String theory seminar
DESY/University of Hamburg
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BURKE INSTITUTE

Caltech

Emergent $\mathcal{N}=4$ SUSY from $\mathcal{N}=1$

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Based mostly on arXiv:2302.06622 [MJK, Craig Lawrie, Ki-Hong Lee, Jaewon Song]

4d SCFTs

Put a 4d SCFT on a curved manifold:

- ➡ The conformal symmetry becomes anomalous and characterized by two quantities $\Rightarrow$ central charges a & c

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4 + \cdots$$

4d SCFTs

Put a 4d SCFT on a curved manifold:

➡ The conformal symmetry becomes anomalous and characterized by two quantities $\Rightarrow$ central charges a & c

UV Theory

a_{UV} , c_{UV}

RG flow

IR Theory

a_{IR} , c_{IR}

$$\langle T^{\mu}_{\mu} \rangle = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4 + \cdots$$

a-theorem : $a_{IR} < a_{UV}$

(a is monotonically decreasing along the RG flow)

[Komargodski,Schwimmer]

4d SCFTs

Put a 4d SCFT on a curved manifold:

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$$\langle T^{\mu}_{\mu} \rangle = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4 + \cdots$$

c mostly decreases along the flow, but not necessarily.

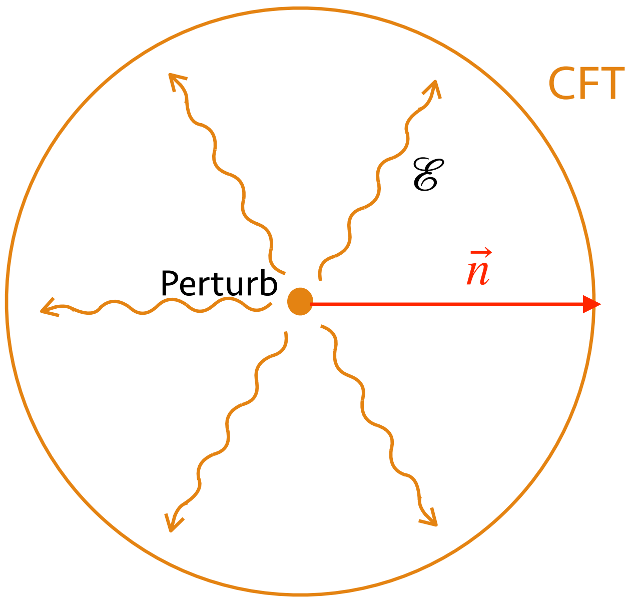
a-theorem : $a_{IR} < a_{UV}$

(a is monotonically decreasing along the RG flow)

[Komargodski,Schwimmer]

The ratio a/c is bounded

➡ Unitarity : $\frac{1}{3} < \frac{a}{c} < \frac{31}{18}$ [Hofman,Maldacena]



Scattering event

- Energy flux through the sphere @infinity:

$$\langle \mathcal{E}(\hat{n}) \rangle > 0$$

- The $\hat{n}$ dependence coefficient for $\mathcal{E} \propto \left(\frac{c-a}{c}\right)$

If $a = c$: The flux in all direction is the same.
The energy/charge propagates isotropically!

The ratio a/c is bounded

- Unitarity : $\frac{1}{3} < \frac{a}{c} < \frac{31}{18}$ [Hofman,Maldacena]
- For supersymmetric theories, the bound gets narrower:

$\mathcal{N}=0$ **SCFTs**	free scalar	$\frac{1}{3} < \frac{a}{c} < \frac{31}{18}$
$\mathcal{N}=1$ **SCFTs**	free chiral	$\frac{1}{2} < \frac{a}{c} < \frac{3}{2}$
$\mathcal{N}=2$ **SCFTs**	free hyper	$\frac{1}{2} < \frac{a}{c} < \frac{5}{4}$
$\mathcal{N}=3,4$ **SCFTs**		$a=c$

free vector

[Aharony,Evtikhiev]

Supersymmetric field theories are highly constraining

- Non-renormalization theorems
- Certain protected quantities are exactly computable
- Rich mathematical structures

Supersymmetric field theories are highly constraining

- Traditionally regarded as a high-energy symmetry in the UV
- Can arise as an emergent symmetry in the IR
- In fact, learned lot about RG using SUSY

IR duality, conformal manifolds, symmetry enhancement, dangerously irrelevant operators, non-commuting flows, ...

Emergent Supersymmetry

- ➡ In two-dimensions, supersymmetry has been shown to emerge in the dilute Ising model at the tri-critical point. [Friedan,Qiu,Shenker'85]
- ➡ This has been extended to quantum critical points of higher-dimensional lattice models. [Lee'06]
- ➡ 4d SYM is suggested to arise from strong-coupling dynamics in the low energy limit of a non-supersymmetric gauge theory. [Kaplan'84]
- ➡ Found $\mathcal{N}=1$ theories in the $\mathcal{N}=1$ preserving conformal manifold of 4d $\mathcal{N}=4$ SYM. [Leigh-Strassler'95]
- ➡ Supersymmetry can emerge at the edges of a topological superconductor, that can be potentially realized experimentally. [Grover,Sheng,Vishwanath'13]

SUSY enhancement via RG flow

➡ One can begin with minimal supersymmetry and flow to an enhanced supersymmetry in the IR.

➡ Several known cases where an $\mathcal{N}=1$ theory flows to an $\mathcal{N}=2$ theory. This provides $\mathcal{N}=1$ Lagrangian descriptions for the $\mathcal{N}=2$ non-Lagrangian theories. [Gadde,Razamat,Willet][Maruyoshi,Song][Razamat,Zafrir][Zafrir]

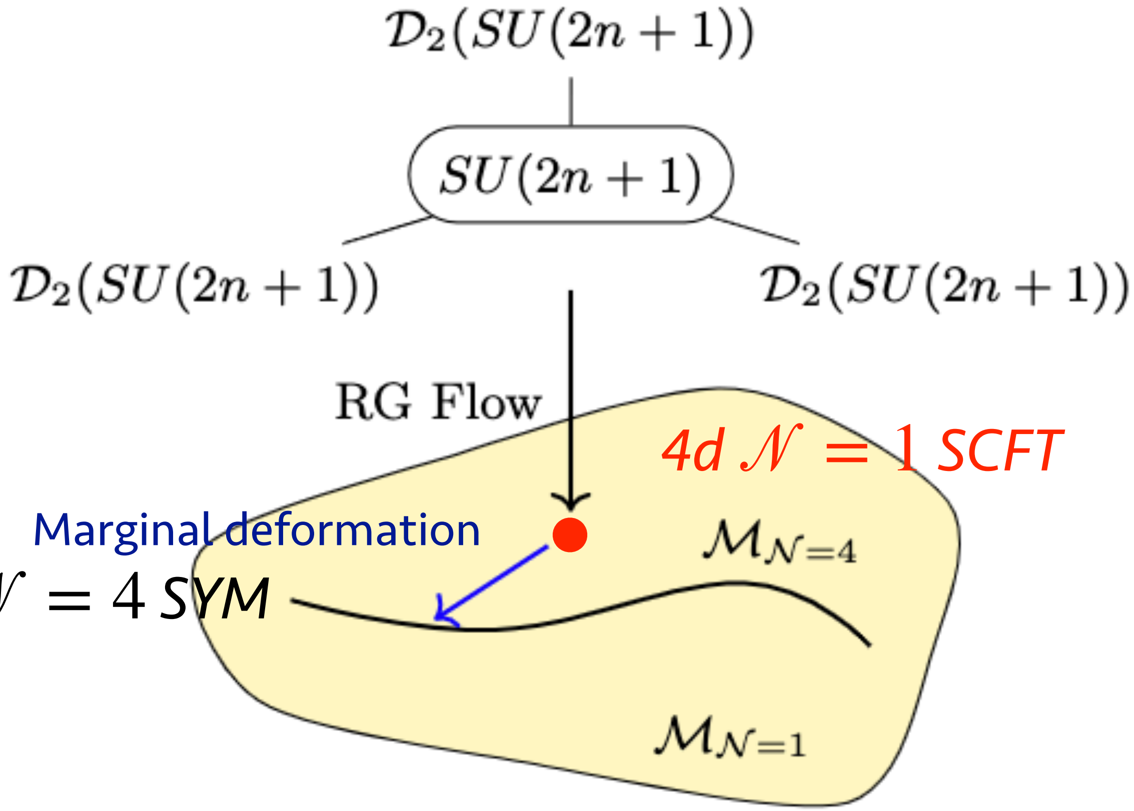
➡ Not only an interesting phenomenon by itself, but also provides a powerful tool to analyze non-perturbative dynamics of the IR fixed point. *(which often has no Lagrangian description with the full extended supersymmetry manifest)*

➡ Supersymmetry enhancement can be thought of as another example of **IR duality**!

Using the RG flow:

Construct an $\mathcal{N}=1$ non-Lagrangian theory that is dual to the $\mathcal{N}=4$ SYM theory!

4d $\mathcal{N}=1$ non-Lagrangian theory



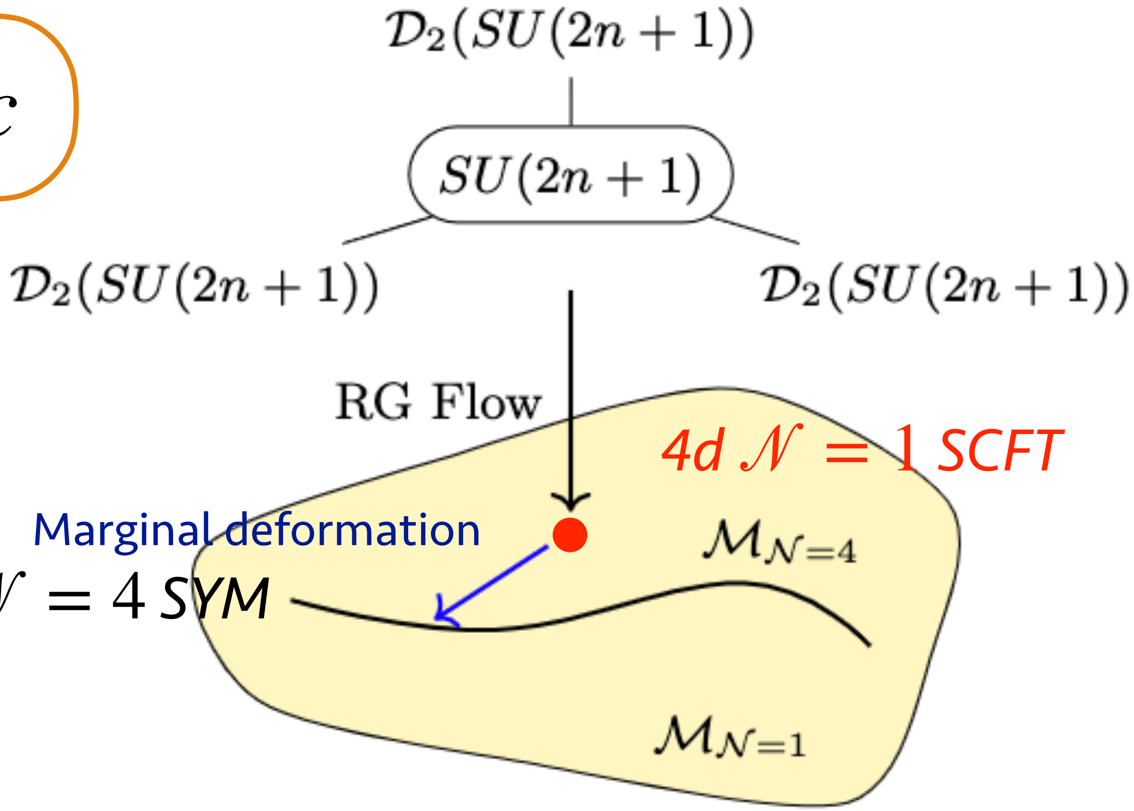
Conformal manifold of 4d $\mathcal{N}=4$ SYM

$\mathcal{N}=1$ preserving conformal manifold

[MJK,Lawrie,Lee,Song'23]

4d $\mathcal{N} = 1$ non-Lagrangian theory

This theory has $a = c$

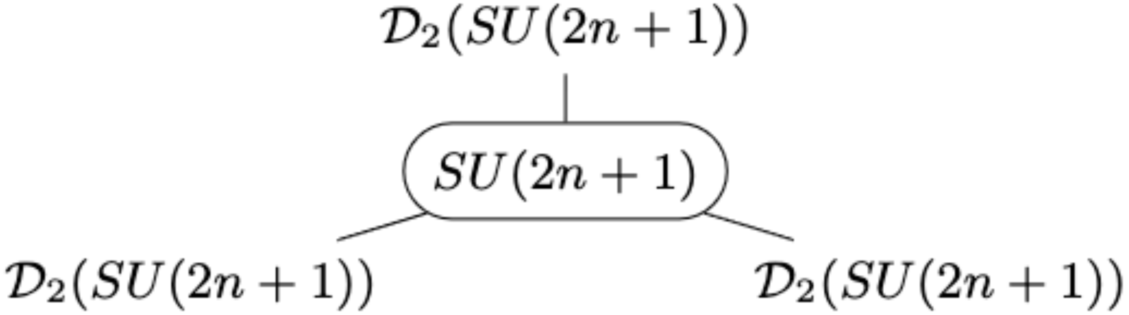


$\mathcal{N} = 1$ preserving conformal manifold

[MJK,Lawrie,Lee,Song'23]

[MJK,Lawrie,Lee,Song'21][MJK,Lawrie,Lee,Song'23]

Construct this $\mathcal{N}=1$ theory with $a=c$

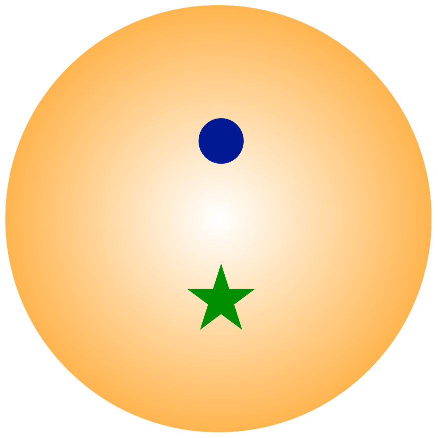


$\mathscr{D}_p(G)$ theory

[Cecotti,del Zotto][Cecotti,del Zotto,Giacomelli]
[Xie][Wang,Xie]

- A 4d $\mathcal{N} = 2$ SCFT of Argyres-Douglas type with a flavor symmetry (at least) G.

- Class $\mathcal{S}$ description:



Regular puncture (flavor symmetry G)

Irregular puncture (parametrized by p)

Extra/enhanced symmetry is due to the irregular puncture

Flavor central charge:

$$k_G = \frac{2(p-1)}{p} h_G^\vee$$

$$-2\mathrm{Tr}(R_{\mathcal{N}=2} T^a T^b) = k_G \delta^{ab}$$

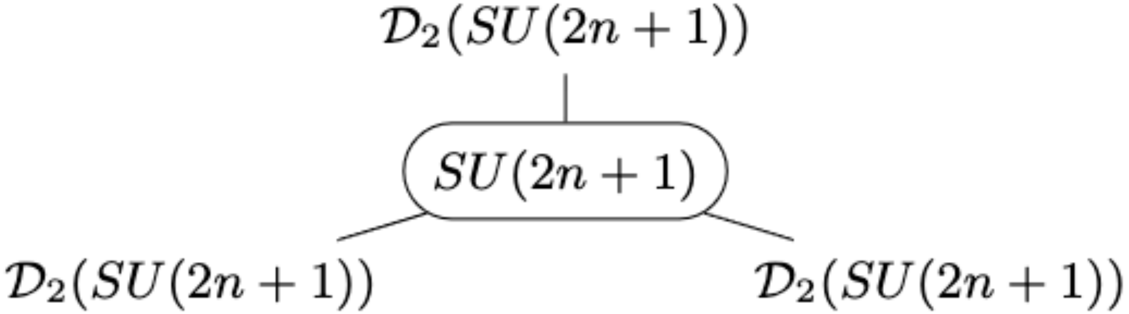
($k_G = h_G^\vee$ if $p = 2$)

behaves as a fractional amount of an adjoint matter

G	$SU(N)$	$SO(2N)$	E_6	E_7	E_8
No additional symmetry	$(p, N) = 1$	$p \notin 2\mathbb{Z}_{>0}$	$p \notin 3\mathbb{Z}_{>0}$	$p \notin 2\mathbb{Z}_{>0}$	$p \notin 30\mathbb{Z}_{>0}$

[MJK,Lawrie,Lee,Song'23]

The dual theory is built out of 3 copies of $\mathscr{D}_2(SU(2n+1))$, gauging the diagonal flavor symmetry group via $\mathcal{N}=1$ gauge multiplet.



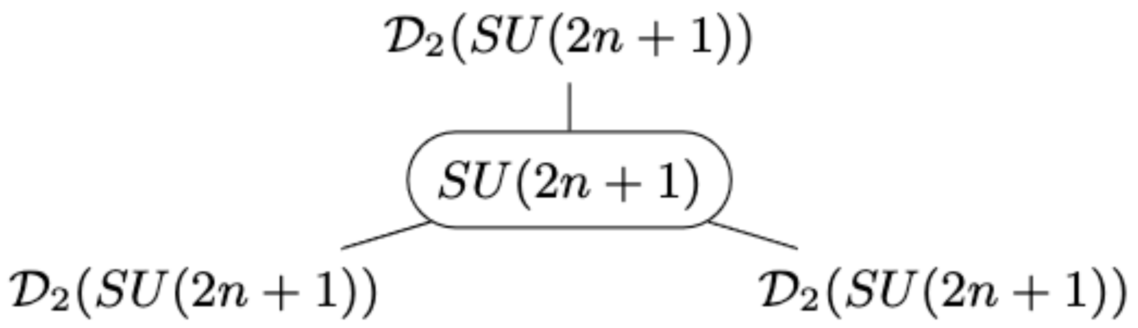
This gives an asymptotic free gauge theory that flows to a point on the conformal manifold of $\mathcal{N}=4$ SYM with $G=SU(2n+1)$.

How do we verify this novel duality?

- Matching anomalies
- Matching chiral operators
- Matching superconformal indices

Diagonal gauging

[MJK,Lawrie,Lee,Song'21]
[MJK,Lawrie,Lee,Song'23]



➡ The one-loop β-function coefficient for the gauge coupling:

$$\beta_g \sim -\mathrm{Tr}RGG \sim -\frac{3}{2}(2n+1) < 0$$: IR strongly coupled → SCFT IR fixed point

Asymptotically free!

➡ $\mathscr{D}_2(G)$ behaves like a half of an adjoint chiral multiplet in terms of one-loop β-function contribution: $N_f = 3/2\ N_c$.

Symmetries: R and Flavor

[MJK,Lawrie,Lee,Song'21]
[MJK,Lawrie,Lee,Song'23]

$$\mathscr{D}_2(SU(2n+1)) \quad : \qquad SU(2n+1) \times SU(2) \times U(1)$$

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$$\mathscr{D}_2(SU(2n+1)) \ : \ SU(2n+1) \times U(1)_R \times U(1)_{F_1}$$

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Gauged

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Gauged

Combined:
$G \times U(1)_{R_0} \times U(1)_F^3$

Generators:

$$R_0 = \frac{1}{3} r + \frac{4}{3} I_3 \,, \quad F = -r + 2 I_3$$

$U(1)_R$ charge of the $\mathcal{N}=2$ R-symmetry

$SU(2)_R$ Cartan of the $\mathcal{N}=2$ R-symmetry

Symmetries: R and Flavor

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Gauged

Combined:
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Upon gauging:

- *Only the anomaly-free combinations are preserved!*
- *In the IR, the putative $U(1)_F^3$ has an ABJ anomaly*
- *Broken to the anomaly-free $U(1)_{\mathscr{F}}^2$*

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$$R_0 = \frac{1}{3}r + \frac{4}{3}I_3 \, , \quad F = -r + 2I_3$$

$U(1)_R$ charge of the $\mathcal{N} = 2$ R-symmetry

$SU(2)_R$ Cartan of the $\mathcal{N} = 2$ R-symmetry

$\mathcal{N} = 1$ SCFT at the IR fixed point

➡ Need: U(1) R-symmetry: $\mathrm{Tr} R T^a T^b = 0\,.$

➡ In a supersymmetric theory, conformal anomalies (i.e. central charges) **fixed** by the trace anomalies of the R-symmetry:

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right), \quad c = \frac{1}{32}\left(9\mathrm{Tr}R^3 - 5\mathrm{Tr}R\right).$$

[Anselmi,Freedman,Grisaru,Johansen]

➡ The R-charge (or mixing parameters ϵ_i) is determined **uniquely** by a-maximization:

$$\frac{\partial a}{\partial \epsilon_i} = 0, \quad \frac{\partial^2 a}{\partial \epsilon_i \epsilon_j} < 0\,.$$

[Intriligator,Wecht]

➡ Check if the theory is **unitary** upon RG flow via a-maximization.

Symmetries: R and Flavor

[MJK,Lawrie,Lee,Song'21]
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Anomaly-free: $G \times U(1)_R \times U(1)_{\mathscr{F}}^2$

$(\mathscr{F}_1 \equiv F_2 - F_1, \quad \mathscr{F}_2 \equiv F_3 - F_2)$

where $R = R_0 + \sum_{i=1}^{3} \epsilon_i F_i$

(Anomaly $\sim \mathrm{Tr} RGG$*)*

$\mathcal{N}=1$ SCFT at the IR fixed point

[MJK,Lawrie,Lee,Song'21]
[MJK,Lawrie,Lee,Song'23]

- The R-symmetry this theory is $R = R_0 + \sum_{i=1}^{3} \epsilon_i F_i\,.$
- Using the anomaly-free condition,

$$0 = \mathrm{Tr}\, RGG = h^\vee_G + \sum_{i=1}^{3}\left(\left(\frac{1}{3}-\epsilon_i\right)\mathrm{Tr}_i rGG + \left(\frac{4}{3}+2\epsilon_i\right)\mathrm{Tr}_i I_3\right)$$

$$\longrightarrow \quad 6 - \sum_{i=1}^{3}\left(1-3\epsilon_i\right) = 0$$

- Now a-maximization fixes the mixing parameters ϵ_i :

$$a(\epsilon_1,\epsilon_2,\epsilon_3) = \frac{d}{32}\left(13 - 9\sum_{i=1}^{3}\epsilon_i^2(\epsilon_i+2)\right) \quad \longrightarrow \quad \epsilon := \epsilon_1 = \epsilon_2 = \epsilon_3 = -\frac{1}{3}\,.$$

- The central charges: $a = c = \frac{1}{4}\dim(SU(2n+1))$

$\mathcal{N} = 1$ SCFT at the IR fixed point

[MJK,Lawrie,Lee,Song'21]
[MJK,Lawrie,Lee,Song'23]

- The R-symmetry this theory is $R = R_0 + \sum_{i=1}^{3} \epsilon_i F_i$.
- Using the anomaly-free condition,

$$0 = \mathrm{Tr}\, RGG = h_G^\vee + \sum_{i=1}^{3} \left(\left(\frac{1}{3} - \epsilon_i \right) \mathrm{Tr}_i rGG + \left(\frac{4}{3} + 2\epsilon_i \right) \mathrm{Tr}_i I_3 \right)$$

$$\longrightarrow \quad 6 - \sum_{i=1}^{3} (1 - 3\epsilon_i) = 0$$

- Now a-maximization fixes the mixing parameters ϵ_i :

$$a(\epsilon_1, \epsilon_2, \epsilon_3) = \frac{d}{32} \left(13 - 9 \sum_{i=1}^{3} \epsilon_i^2 (\epsilon_i + 2) \right) \quad \longrightarrow \quad \epsilon := \epsilon_1 = \epsilon_2 = \epsilon_3 = -\frac{1}{3} .$$

- The central charges: $a = c = \frac{1}{4} \dim(SU(2n+1))$ *Does this look familiar?*

Matching anomalies

[MJK,Lawrie,Lee,Song'23]

➡ The anomaly polynomial of the IR theory is

$$I_6 = \frac{1}{6}k_{RRR}c_1(R)^3 + \sum_{\alpha=1}^{2}\frac{1}{6}k_{RR\mathcal{F}_\alpha}c_1(R)^2c_1(\mathcal{F}_\alpha) + \sum_{\alpha,\beta=1}^{2}\frac{1}{6}k_{R\mathcal{F}_\alpha\mathcal{F}_\beta}c_1(R)c_1(\mathcal{F}_\alpha)c_1(\mathcal{F}_\beta)$$
$$+ \sum_{\alpha,\beta,\gamma=1}^{2}\frac{1}{6}k_{\mathcal{F}_\alpha\mathcal{F}_\beta\mathcal{F}_\gamma}c_1(\mathcal{F}_\alpha)c_1(\mathcal{F}_\beta)c_1(\mathcal{F}_\gamma) - \frac{1}{24}k_Rc_1(R)p_1(T) - \sum_{\alpha=1}^{2}\frac{1}{24}k_{\mathcal{F}_\alpha}c_1(\mathcal{F}_\alpha)p_1(T)\,,$$

with $k_{RRR} = \frac{8d}{9}\,,\quad k_{R\mathcal{F}_\alpha^2} = -\frac{2d}{3}\,,\quad k_{R\mathcal{F}_1\mathcal{F}_2} = \frac{d}{3}\,,\quad k_{\mathcal{F}_1^2\mathcal{F}_2} = -k_{\mathcal{F}_1\mathcal{F}_2^2} = d\,,\quad a = c = \frac{1}{4}d$

where $d = \dim(SU(2n+1)) = 4n(n+1)\,.$

Match those of $\mathcal{N} = 4$ SYM with $G = SU(2n+1)$!

$c_1(R)$: the 1st Chern class of the superconformal R-symmetry bundle
$p_1(T)$: the 1st Pontryagin class of the tangent bundle to the 4d spacetime
$c_1(\mathcal{F}_\alpha)$: the 1st Chern class of the bundles associated to each $U(1)_{\mathcal{F}_a}$

Matching anomalies

[MJK,Lawrie,Lee,Song'23]

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$$+ \sum_{\alpha,\beta,\gamma=1}^{2}\frac{1}{6}k_{\mathcal{F}_\alpha\mathcal{F}_\beta\mathcal{F}_\gamma}c_1(\mathcal{F}_\alpha)c_1(\mathcal{F}_\beta)c_1(\mathcal{F}_\gamma) - \frac{1}{24}k_R c_1(R)p_1(T) - \sum_{\alpha=1}^{2}\frac{1}{24}k_{\mathcal{F}_\alpha}c_1(\mathcal{F}_\alpha)p_1(T)\,,$$

with $k_{RRR} = \frac{8d}{9}, \quad k_{R\mathcal{F}_\alpha^2} = -\frac{2d}{3}, \quad k_{R\mathcal{F}_1\mathcal{F}_2} = \frac{d}{3}, \quad k_{\mathcal{F}_1^2\mathcal{F}_2} = -k_{\mathcal{F}_1\mathcal{F}_2^2} = d\,, \quad a = c = \frac{1}{4}d$

where $d = \dim(SU(2n+1)) = 4n(n+1)\,.$

Match those of $\mathcal{N} = 4$ SYM with $G = SU(2n+1)$!

The 't Hooft anomalies are invariant on a conformal manifold

⟶ *The IR theory lives in the same $\mathcal{N} = 1$ preserving conformal manifold as $\mathcal{N} = 4$ SYM!*

How do we verify this novel duality?

- Matching anomalies ✓
- Matching chiral operators
- Matching superconformal indices

Matching chiral operators

[MJK,Lawrie,Lee,Song'23]

➡ Compare the gauge-invariant operator spectrum of dual theories.

➡ The single-trace chiral operators of $\mathcal{N} = 4$ SYM: $\mathrm{Tr}\,\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}$.

($\phi_{i_k\in\{1,2,3\}}$: adjoint chirals)

➡ Each $\mathscr{D}_2(SU(2n+1))$ has the moment map operator μ in the adjoint of G with dimension 2 and carries R-charges $r = 0,\ I_3 = 1$.

➡ Under the RG flow, the moment map operators in the IR becomes

$$\Delta_{\mathrm{IR}}(\mu) = \frac{3}{2}R = \frac{3}{2}\left(\frac{4}{3} + 2\epsilon\right) = 1$$

➡ Upon gauging $\mu_1,\ \mu_2,\ \mu_3$, only the singlet part survives:

$$\mathrm{Tr}\,\mu_{i_1}\mu_{i_2}\cdots\mu_{i_k} \qquad (i_k \in \{1,2,3\})$$

Matching chiral operators

[MJK,Lawrie,Lee,Song'23]

➡ Each $\mathcal{N}=2$ multiplet containing a Coulomb branch operator has two $\mathcal{N}=1$ chiral multiplets, whose components are u (the scalar primary) and its $\mathcal{N}=2$ descendant Q^2u.

➡ Each $\mathcal{D}_2(SU(2n+1))$ has Coulomb branch operators of scaling dimensions $\Delta_{CB}=\frac{r}{2}=\left\{\frac{3}{2},\frac{5}{2},\cdots,\frac{2n+1}{2}\right\}$ and their superpartners $(\Delta, R_{\mathcal{N}=2}, I_3)=(\Delta_{CB}+1, r-2, 1)$.

➡ Upon the RG flow, these operators in the IR become

$$\Delta_{IR}(u)=(1-3\epsilon)\,\Delta_{CB}(u)\,,\quad \Delta_{IR}(Q^2u)=1+6\epsilon+(1-3\epsilon)\,\Delta_{CB}(u)\,.$$

➡ With $\epsilon=-1/3$ and putting them together, $\Delta_{IR}=\{2,3,\cdots,2n+1\}$.

➡ The Casimir operator of $\mathcal{N}=4$ SYM is $\mathrm{Tr}(\phi_i)^k$, which match this!

Matching chiral operators

[MJK,Lawrie,Lee,Song'23]

➡ The operator spectrum matches as

$\mathcal{N}=4$ **SYM**	$\mathcal{N}=1$ **dual theory**
$\mathrm{Tr}\,\phi_{i_1}\phi_{i_2}\cdots\phi_{i_k}$	$\mathrm{Tr}\,\mu_{i_1}\mu_{i_2}\cdots\mu_{i_k}$
$\mathrm{Tr}(\phi_i)^k$	$(u_i, Q^2 u_i)$

Superfluous looking chiral operators are removed via relations

The adjoint part of the square of the moment map operator is vanishing

$$\mu^2\big|_{\text{adj}} = 0\,, \quad \mathrm{Tr}\mu^k = 0\,.$$

Because the Higgs branch of this theory is given by a nilpotent orbit

Removes superfluous Casimir operators in the spectrum

Matching conformal manifolds

[MJK,Lawrie,Lee,Song'23]

- The $\mathcal{N} = 1$ dual theory has 5 marginal operators:
  - 3 from each Coulomb branch operators of dimension $3/2$.
  - 2 formed from moment maps: $\mathrm{Tr}\,\mu_1\mu_2\mu_3$ and $\mathrm{Tr}\,\mu_1\mu_3\mu_2$.
  - Two are marginally irrelevant: it breaks $U(1)^2$ symmetry. They combine with the broken flavor symmetry currents to form a long multiplet and becomes non-BPS. [Green,Komargodski,Seiberg,Tachikawa,Wecht]
- $\mathcal{N} = 4$ SYM has 11 marginal operators $\mathrm{Tr}\,\phi_i\phi_j\phi_k$:
  - 8 marginally irrelevant: recombine with the generators of SU(3) flavor symmetry broken at a generic point of the conformal manifold
- The conformal manifold is 3 (complex) dimensional and **matching**!

Matching conformal manifolds

[MJK,Lawrie,Lee,Song'23]

➡ Move to the $U(1)^2$-preserving sub-locus in the conformal manifold:

- The off-diagonal generators of SU(3) current combine with marginal operators to become long multiplets and become irrelevant.
- This removes 6 out of 11 from the $\mathcal{N} = 4$ SYM side.

➡ Hence they both give 5 (complex) dimensions, **matching**!

How do we verify this novel duality?

- Matching anomalies ✓
- Matching chiral operators ✓
- Matching superconformal indices

Matching superconformal indices

- For $\mathscr{D}_2(SU(2n+1))$ with $n > 1$, we don't know the full index.
- The only case computable: $\mathscr{D}_2(SU(3))$. [Agarwal,Maruyoshi,Song]
- Using this, we can compute the superconformal index of the $\mathcal{N}=1$ dual theory: $I = \mathrm{Tr}(-1)^F t^{3(R+2j_2)} y^{2j_1} \Pi_i v_i^{f_i}$ [MJK,Lawrie,Lee,Song'22]
- This matches that of $\mathcal{N}=4$ SYM with $G = SU(3)$:

$$\begin{aligned}\widehat{I}^{\mathfrak{su}_3} &\equiv (1-t^3y)(1-t^3/y)(I^{\mathfrak{su}_3}-1)\\ &= t^4\chi_{\mathbf{6}}^{\mathfrak{su}_3} - t^5\chi_{\mathbf{2}}^{\mathfrak{su}_2}\chi_{\mathbf{3}}^{\mathfrak{su}_3} + t^6(\chi_{\mathbf{10}}^{\mathfrak{su}_3} - \chi_{\mathbf{8}}^{\mathfrak{su}_3} + 1) - t^7\chi_{\mathbf{2}}^{\mathfrak{su}_2}(\chi_{\mathbf{6}}^{\mathfrak{su}_3} - \chi_{\bar{\mathbf{3}}}^{\mathfrak{su}_3}) + t^8(\chi_{\mathbf{15}'}^{\mathfrak{su}_3} - \chi_{\mathbf{15}}^{\mathfrak{su}_3} + \chi_{\bar{\mathbf{6}}}^{\mathfrak{su}_3} + 2\chi_{\mathbf{3}}^{\mathfrak{su}_3})\\ &\quad - t^9\chi_{\mathbf{2}}^{\mathfrak{su}_2}(\chi_{\mathbf{10}}^{\mathfrak{su}_3} + 1) + t^{10}(\chi_{\mathbf{3}}^{\mathfrak{su}_2}\chi_{\bar{\mathbf{3}}}^{\mathfrak{su}_3} + \chi_{\mathbf{21}}^{\mathfrak{su}_3} - \chi_{\overline{\mathbf{15}}}^{\mathfrak{su}_3} + 2\chi_{\mathbf{6}}^{\mathfrak{su}_3} - 2\chi_{\bar{\mathbf{3}}}^{\mathfrak{su}_3}) + \cdots,\end{aligned}$$

[MJK,Lawrie,Lee,Song'23]

  - $\chi_{\mathbf{R}}^{\mathfrak{su}_2} = \chi_{\mathbf{R}}^{\mathfrak{su}_2}(y)$ *is the character of the representation* $\mathbf{R}$ *in Lorentz spin* j_1.
  - *The* $U(1)^2$ *flavor symmetry enhances to* $SU(3)$ *at certain points of the conformal manifold* $\longrightarrow$ *Each term is written in terms of* $\chi_{\mathbf{R}}^{\mathfrak{su}_3}$ *of the enhanced flavor*

Matching Schur index

➡ If there existed the full index for $\mathscr{D}_2(SU(2n+1))$ with $n \geq 1$, the index of the dual theory would be

$$I(p,q) = \int [dz] I_{vec}(z) \prod_{i=1}^{3} I^{\mathscr{D}_2(SU(2n+1))}(z) \Big|_{t \to (pq)^{\frac{2}{3}+\epsilon_i}}, \quad \epsilon_i = -\frac{1}{3}.$$

➡ For $\mathscr{D}_2(SU(2n+1))$ with $n \geq 1$, the Schur limit of the index is known:

$$I_S^{\mathscr{D}_2(SU(2n+1))}(q;z) = \mathrm{PE}\left[\frac{q}{1-q^2}\chi_{\mathbf{adj}}(z)\right]$$

[Xie,Yan,Yau][Song,Xie,Yan]

➡ This is identical to that of a free hypermultiplet upon rescaling $q \to q^2$.

➡ The index of $\mathcal{N}=4$ SYM: $I^{\mathcal{N}=4}(p,q) = \int [dz] I_{vec}(z) I_{chi}(z)^3$,

where the index for the adjoint chiral is $I_{chi}(z) = \mathrm{PE}\left[\frac{(pq)^{1/3} - (pq)^{2/3}}{(1-p)(1-q)}\chi_{\mathrm{adj}}(z)\right].$

Matching Schur index

➡ Take the Schur limit: $q = \mathfrak{t} = (pq)^{\frac{1}{3}}$ or equivalently $p \to q^2$ [Buican,Nishinaka]

$$I^{dual}(p,q) = \int [dz] I_{vec}(z) \prod_{i=1}^{3} \mathrm{PE}\left[\frac{q}{1-q^2}\chi_{\mathbf{adj}}(z)\right]\Big|_{q,\mathfrak{t}\to(pq)^{1/3}}$$

[MJK,Lawrie,Lee,Song'23]

$$= \int [dz] I_{vec}(z) \mathrm{PE}\left[\frac{(pq)^{1/3}-(pq)^{2/3}}{(1-p)(1-q)}\chi_{\mathrm{adj}}(z)\right]^3\Big|_{p\to q^2} = I^{\mathcal{N}=4}(p,q)$$

➡ The two Schur indices match!

How do we verify this novel duality?

- Matching anomalies ✓
- Matching chiral operators ✓
- Matching superconformal indices ✓

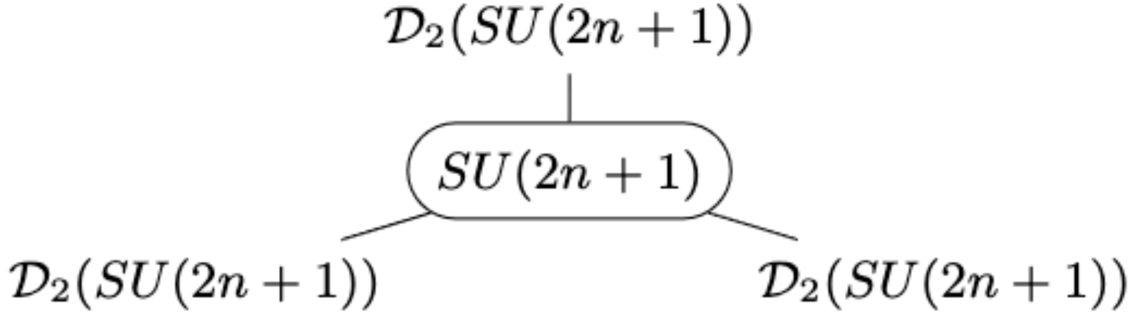
Voila!

What if we consider $\mathscr{D}_{p\geq 2}(G \neq SU(2n+1))$?

- Construct $\mathcal{N} = 1{,}2$ SCFTs with $a = c$ in this fashion!

Now we are familiar with our dual theory:

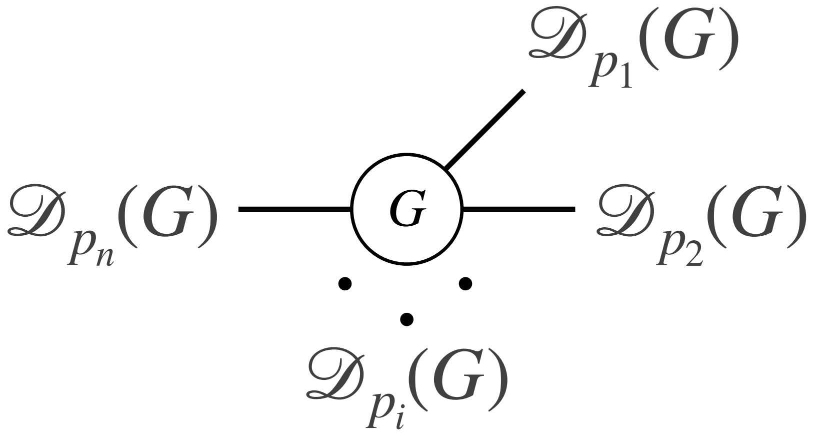
The dual theory is built out of 3 copies of $\mathscr{D}_2(SU(2n+1))$, gauging the diagonal flavor symmetry group via $\mathcal{N}=1$ gauge multiplet.



Actually, this is a special example of $\mathcal{N}=1$ theory with $a=c$!
Shall we construct them?

Gauging/gluing $\mathscr{D}_p(G)$ theories

➡ A collection of $\mathscr{D}_p(G)$ theories can be gauged together by their common flavor symmetry G.



➡ To obtain an $\mathcal{N}=2$ SCFT upon gauging, the beta function for the gauge coupling has to vanish.

$$\beta_G = 0 \iff \sum_{i=1}^{n} k_i = 4h_G^\vee$$

flavor central charge of $\mathscr{D}_{p_i}(G)$: $k_i = \frac{2(p_i-1)}{p_i}h_G^\vee$

$$\Longrightarrow \sum_{i=1}^{n} \frac{1}{p_i} = n-2$$

➡ Very restrictive — there are only 4 solutions:

the $\{p_i\}$ sets $(2,2,2,2),\ (3,3,3),\ (2,4,4),\ (2,3,6)$.

$\hat{\Gamma}(G)$ theory with $a = c$

[MJK,Lawrie,Song]

(p_1, p_2, p_3, p_4)	$\widehat{\Gamma}(G)$	Quivers via gauging $\mathcal{D}_p(G)$s	$a = c$
$(2,2,2,2)$	$\widehat{D}_4(G)$	$\begin{array}{ccccc} & & \mathcal{D}_2(G) & & \\ & & \vert & & \\ \mathcal{D}_2(G) & - & G & - & \mathcal{D}_2(G) \\ & & \vert & & \\ & & \mathcal{D}_2(G) & & \end{array}$	$\frac{1}{2}\dim(G)$
$(1,3,3,3)$	$\widehat{E}_6(G)$	$\begin{array}{ccccc} & & \mathcal{D}_3(G) & & \\ & & \vert & & \\ \mathcal{D}_3(G) & - & G & - & \mathcal{D}_3(G) \end{array}$	$\frac{2}{3}\dim(G)$
$(1,2,4,4)$	$\widehat{E}_7(G)$	$\begin{array}{ccccc} & & \mathcal{D}_2(G) & & \\ & & \vert & & \\ \mathcal{D}_4(G) & - & G & - & \mathcal{D}_4(G) \end{array}$	$\frac{3}{4}\dim(G)$
$(1,2,3,6)$	$\widehat{E}_8(G)$	$\begin{array}{ccccc} & & \mathcal{D}_2(G) & & \\ & & \vert & & \\ \mathcal{D}_3(G) & - & G & - & \mathcal{D}_6(G) \end{array}$	$\frac{5}{6}\dim(G)$

➡ The theory has $a = c$ when

$$\gcd(h_G^\vee, \alpha_\Gamma) = 1$$

- $h_G^\vee$ the dual Coxeter number of G
- α_Γ the largest comark associated to the affine Dynkin diagram $\hat{\Gamma}$

while $\Gamma = D_4, E_6, E_7, E_8$.

➡ Such a theory has **no flavor symmetry**.

$\hat{\Gamma}(G)$ theory with $a = c$

[MJK,Lawrie,Song]

$\widehat{\Gamma}(G)$	$a = c$
$\widehat{D}_4(SU(2\ell+1))$	$2\ell(\ell+1)$
$\widehat{E}_6(SU(3\ell\pm 1))$	$2\ell(3\ell\pm 2)$
$\widehat{E}_6(SO(6\ell))$	$2\ell(6\ell+1)$
$\widehat{E}_6(SO(6\ell+4))$	$2(2\ell+1)(3\ell+2)$
$\widehat{E}_7(SU(4\ell\pm 1))$	$6\ell(2\ell\pm 1)$
$\widehat{E}_8(SU(6\ell\pm 1))$	$10\ell(3\ell\pm 1)$

➡ The theory has $a = c$ when

$$\gcd(h^\vee_G, \alpha_\Gamma) = 1$$

- $h^\vee_G$ the dual Coxeter number of G
- α_Γ the largest comark associated to the affine Dynkin diagram $\hat{\Gamma}$

while $\Gamma = D_4, E_6, E_7, E_8$.

➡ The theory has (at least) 1 exactly marginal coupling.

➡ The theory has 1-form symmetry given by the center of G.

Connection to $\mathcal{N} = 4$ SYM

[MJK,Lawrie,Song]

The Schur index of $\hat{\Gamma}(G)$ theory without any flavor symmetry

$$\Vert$$

Up to rescaling fugacities

The Schur index of $\mathcal{N} = 4$ SYM

$$I_{\widehat{\Gamma}(G)}(q) = I_G^{\mathcal{N}=4}(q^{\alpha_\Gamma}, q^{\alpha_\Gamma/2-1})$$

➡ This is a superset of theories with $a = c$:

$$\begin{gathered}\widehat{\Gamma}(SU(N)) \quad \text{with} \ \gcd(\alpha_\Gamma, N) = 1\,, \\ \widehat{E}_6(SO(2N))\,, \quad \widehat{D}_4(E_6)\,, \quad \widehat{E}_6(E_7)\,, \quad \widehat{E}_7(E_6)\,, \\ \widehat{D}_4(E_8)\,, \quad \widehat{E}_6(E_8)\,, \quad \widehat{E}_7(E_8)\,, \quad \widehat{E}_8(E_8)\,.\end{gathered}$$

For $\hat{E}_6(SU(2))$, this relation follows from a graded vector space isomorphism between the associated vertex operator algebras [Buican,Nishinaka]

Schur index for $\Gamma = D_4, E_6, E_7, E_8$

- For the theories with $a = c$, the $\mathscr{D}_p(G)$ theories needed does not carry any extra flavor symmetry. The Schur index of a $\mathscr{D}_p(G)$:

$$I_{\mathscr{D}_p(G)}(q,\vec{z}) = \mathrm{PE}\left[\frac{q-q^p}{(1-q)(1-q^p)}\chi^G_{\mathrm{adj}}(\vec{z})\right]$$

[Song,Xie,Yan][Kac,Wakimoto]

- The Schur Index of the $\hat{\Gamma}(G)$ theory with $a = c$ is then

$$I_{\hat{\Gamma}(G)}(q) = \int [d\vec{z}\,]\,\mathrm{PE}\left[\frac{q+q^{\alpha_\Gamma-1}-2q^{\alpha_\Gamma}}{(1-q)(1-q^{\alpha_\Gamma})}\chi^G_{\mathrm{adj}}(\vec{z})\right]$$

[MJK,Lawrie,Song]

- For $\hat{D}_4(G)$ theory, the Schur index can be written in terms of MacMahon's generalized 'sum-of-divisor' function, which is quasi-modular:

$$I_{\widehat{D}_4(SU(2k+1))}(q) = q^{-k(k+1)}A_k(q^2)$$

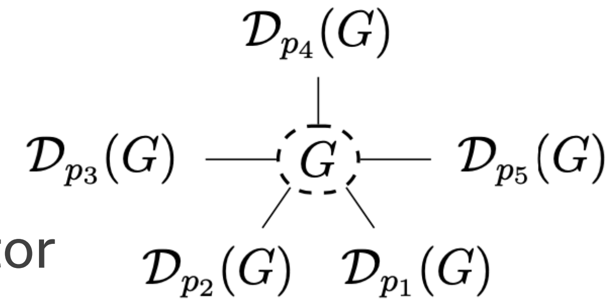
$$I^{\mathcal{N}=4}_{SU(2k+1)}(q) = q^{-\frac{k(k+1)}{2}}A_k(q)$$

$$A_k(q) = \sum_{0<m_1<m_2\cdots<m_k}\frac{q^{m_1+\cdots m_k}}{(1-q^{m_1})^2\cdots(1-q^{m_k})^2}$$

4d $\mathcal{N}=1$ SCFTs with $a=c$

[MJK,Lawrie,Lee,Song'21]

- Now do the similar construction for $\mathcal{N}=1$ theories with $a=c$.
- We have more sets of $\{p_i\}$ satisfying the bound:



$\beta > 0$: IR-free → Decouples to each $\mathscr{D}_p(G)$ + free vector

$\beta = 0$: conformal gauging (as before)

$\beta < 0$: IR strongly coupled → SCFT IR fixed point ← Asymptotically-free

$$\beta \leq 0 \ : \quad \sum_{i=1}^{n} \frac{1}{p_i} \geq n-3$$

Conditions on p

[MJK,Lawrie,Lee,Song'21]

➡ $\beta < 0 \;:\; \displaystyle\sum_{i=1}^{n} \frac{1}{p_i} > n-3$

Not all such gaugings flow to interacting SCFTs

p_1	p_2	p_3	p_4	p_5
$\cancel{1}$	$\cancel{1}$	$\cancel{1}$	$\cancel{1}$	$\cancel{p_5}$
1	1	1	p_4	p_5
1	1	p_3	p_4	p_5
1	2	2	p_4	p_5
1	2	3	≤ 6	p_5
1	2	3	7	≤ 41
1	2	3	8	≤ 23
1	2	3	9	≤ 17

p_1	p_2	p_3	p_4	p_5
1	2	3	10	≤ 14
1	2	3	11	≤ 13
1	2	4	4	p_5
1	2	4	5	≤ 19
1	2	4	6	≤ 11
1	2	4	7	≤ 9
1	2	5	5	≤ 9
1	2	5	6	≤ 7

p_1	p_2	p_3	p_4	p_5
1	3	3	3	p_4
1	3	3	4	≤ 11
1	3	3	5	≤ 7
1	3	4	4	≤ 5
2	2	2	2	p_5
2	2	2	3	3
2	2	2	3	4
2	2	2	3	5

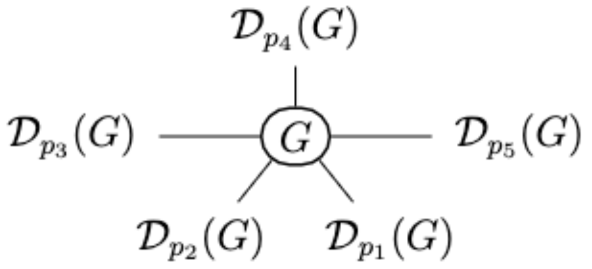
This is what gave the dual theory to $\mathcal{N} = 4$ SYM!

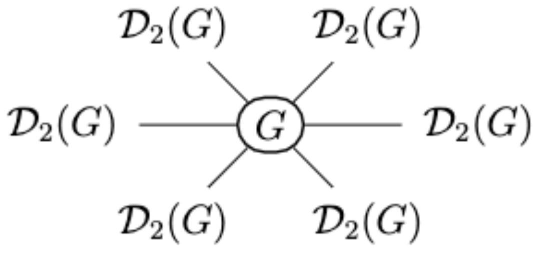
➡ $\beta = 0 \;:\; \displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = n-3$

p_1	p_2	p_3	p_4	p_5	p_6
1	1	2	2	7	42
1	1	2	3	8	24
1	1	2	3	9	18
1	1	2	3	10	15
1	1	2	3	12	12
1	1	2	4	5	20

p_1	p_2	p_3	p_4	p_5	p_6
1	1	2	4	6	12
1	1	2	4	8	8
1	1	2	5	5	10
1	1	2	6	6	6
1	1	3	3	4	12
1	1	3	3	6	6

p_1	p_2	p_3	p_4	p_5	p_6
1	1	3	4	4	6
1	1	4	4	4	4
1	2	2	3	3	3
1	2	2	2	4	4
1	2	2	2	3	6
2	2	2	2	2	2





➡ Surprising fact: when $gcd(p_i, h_G^\vee) = 1$, we always have $a = c$!

Including adjoint matter

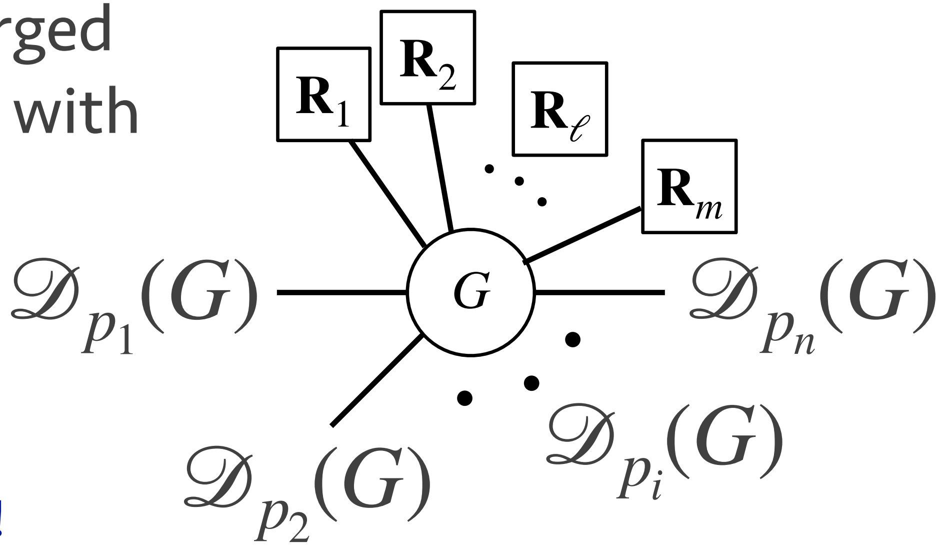
[MJK,Lawrie,Lee,Song'21]

➢ Include m number of chiral multiplets charged under G in the representation $\mathbf{R}_{\ell=1,\cdots m}$ of G with R-charge R_ℓ. the Dynkin index of the representation

$$\beta \leq 0 \ : \quad \sum_{i=1}^{n} \frac{p_i - 1}{p_i} + \sum_{\ell=1}^{m} \frac{I(\mathbf{R}_\ell)}{h_G^\vee} \leq 3$$

Can **only** be satisfied with $m = 0,1,2,3$.

$\Longrightarrow$ Can include up to **three** adjoint chirals!



$$\frac{\dim(G)}{h_G^\vee} = \frac{48(a_i - c_i)}{-\frac{p_i - 1}{p_i} h_G^\vee} = \frac{\dim(\mathbf{R}_\ell)}{I(\mathbf{R}_\ell)}$$

If $\gcd(h_G^\vee, \alpha_\Gamma) = 1$

If the matter is in the adjoint representation

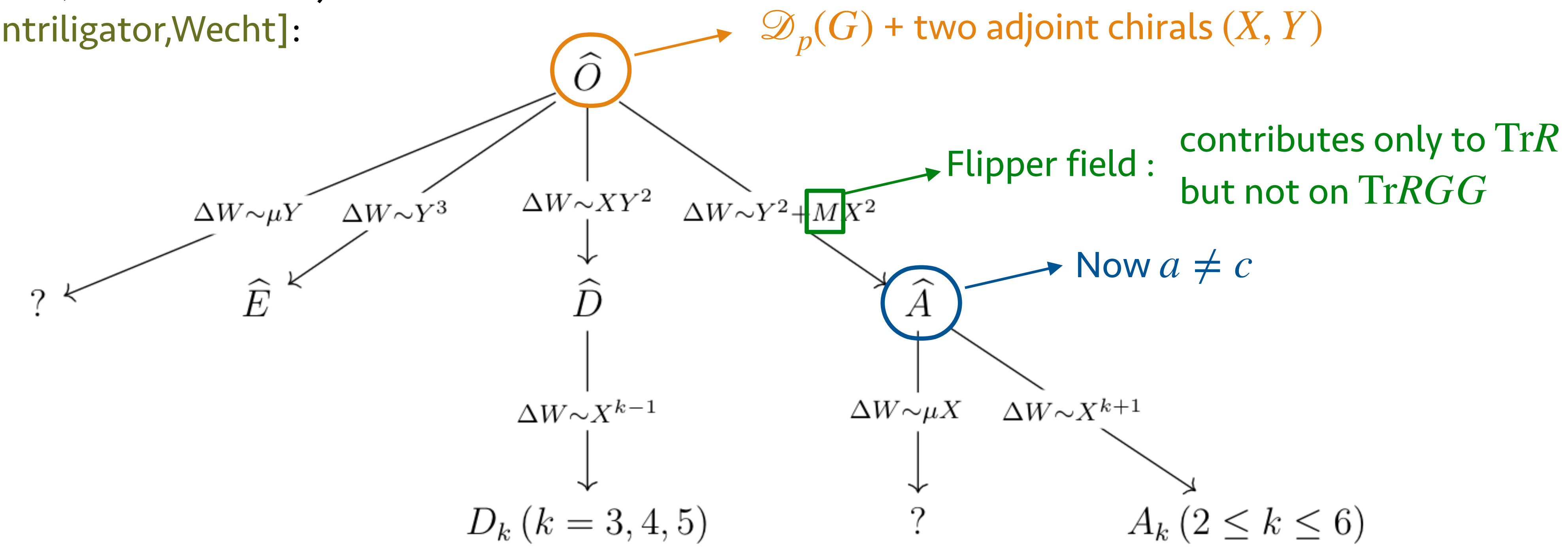
$$16\,(a - c) = \frac{\dim(G)}{h_G^\vee} \,\mathrm{Tr}\, RGG = 0$$

then the anomaly cancellation guarantees $a = c$.

Deform by superpotentials:

Landscape of $\mathcal{N} = 1$ theories with $a = c$

Inspired from SQCD with two adjoint chirals analysis of [Intriligator,Wecht]:



[MJK,Lawrie,Lee,Song, To appear]

RG flows from $\mathcal{N}=1$ $a=c$ theories to $\mathcal{N}=4$ SYM

- It turns out many (not all) of the $a=c$ theories we consider can be deformed so that it RG flows to the $\mathcal{N}=4$ SYM theory!
- Upon deforming $\mathscr{D}_p(G)$ via its relevant operator of scaling dimension $\Delta = \frac{p+1}{p}$, it flows to a theory of $|G|$ free chirals.
  [Bolognesi,Giacomelli,Konishi][Xie,Yan]
- By deforming our $\mathcal{N}=1,2$ $a=c$ SCFT using this operator, we can effectively replace the $\mathscr{D}_p(G)$ via an adjoint chiral in G.
- Once we reach 3 adjoint chirals and nothing else, we get a theory that is in the **same conformal manifold as $\mathcal{N}=4$ SYM**!

Holographic outlook

- ➡ The aforementioned class $\mathcal{S}$ description paves the way to a holographic dual of our $\mathcal{N}=1$ theory in the UV.

[MJK,Lawrie,Song'21[MJK,Lawrie,Lee,Song'21'22]

- ➡ The putative holographic duals of $\mathcal{N}=1$,2 SCFTs with $a=c$ should have vanishing $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ correction term, whose coefficient is proportional to $(c-a)$, in the supergravity action.

[Anselmi,Kehagias]

- ➡ Our novel RG flow suggests that there is a domain wall solution interpolating the $\mathcal{N}=1$ UV theory and the IR $\mathcal{N}=4$ SYM, which admits ``miraculous cancellations,'' and therefore sheds light on this seemingly fine-tuned coefficient. [MJK,Lawrie,Lee,Song'23]

Thank you for listening!