum_{\Delta_n>1} b_{n,2} \frac{4C_n}{\Delta_n - 1} - (2\mathbb{B})^2(2-\mathbb{F})\,(1+\log 4) + 4\,\mathbb{B}^2$$
$$- (2\mathbb{B})^2 \left[\int_{\delta_x}^{\frac{1}{2}} dx \frac{(2x-3)((x-1)x+1)\delta f(x)}{(x-1)x^3} dx + \frac{3}{2}(3-\mathbb{F}) \log \delta_x \right]$$

A second sum rule!

Cusp

$$-\frac{1}{2}(\mathbb{C}+\mathbb{B}) = -(2\mathbb{B})\sum_{\Delta_n>1} b_{n,2}\,\frac{4C_n}{\Delta_n-1} - (2\mathbb{B})^2(2-\mathbb{F})\,(1+\log 4) + 4\,\mathbb{B}^2$$
$$-\,(2\mathbb{B})^2\left[\int_{\delta_x}^{\frac{1}{2}} dx\,\frac{(2x-3)((x-1)x+1)\delta f(x)}{(x-1)x^3}\,dx + \frac{3}{2}(3-\mathbb{F})\log\delta_x\right]$$



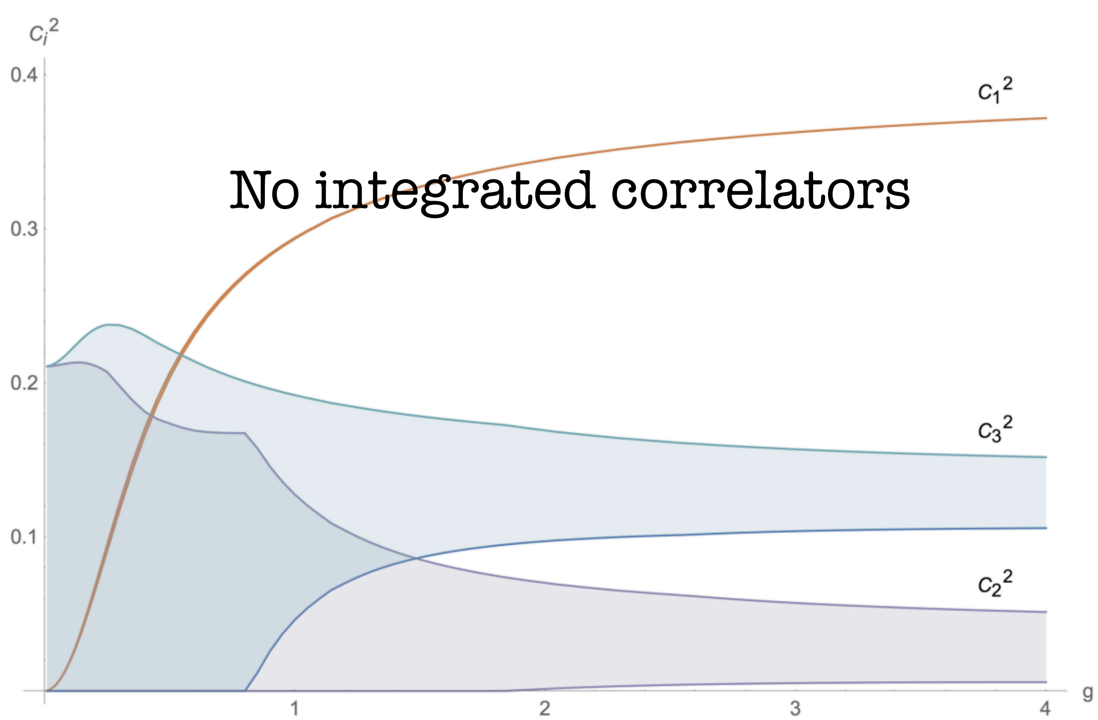
$$\frac{1}{\mathbb{B}}\sum_{\Delta_n>1} b_{n,2}\frac{C_n}{\Delta_n-1} = \int_{\delta_x}^{\frac{1}{2}} dx\,\frac{(x-2)\delta f(x)}{x^3} - (3-\mathbb{F})\log(\delta_x) + (\mathbb{F}-2)\,\log(2) + 1.$$

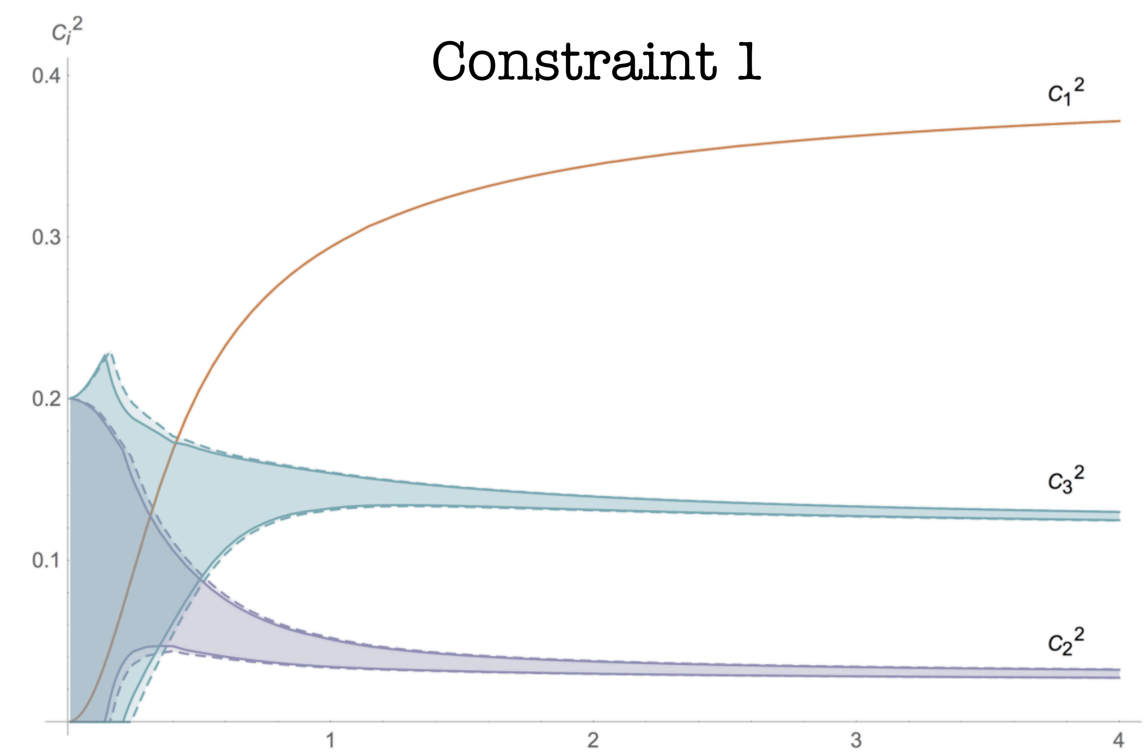
We just eliminate the Wilson coefficients from the two relations, which finally gives the constraint on the integrated 4-point function

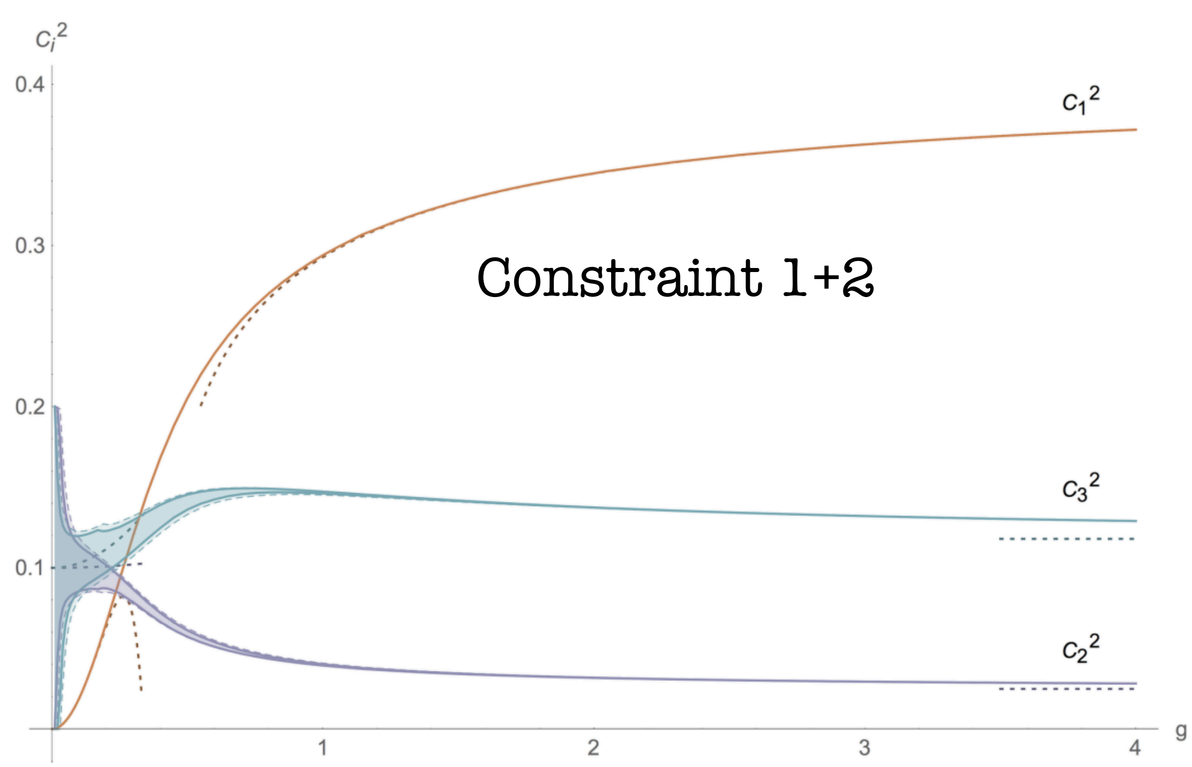
Comments and outlook

A new method to extract information for bootstrability, orthogonal to localization. It should also work far from BPS

This was very effective in the context of the single-correlator bootstrap, numerically as well as analytically at weak coupling







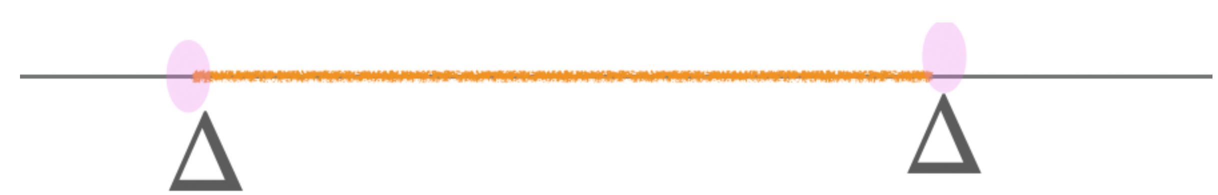
Interesting to see how far can we shrink the bounds…
is there any hope of convergence?

Together with bootstrap in multi-correlator channels,
[AC,Gromov,Julius,Preti Sokolova, to appear]
integrated correlators seem avenue for dramatic improvement…

In principle we have access to **a lot of information**:

Integrated n-point functions … (cf. multi-point bootstrap [Barrat, Liendo, Peveri '22])

Integrated non-BPS 4-pt functions…



Integrated local correlators from the γ - deformation…

Further questions

Conformal perturbation theory: we only obtained sum rules for $b_{n,2}$ - can we actually fix all coefficients?

Does this give any further constraints? or at least allow to compute interesting observables? (e.g. multi-cusp configurations…)

Any use for the method in non susy defect context..?

[Gimenez-Grau, Lauria, Liendo, van Vliet '22]

Formally everything should work for defects breaking global symmetries - except $\mathbb{B}$ and $\mathbb{C}$ are unknown.

This is a way to fix them - however $\mathbb{B}$ also has further interpretation as bulk-defect CFT data so one of the two identities - at least - could be constraining. Is there more that one can do?

Thank you for listening