Notes on QFT in curved spacetime

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Abstract

Here we compile notes on quantum field theory in curved spacetime and the Hawking effect.

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Chapter 1

The Klein Gordon Field

1.1 Overview

A spacetime is considered to contain a black hole if the entire spacetime is not contained in the causal past of future null infinity. In other words, there exists a region in such a spacetime from which even light cannot escape. The simplest example of a black hole is the spherically symmetric Schwartzschild vacuum solution to the Einstein's equation. Here, despite there being any matter around, the spacetime exhibits a singularity (at $r = 0$) and an event horizon (at $r = r_s = 2GM$). Schwartzschild black hole is essentially an eternal black hole, a classical solution that is stationary. An eternal black hole left alone will stay the same for eternity. However, interesting phenomena arise when one attempts to be a bit more realistic and tries to include matter fields interacting quantum mechanically in such a spacetime. Hawking realized in 1976 that in such a quantum mechanical setting black holes must emit thermal radiation, and will eventually evaporate away! This astounding observation has led to a flurry of research, especially to explain an apparent paradox that results from such a behavior — is the evaporation of black holes due to Hawking radiation in contradiction with the laws of quantum mechanics that necessitate unitary (information-preserving) evolution of pure states?

In these notes we review the framework of QFT in curved spacetime which will help us understand the Hawking effect. The central idea is that notion of particles becomes ill-defined in curved spacetimes. As we will review below, in the flat spacetime, Poincaré symmetry allows us to formulate QFT from the perspective of global inertial observers, and a notion of particle can be defined that all such observers agree upon. However, in curved spacetime, these ideas do not generalize and as a result two observers need not agree on what they call, for example, a "zero-particle" vacuum state. One must then carefully (re-)formulate QFT in a basis independent fashion that is not tied to specifics of Minkowski space. It might appear a daunting task to consider QFT in curved spacetime since already in flat spacetimes field theories, such as a non-abelian gauge theory, are challenging enough. Fortunately, they key physics behind the Hawking effect has little to do with non-linearity of the quantum field and this effect can be analyzed by considering linear (free) scalar fields with the classical solutions obeying the Klein-Gordon equation. These notes are based on books by Wald [1] and Carroll [2].

1.2 The Klein-Gordon Field

Our goal in this chapter is to reformulate the QFT for linear scalar field in a coordinate invariant way. We begin with recalling the classical action for Klein-Gordon field in Minkowski spacetime:

$$S = -\frac{1}{2}\int \mathrm{d}^4x \left(\partial_a\phi\partial^a\phi + m^2\phi^2\right) \tag{1.1}$$

where we follow the $(-,+,+,+)$ prescription for the metric signature. The classical equations of motion are

$$\left(\Box - m^2\right)\phi = 0\,. \tag{1.2}$$

For flatspace time we can introduce a global inertial coordinate system. Different inertial coordinate systems will be related to each other via spacetime translations and Lorentz transformations. In this inertial system, we can write the action as

$$S = \int \mathrm{d}t\,\mathcal{L}\,, \qquad \mathcal{L} = \frac{1}{2}\int \mathrm{d}^3x \left(\dot{\phi}^2 - \left(\vec{\nabla}\phi\right)^2 - m^2\phi^2\right). \tag{1.3}$$

For convenience, we will replace $\mathbb{R}^3$ by a three-torus T^3 with side length L and impose periodic boundary condition on the scalar field. This allows us to decompose the space-integral above in Fourier modes

$$\phi(t,\boldsymbol{x}) = L^{-3/2}\sum_{\boldsymbol{k}}\phi_{\boldsymbol{k}}(t)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,, \qquad \boldsymbol{k} = \frac{2\pi}{L}(n_1,n_2,n_3)\,, \tag{1.4}$$

with the inverse Fourier transform given by

$$\phi_{\boldsymbol{k}}(t) = L^{-3/2}\int \mathrm{d}^3x\,\phi(t,\boldsymbol{x})e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,, \tag{1.5}$$

and express the Lagrangian in terms of these modes

$$\mathcal{L} = \sum_{\boldsymbol{k}}\frac{1}{2}|\dot{\phi}_{\boldsymbol{k}}|^2 - \frac{1}{2}\omega_{\boldsymbol{k}}^2|\phi_{\boldsymbol{k}}|^2\,, \tag{1.6}$$

where

$$\omega_{\boldsymbol{k}} \equiv \boldsymbol{k}^2 + m^2\,. \tag{1.7}$$

Thus, the linear scalar field is equivalent to collection of (countably) infinitely many decoupled harmonic oscillators characterized by frequencies $\omega_{\boldsymbol{k}}$. We must then figure out how to deal with the case of infinite oscillators.

1.3 Classical phase-space and the symplectic structure

Before we consider the quantum theory, let us look closely at the mathematical structure of the classical solutions. Consider an n-dimensional classical system specified by positions $\{q_i\}$

and momenta $\{p_i\}$. The classical dynamics is governed by the Hamiltonian via the Hamilton's equations of motion

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}\,, \tag{1.8}$$

The positions and momenta comprise a $2n$-dimensional Manifold $\mathcal{M}$. It will be convenient to express the $2n$ coordinates $\{q_i, p_i\}$ as $y = (q_1, \ldots q_n; p_1, \ldots p_n)$, such that Eq. (1.8) becomes

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}t} = \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \frac{\partial H}{\partial y^\nu}\,, \qquad \Omega^{\mu\nu} = \begin{bmatrix} 0 & 1_{n\times n} \\ -1_{n\times n} & 0 \end{bmatrix}_{\mu\nu} . \tag{1.9}$$

For the reasons mentioned above, we will consider free, linear theories. Hence, we limit our discussion to Hamiltonians that is a quadratic function on y^μ which will lead to linear evolution equations of the coordinates y^μ. Each point in the phase-space $\mathcal{M}$ can be thought of as representing initial data for the Hamilton's equations in Eq. (1.8) and gives rise to a unique solution. One can then identify $\mathcal{M}$ with the manifold of solution space $\mathcal{S}$ which contains elements $y_i(t)$ (here the i index labels the solution with $y_i^\mu(t)$ being the components). Limiting to quadratic Hamiltonians gives rise to linear evolution equations of the positions and momenta, such the solution space acquires a natural vector space structure – linear combinations of solutions to Hamilton's equations are also solutions. However, this is also true of any general, coupled linear first order differential equations. The special property of Hamilton's equations is that they allow us to define a *symplectic product* of two solutions $y_1(t)$ and $y_2(t)$ that is conserved over the course of evolution:

$$s(t) \equiv \Omega(y_1(t), y_2(t)) = \sum_{\alpha\beta} \Omega_{\alpha\beta} y_1^\alpha y_2^\beta\,, \tag{1.10}$$

where $\Omega_{\alpha\beta}$ is the inverse of $\Omega^{\mu\nu}$ in Eq. (1.8). It is a straighforward exercise to check using Eq. (1.9) that for a quadratic Hamiltonian,

$$H(t; y) \equiv \frac{1}{2} \sum_{\mu,\nu} K_{\mu\nu}(t) y^\mu y^\nu\,, \tag{1.11}$$

the symplectic product of solutions $y_{1,2}(t)$ in Eq. (1.10) is time-independent. In proving this one makes use of the antisymmetric property of $\Omega_{\alpha\beta}$. Thus the vector space of solutions $\mathcal{S}$ is now endowed with a symplectic structure $\Omega : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ that is conserved and is independent of the initial time $t = 0$. This is the basic mathematical structure required for construction of the quantum theory.

Next, observables are maps $f : \mathcal{M} \to \mathbb{R}$ that are functions of positions and momenta $f\big(\{q_i\}, \{p_i\}\big)$. Specifically we will be interested in linear observables, where f's are linear combinations of q_i and p_i:

$$f\big((\{q_i\}, \{p_i\})\big) = \sum_i \alpha_i q_i + \beta_i p_i\,. \tag{1.12}$$

Note that the coordinates q_i and p_i are themselves observables. Limiting to linear observables is big simplification since later on in generalizing to QFT, these observables will become distributions and there are technical challenges involved with dealing with products of distributions. Then, all the linear observables can be expressed in terms of the *fundamental observables* $\Omega(y, \cdot)$, where the empty slot is a place-holder for the argument of the observable f. (We have dropped the t argument of y as its product with another solution is conserved). To see this, we can go back to the basis $\{q_i, p_i\}$ and write Eq. (1.10) as

$$\Omega(y_1, y_2) = \sum_\mu \left(p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\mu}\right), \tag{1.13}$$

such that the function f in Eq. (1.12) is given by

$$f(y) = \Omega\big((-\beta_1, -\beta_2, \ldots, -\beta_n; \alpha_1, \ldots \alpha_n)\,,\; y\,\big) \tag{1.14}$$

The Poisson brackets of positions and momenta q_i and p_i,

$$\{q_i, q_j\} = \{p_i, p_j\} = 0\,, \qquad \{q_i, p_j\} = \delta_{ij}\,, \tag{1.15}$$

now in terms of $\Omega(y, \cdot)$ become

$$\big\{\Omega(y_1, \cdot), \Omega(y_2, \cdot)\big\} = -\Omega(y_1, y_2)\,. \tag{1.16}$$

The advantage of expressing the position and momentum observables in terms of $\Omega(y, \cdot)$ is that the above equation holds independently of the choice of the coordinates on the phase space $\mathcal{M}$.

1.4 Quantum theory of infinite oscillators

To construct the quantum theory we will first have to choose a Hilbert space $\mathcal{F}$ of states and hermitian operators $\hat{f}_i : \mathcal{F} \to \mathcal{F}$ that correspond to classical observables f_i. The Poisson bracket relations in the classical theory now become commutation relations on quantum operators. The canonical commutation relations are now given by

$$\big[\hat{\Omega}(y_1, \cdot), \hat{\Omega}(y_2, \cdot)\big] = -\mathrm{i}\Omega(y_1, y_2)\,, \tag{1.17}$$

where the right hand side $\Omega(y_1, y_2)$ is a number and $\hat{\Omega}(y_i, \cdot)$ are hermitian operators corresponding to the classical observables $\Omega(y_i, \cdot)$.

Let us recall that Hamiltonian for a 1 dimensional simple harmonic oscillator oscillating with frequency ω_i is given by

$$H = \frac{1}{2}p_i^2 + \frac{1}{2}\omega_i^2 q_i^2\,. \tag{1.18}$$

We can directly start with canonical commutation relations in terms of $\hat{q}$ and $\hat{p}$ operators and rewrite the above result as the Hamiltonian operator in the quantum theory. To proceed further we then introduce the non-hermitian annihilation operator

$$a_i = \sqrt{\frac{\omega_i}{2}}q_i + \mathrm{i}\sqrt{\frac{1}{2\omega_i}}p_i\,, \tag{1.19}$$

with the spectrum given by

$$|n\rangle = \frac{1}{\sqrt{n!}}\big(a_i^\dagger\big)^n|0\rangle\,, \tag{1.20}$$

where the n^{th} state satisfies $H|n\rangle = (n+1/2)\omega_i$. Thus, in order to construct the Hilbert space $\mathcal{F}$ of the scalar field, we might simply consider taking tensor product of Hilbert spaces for each of the oscillator mode. For example, for n decoupled oscillators the Hilbert space of the combined system can be taken to be

$$\mathcal{F} = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n\,. \tag{1.21}$$

where $\mathcal{F}_i$ is the Hilbert space of a single oscillator with frequency ω_i. While this is okay for finite number of oscillators, it turns out that the product above does not generalize suitably to the case of infinite number of oscillators. Such a generalization turns out to yield a Hilbert space that is *too large*. Similar to how a collection of an infinite string of binary digits 0 and 1 is uncountable, a Hilbert space that includes all states of countably infinite oscillators has an uncountable dimension. We would rather start differently and consider a minimalist approach of finding a Hilbert space that yields sensible results for observables that we are interested in. See the introductory discussion in Ref. [3] for more details.

Thus, we will consider an alternative construction that yields a Hilbert space that coincides with Eq. (1.21) for finitely many oscillators, but can be straightforwardly generalized to the infinite case. The only technical difference between this construction we are about to describe and the one above is that the Hamiltonians differ in the two cases by an additive constant. Essentially, the construction below will result in a Hamiltonian which sets the vacuum energy to zero, as opposed to $\frac{1}{2}\omega_i$ for the one above. This additive constant, though infinite in the case of infinitely many oscillators, is not a cause for concern.

The starting point of the alternative construction is to consider *all* the frequencies $\{\omega_i\}$ at once via the solution space $\mathcal{S}$, instead of first considering all the resonances $\omega_i, 2\omega_i, \ldots$ of a given oscillator and then the tensor product as in Eq. (1.21). In the language of describing the classical system above, we consider $2n$-dimensional phase-space of positions and momenta (which are equivalent to the scalar field value and its derivative). Note that in the classical theory the solutions $y \in \mathcal{S}$ are real. For example, for initial conditions $(q,p) = (0,a)$ at $t=0$, the classical solution is given by,

$$q(t) = \frac{a}{\omega}\sin(\omega t)\,, \qquad p(t) = \dot{q}(t) = a\cos(\omega t)\,. \tag{1.22}$$

The first step towards constructing the Hilbert space is to complexify the solution space $\mathcal{S}$ to $\mathcal{S}^{\mathbb{C}}$. Thus, we will also allow for complex solutions in $\mathcal{S}^{\mathbb{C}}$, e.g. $y(t) = ae^{\pm \mathrm{i}\omega t}$. On this $2n$-dimensional complex vector space, we define the map $(,) : \mathcal{S}^{\mathbb{C}} \times \mathcal{S}^{\mathbb{C}} \to \mathbb{C}$

$$\big(y_1, y_2\big) \equiv -\mathrm{i}\Omega\big(\bar{y}_1, y_2\big)\,. \tag{1.23}$$

where $\bar{y}_1$ is the complex conjugate of the solution y_1. Recall that a Hilbert space is a vector space which is complete in the norm associated to an inner product. A property of inner product is that it be positive definite, i.e. $\langle\psi,\psi\rangle \geq 0$. The map above in Eq. (1.23) however fails to be positive definite for "negative frequency solutions". For example, for $y_-(t) = (ae^{+\mathrm{i}\omega t}, \mathrm{i}\omega a e^{+\mathrm{i}\omega t})$ for $\omega > 0$, using Eq. (1.13) we have

$$\begin{aligned}(y_-, y_-) &= -\mathrm{i}\Big(\big(-\mathrm{i}\omega a e^{-\mathrm{i}\omega t}\big)ae^{+\mathrm{i}\omega t} - \big(+\mathrm{i}\omega a e^{+\mathrm{i}\omega t}\big)\big(ae^{-\mathrm{i}\omega t}\big)\Big) \\ &= -2\omega a^2\,. \end{aligned} \tag{1.24}$$

On the other hand positive frequency solutions $y_+ = (ae^{-\mathrm{i}\omega t}, -\mathrm{i}\omega a e^{-\mathrm{i}\omega t})$ have positive norm, and are orthogonal to negative frequency solutions:

$$(y_+, y_+) = +2\omega a^2\,, \qquad (y_+, y_-) = 0\,. \tag{1.25}$$

Thus, we can restrict our attention to positive frequency solutions and use Eq. (1.23) to define an inner product. Then we can consider Hilbert space completion of in the associated norm to obtain a complex Hilbert space $\mathcal{H}$. Thus $\mathcal{H}$ only consists of positive frequency solutions of $\mathcal{S}^{\mathbb{C}}$. Then the Hilbert space that we've been seeking for the complete set of decoupled oscillators is the *symmetric Fock space* associated with $\mathcal{H}$:

$$\mathcal{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty}\Big(\bigotimes_s^n \mathcal{H}\Big) \tag{1.26}$$

We represent elements of $\mathcal{H}$ as ξ^a which are normalized to 1. These elements are simply the solutions

$$\xi_i(t) = \frac{1}{\sqrt{2\omega_i}}e^{-\mathrm{i}\omega_i t}\,. \tag{1.27}$$

The elements of the symmetric Fock space are then written as

$$\Psi = \big(\psi, \psi^{a_1}, \psi^{a_1 a_2}, \ldots, \psi^{a_1\ldots a_n}, \ldots\big)\,, \qquad \psi^{a_1\ldots a_n} = \psi^{(a_1\ldots a_n)}\,. \tag{1.28}$$

We are considering symmetrized products since we are dealing with bosons. For $n = 0$, we simply have complex numbers such that $\psi \in \mathbb{C}$ in the above equation. The solutions with $n > 1$ can be interpreted as "multi-particle" state. Now, recall that elements of $\mathcal{H}$ are simply positive frequency solutions in $\mathcal{S}^{\mathbb{C}}$. For each $\xi^a \in \mathcal{H}$ we have a corresponding negative frequency solution $\bar{\xi}_a$. In this index notation in which the inner products are written as $(\xi,\eta) = \bar{\xi}_a\eta^a$. We can equivalently define the norm on the negative frequency solutions to be opposite of Eq. (1.23), which defines the conjugate Hilbert space $\mathcal{H}$.

Now, we define the annihilation operator $a(\bar{\xi}_a) : \mathcal{F}_s(\mathcal{H}) \to \mathcal{F}_s(\mathcal{H})$ associated with $\bar{\xi}$ as

$$a(\bar{\xi})\Psi = \big(\bar{\xi}_a\psi^a, \sqrt{2}\bar{\xi}_a\psi^{aa_1}, \sqrt{3}\bar{\xi}_a\psi^{(aa_1a_2)}, \ldots\big)\,. \tag{1.29}$$

Likewise, the creation operator associated with $\xi^a \in \mathcal{H}$, $a^\dagger(\xi^a) : \mathcal{F}_s(\mathcal{H}) \to \mathcal{F}_s(\mathcal{H})$ is defined via

$$a^\dagger(\xi)\Psi = \big(0, \psi\xi^{a_1}, \sqrt{2}\xi^{(a_1}\psi^{a_2)}, \sqrt{3}\xi^{(a_1}\psi^{a_2a_3)}, \ldots\big)\,. \tag{1.30}$$

Notice that the annihilation operator removes the entry ψ for $n = 0$ in Ψ, where as the creation operator does not have the $n = 0$ entry in its range – as we would have expected. It is a fun exercise to verify that

$$\left[a(\bar{\xi}), a^\dagger(\eta)\right] = (\xi, \eta)\mathbb{I}\,. \tag{1.31}$$

Where the right hand side is the number (ξ, η) times the identity operator. This can be proved using the relation

$$\bar{\xi}_a \eta^{(a}\psi^{a_1 \ldots a_n)} = \frac{1}{n}\left[\bar{\xi}_a \eta^a \psi^{(a_1 \ldots a_n)} + (n-1)\eta^{(a_1}\bar{\xi}_a \psi^{(a a_2 \ldots a_n))}\right]. \tag{1.32}$$

Having defined the creation and annihilation operators associated with elements of $\mathcal{H}$, the Heisenberg picture position and momentum operators on $\mathcal{F}_s(\mathcal{H})$ are given by

$$q_{iH}(t) = \xi_i(t) a_i + \bar{\xi}_i(t) a_i^\dagger \tag{1.33}$$

$$p_{iH}(t) = \frac{\mathrm{d}}{\mathrm{d}t} q_{iH}(t)\,. \tag{1.34}$$

where $a_i = a(\bar{\xi}_i)$ associated with the oscillator with frequency ω_i. Note that $q_{iH}(t)$ and $p_{iH}(t)$ inherit their time dependence from the pre-factors $\xi_i(t)$ whereas the operators $a_i, a_i^\dagger$ are associated with the solution ξ_i which exists for all times, and hence are time-independent. It is straightforward to check using Eq. (1.32) that these operators satisfy canonical commutation relations of position and momentum operators.

We can also express the fundamental linear observables $\hat{\Omega}(y, \cdot)$ acting on states in $\mathcal{F}_s(\mathcal{H})$ in terms of creation and annihilation operator. For each $y \in \mathcal{S}$ the Schrödigner picture operator representing the classical observable $\Omega(y, \cdot)$ is given by

$$\hat{\Omega}(y, \cdot) = \mathrm{i}a(y_-) - \mathrm{i}a^\dagger(y_+)\,, \tag{1.35}$$

where $y_\pm$ are the positive and negative frequency parts of the solution $y(t)$ at $t = 0$. We can verify this by noting that from Eqs. (1.12) and (1.14), $q_{iH}(t)$ in Eq. (1.33) is the observable corresponding to setting p_i^{th} component of $y(t)$ equal to one at $t = 0$. So we begin with a solution $\psi(t)$ such that $\dot{\psi}(t = 0) = 1$. Hence, $\psi = 1/\omega_i \sin \omega_i t$. Thus, the positive and negative frequency parts are given by

$$\psi^+(t) = \frac{\mathrm{i}}{2\omega_i} e^{-\mathrm{i}\omega_i t}\,, \qquad \psi^-(t) = -\frac{\mathrm{i}}{2\omega_i} e^{+\mathrm{i}\omega_i}\,. \tag{1.36}$$

Because $a(\bar{\xi}_i)$ and $a^\dagger(\xi_i)$ are linear in $\bar{\xi}_a$ and ξ^a, we simply have

$$a(\psi^-(t = 0)) = -\frac{\mathrm{i}}{\sqrt{2\omega_i}} a_i\,, \qquad a^\dagger(\psi^+(t = 0)) = \frac{\mathrm{i}}{\sqrt{2\omega_i}} a_i^\dagger\,, \tag{1.37}$$

such that

$$\begin{aligned}\hat{\Omega}(\psi, \cdot) &= \frac{1}{\sqrt{2\omega_i}} a_i + \frac{1}{\sqrt{2\omega_i}} a_i^\dagger \\ &= q_{iH}(t = 0)\,.\end{aligned} \tag{1.38}$$

Accordingly, the Heisenberg picture operators are given by

$$\hat{\Omega}_{\rm H}(y,\cdot) = {\rm i}a\big(y_{t-}\big) - {\rm i}a^\dagger(y_{t+})\,. \tag{1.39}$$

Here the time dependence arises from the solution y_t whose initial data at $t=0$ is $y(-t)$.

We close this section by using the above Hilbert space construction for infinite, decoupled oscillators to write down the quantum theory of the scalar field. We first note that because we are considering a real scalar field the modes for $\boldsymbol{k}$ and $-\boldsymbol{k}$ are related as

$$\bar{\phi}_{\boldsymbol{k}} = \phi_{-\boldsymbol{k}}\,, \tag{1.40}$$

and secondly that in Eq. (1.6) we have two independent sets of oscillators corresponding to "positions" $\sqrt{2}\Re(\phi_{\boldsymbol{k}})$ and $\sqrt{2}{\rm Im}(\phi_{\boldsymbol{k}})$. The factor of $\sqrt{2}$ arises since the sum runs over both $\boldsymbol{k}$ and $-\boldsymbol{k}$. Both sets of oscillators have solutions that can be written in terms of annihilation and creation operators defined above. It is convenient to work with the combination

$$a_{\boldsymbol{k}} = \frac{1}{\sqrt{2}}\big(b_{\boldsymbol{k}} + {\rm i}c_{\boldsymbol{k}}\big)\,, \tag{1.41}$$

where $b_{\boldsymbol{k}}$ are annihilation operators associated with $\sqrt{2}\Re(\phi_{\boldsymbol{k}})$ oscillators and $c_{\boldsymbol{k}}$ for $\sqrt{2}{\rm Im}(\phi_{\boldsymbol{k}})$. These two sets of oscillators are precisely what we saw above and we have simply relabeled the frequencies $\omega_i \to \omega_{\boldsymbol{k}}$. Then it follows that the scalar field $\hat{\phi}(t,\boldsymbol{x})$ has the formal solution:

$$\hat{\phi}(t;\boldsymbol{x}) = \sum_{\boldsymbol{k}} \Big(\psi_{\boldsymbol{k}}(t,\boldsymbol{x})a_{\boldsymbol{k}} + \bar{\psi}_i(t,\boldsymbol{x})a^\dagger_{\boldsymbol{k}}\Big)\,, \tag{1.42}$$

where in analogy to Eq. (1.27), $\psi_{\boldsymbol{k}}$ and $\bar{\psi}_{\boldsymbol{k}}$ are the normalized positive and negative frequency plane wave solutions to the Klein-Gordon equation:

$$\psi_{\boldsymbol{k}} \equiv \frac{1}{L^{3/2}\sqrt{2\omega_{\boldsymbol{k}}}} e^{+{\rm i}\boldsymbol{k}\cdot\boldsymbol{x} - {\rm i}\omega_{\boldsymbol{k}}t}\,. \tag{1.43}$$

The new distinction from Eq. (1.27) is that they carry an additional $\boldsymbol{x}$ dependence.

Similarly, our fundamental observables on the Klein-Gordon field are given by

$$\begin{aligned} \hat{\Omega}\big(\psi,\cdot\big) &= {\rm i}a(\psi_-) - {\rm i}a^\dagger(\psi_+)\,, \\ \hat{\Omega}_{\rm H}\big(\psi,\cdot\big) &= {\rm i}a(\psi_{t-}) - {\rm i}a^\dagger(\psi_{t+})\,. \end{aligned} \tag{1.44}$$

Here ψ is a generic solution to the Klein-Gordon equation.

We can now ask how the above mode decomposition looks like for another observer within the family of global inertial observers. The two observers relate their coordinate systems via Lorentz transformations. Suppose the coordinates of two observers are related as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ and the unprimed observer is interested in performing measurement $\hat{\Omega}(\psi,\cdot)$ using a solution ψ of the Klein-Gordon equation. We can decompose the solution into positive and negative frequencies plane wave basis:

$$\psi(x) = \sum_{\boldsymbol{k}} \big(\alpha_{\boldsymbol{k}}\psi_{\boldsymbol{k}}(x) + \beta_{\boldsymbol{k}}\bar{\psi}_{\boldsymbol{k}}\big)\,, \tag{1.45}$$

such that

$$\hat{\Omega}(\psi,\cdot) = \sum_{\boldsymbol{k}} \left(\alpha_{\boldsymbol{k}}\hat{\Omega}(\psi_{\boldsymbol{k}},\cdot) + \beta_{\boldsymbol{k}}\hat{\Omega}(\bar{\psi}_{\boldsymbol{k}},\cdot)\right) \tag{1.46}$$

$$= \hat{\Omega}\left(\psi',\cdot\right). \tag{1.47}$$

Here ψ' is the solution observed by the primed observer:

$$\psi'(x') = \sum_{\boldsymbol{k}} \left(\alpha_{\Lambda^{-1}\boldsymbol{k}}\psi_{\boldsymbol{k}}(x') + b_{\Lambda^{-1}\boldsymbol{k}}\bar{\psi}_{\boldsymbol{k}}(x')\right). \tag{1.48}$$

In particular, if the unprimed observer finds a one-particle state with momentum $\boldsymbol{k}$ such that only $\alpha_{\boldsymbol{k}} = 1$ and the rest vanishing, then the primed observer will observe a one-particle state with momentum $\Lambda\boldsymbol{k}$ and frequency $\gamma\omega_{\boldsymbol{k}}$. This is of course expected, but we would like to stress that both the observers agree on the notion of particles, and only the momenta and frequencies shift. Accordingly, both will agree on what constitutes a vacuum state and if a solution has positive or negative frequency. This feature, however, is only specific to global inertial observers in flat spacetime, and as we will see in the following sections, it does not hold for curved spacetimes or noninertial observers.

Finally, we make some remarks concerning subtleties associated with the above generalization to the infinite oscillators case. Unlike the case of a finite number of harmonic oscillators, the complexification of the space of real solutions to Klein-Gordon equation and its decomposition into positive and negative frequency solutions is not straightforward procedure as the space of positive frequency solutions cannot be simply identified after complexifying $\mathcal{S}$ to $\mathcal{S}^{\mathbb{C}}$. Furthermore, the solution written above in Eq. (1.42) in the plane-wave basis does not converge. One can nevertheless view it as a formal solution, and make sense of it by "smearing" it with test functions $f : \mathbb{R}^4 \to \mathbb{R}$ with compact support. This procedure of smearing allows us to make sense of the linear observables $\hat{\Omega}(\psi,\cdot)$. However, this problem becomes particularly severe for non-linear functions. Note that in the alternative construction we did not bother writing down the expression of the Hamiltonian. The Hamiltonian, or more generally the energy momentum tensor T^{ab} associated with the Klein-Gordon field is a quadratic function of field operators. Products of distributions are not mathematically well-defined and must be treated with great care. The calculation of expectation value of energy momentum tensor is relevant for "back-reaction" effects which enters the Einstein's equation in the semi-classical picture. We do not review these subtleties here and refer to Ref. [1] for more details.

Chapter 2

The Unruh Effect

The Klein-Gordon equation can be suitably generalized to the case of curved spacetime by using the invariant volume element $\sqrt{-g}\mathrm{d}^4x$ in Eq. (1.1) and possibly including linear couplings of the scalar field with Ricci curvature R. We will restrict to spacetimes where the classical dynamics of the curved spacetime Klein-Gordon equation (with covariant d'Alembertian) is unique to initial values of the field on a Cauchy hypersurface Σ. Given presence of such a hypersurface, the value of field at any location in the spacetime can be back-tracked or forward-tracked to its value on Σ at $t = 0$. The key subtlety, however, is that there is no unique or preferred choice of decomposition of these classical solutions into positive and negative frequencies. Such a choice is in fact closely tied to the choice one makes for defining the inner product in Eq. (1.23) on the space of classical solutions (suitably complexified). For finite dimensional case, it can be shown that different choices of inner products leads to unitary equivalent Hilbert spaces (discussed below). This however no longer continues to be true for infinite dimensional case. Hence, it is natural to ask how QFT formulations in two different bases are related. For the Minkowski case we considered in the previous section we were guided by a preferred choice of the inner product by taking a spacelike Cauchy surface of flat Euclidean space at $t = 0$ and decomposing in the plane-wave basis. In this chapter after an abstract discussion of this problem we will describe the Unruh effect where a noninertial observer finds a dramatically different description of the ordinary Minkowski vacuum state as a thermal, multiparticle state.

2.1 Unitary equivalence

We will restrict ourselves to choice of bases that lead to unitary equivalent descriptions (defined below). This is also relevant for defining the S-matrix where one is interested in relating descriptions via Fock spaces $\mathcal{F}_s(\mathcal{H}_{\mathrm{in}})$ and $\mathcal{F}_s(\mathcal{H}_{\mathrm{out}})$ suitable for describing incoming and outgoing states. The S-matrix in this case only well-defined if these two descriptions are unitary equivalent.

For now, we will keep the notation abstract and demonstrate the working of the machinery via an explicit example of Unruh effect in the following section. For concreteness, consider two unitary equivalent decompositions $\mathcal{H}_1$ and $\mathcal{H}_2$ of the complexified solution space $\mathcal{S}^{\mathbb{C}}$ of the curved space Klein-Gordon equation[1]. We then consider the resulting Fock spaces $\mathcal{F}_1 = \mathcal{F}_s(\mathcal{H}_1)$ and $\mathcal{F}_2 = \mathcal{F}_s(\mathcal{H}_2)$ and the linear operators $\hat{\Omega}_i(\psi, \cdot) : \mathcal{F}_i \to \mathcal{F}_i$. Here we write $\mathcal{S}^{\mathbb{C}} = \mathcal{H}_1 \oplus \bar{\mathcal{H}}_1 = \mathcal{H}_2 \oplus \bar{\mathcal{H}}_2$. The spaces $\mathcal{H}_i$ and $\bar{\mathcal{H}}_i$ define the decomposition of solutions into respective positive and negative

[1] It should be kept in mind that, in line with the remark made earlier, $\mathcal{S}^{\mathbb{C}}$ is not a straightforward complexification of the real solution space $\mathcal{S}$. We will however continue to gloss over this subtlety.

frequencies. The statement that the two descriptions are unitary equivalent is that there exists a unitary operator $U : \mathcal{F}_1 \to \mathcal{F}_2$ such that for all $\psi \in \mathcal{S}^{\mathbb{C}}$,

$$U\hat{\Omega}_1(\psi,)U^{-1} = \hat{\Omega}_2(\psi, \cdot)\,. \tag{2.1}$$

To proceed further, we define the projections $K_1 : \mathcal{S}^{\mathbb{C}} \to \mathcal{H}_1$ and $\bar{K}_1 : \mathcal{S}^{\mathbb{C}} \to \bar{\mathcal{H}}_1$ that project a solution into positive and negative frequency components (according to the inner product defining $\mathcal{H}_1$). Likewise we define K_2 and $\bar{K}_2$ for $\mathcal{H}_2$. These projections are orthogonal such that $K_i\bar{K}_i = \bar{K}_iK_i = 0$. The inner product on $\mathcal{H}_i$ of the complexified space $\mathcal{S}^{\mathbb{C}}$ is defined analogously to Eq. (1.23) as

$$\big(K_i\psi_1, K_i\psi_2\big)_{\mathcal{H}_i} \equiv -\mathrm{i}\Omega\big(\overline{K_i\psi_1}, K_i\psi_2\big)\,. \tag{2.2}$$

Equivalently, the projection operators K_i ensure that this inner product remains positive definite. The above equation may seem somewhat circular in defining the projection operators and the inner products. However, the bottom line is that there are different ways of splitting the complexified solution space on which the above inner product of projections of $\psi_{1,2}$ on $\mathcal{H}_i$ is positive. Equivalently, on the conjugate space $\bar{\mathcal{H}}_i$, the inner product is defined to be negative of above

$$(\bar{K}_i\psi_1, \bar{K}_i\psi_2)_{\bar{\mathcal{H}}_i} \equiv +\mathrm{i}\Omega\big(\overline{\bar{K}_i\psi_1}, \bar{K}_i\psi_2\big)\,. \tag{2.3}$$

Hence the statement of Eq. (2.1) becomes

$$U\Big[\mathrm{i}a_1\big(\bar{K}_1\psi\big) - \mathrm{i}a_1^\dagger\big(K_1\psi\big)\Big]U^{-1} = \mathrm{i}a_2\big(\bar{K}_2\psi\big) - \mathrm{i}a_2^\dagger\big(K_2\psi\big)\,, \tag{2.4}$$

where a_i and $a_i^\dagger$ are annihilation and creation operators of the Fock spaces $\mathcal{F}_s(\mathcal{H}_i)$. Here the key point is that a positive frequency solution corresponding to $\mathcal{H}_1$ decomposition will in general have both positive and negative frequency components in the basis of $\mathcal{H}_2$ and $\bar{\mathcal{H}}_2$ Hilbert spaces. Let us now then define restrictions of the projections K_i and $\bar{K}_i$ on subspaces $\mathcal{H}_j$ for $i \neq j$:

$$\begin{aligned} &A : \mathcal{H}_2 \to \mathcal{H}_1\,, && B : \mathcal{H}_2 \to \bar{\mathcal{H}}_1\,, \\ &C : \mathcal{H}_1 \to \mathcal{H}_2\,, && D : \mathcal{H}_1 \to \bar{\mathcal{H}}_2\,, \end{aligned} \tag{2.5}$$

where A and B are restrictions of K_1 and $\bar{K}_1$ on $\mathcal{H}_2$. Likewise, C and D are restrictions of K_2 and $\bar{K}_2$ on $\mathcal{H}_1$. For example, for $\chi \in \mathcal{H}_1$, $C\chi = K_2\chi$. Accordingly the restrictions on conjugate Hilbert spaces $\bar{\mathcal{H}}_i$ are written as

$$\begin{aligned} &\bar{A} : \bar{\mathcal{H}}_2 \to \bar{\mathcal{H}}_1\,, && \bar{B} : \bar{\mathcal{H}}_2 \to \mathcal{H}_1\,, \\ &\bar{C} : \bar{\mathcal{H}}_1 \to \bar{\mathcal{H}}_2\,, && \bar{D} : \bar{\mathcal{H}}_1 \to \mathcal{H}_2\,, \end{aligned} \tag{2.6}$$

We show these projections schematically in Fig. 2.1.

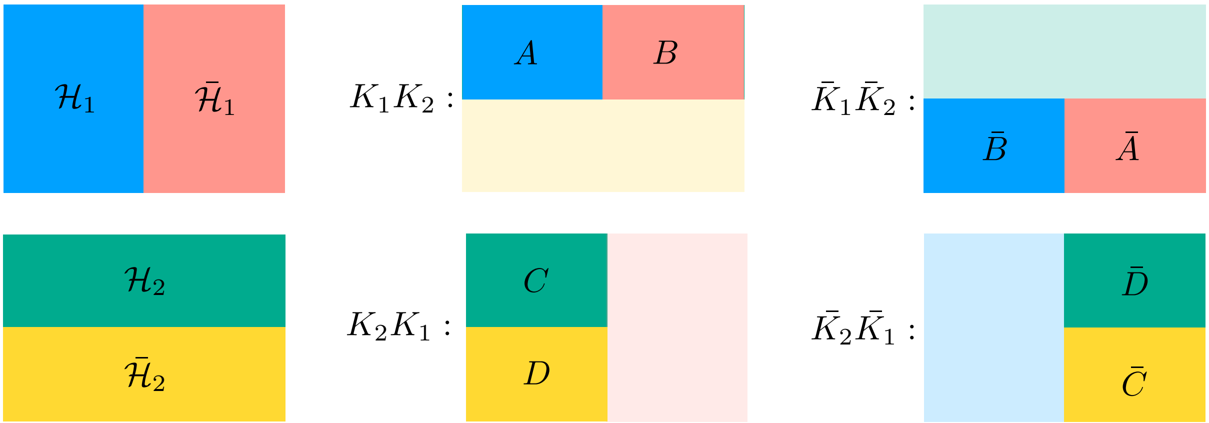

Figure 2.1 Projections in two different Hilbert space decompositions of the complexified solution space.

We can derive properties of these maps and relation amongst them by considering various inner products. For $\chi, \psi \in \mathcal{H}_2$,

$$\begin{aligned}(\psi,\chi)_{\mathcal{H}_2} &= -\mathrm{i}\Omega(\bar{\psi},\chi) \\ &= -\mathrm{i}\Omega(\overline{K_1\psi+\bar{K}_1\psi}, K_1\chi+\bar{K}_1\chi) \\ &= -\mathrm{i}\Omega(\overline{K_1\psi}, K_1\chi) - \mathrm{i}\Omega(\overline{\bar{K}_1\psi}, \bar{K}_1\chi) \\ &= (A\psi, A\chi)_{\mathcal{H}_1} - (B\psi, B\chi)_{\bar{\mathcal{H}}_1} \\ &= (\psi, A^\dagger A\chi)_{\mathcal{H}_2} - (\psi, B^\dagger B\chi)_{\mathcal{H}_2}\,,\end{aligned} \tag{2.7}$$

such that

$$A^\dagger A - B^\dagger B = 1\,. \tag{2.8}$$

Now start with $\chi \in \mathcal{H}_2$ and $\psi \in \bar{\mathcal{H}}_2$:

$$\begin{aligned}0 &= (\psi,\chi)_{\mathcal{H}_2} \\ &= -\mathrm{i}\Omega(\overline{K_1\psi}, K_1\chi) - \mathrm{i}\Omega(\overline{\bar{K}_1\psi}, \bar{K}_1\chi) \\ &= (\bar{B}\psi, A\chi)_{\mathcal{H}_1} - (\bar{A}\psi, B\chi)_{\bar{\mathcal{H}}_1} \\ &= (A^\dagger\bar{B}\psi, \chi)_{\mathcal{H}_2} - (B^\dagger\bar{A}\psi, \chi)_{\mathcal{H}_2}\,,\end{aligned} \tag{2.9}$$

such that

$$A^\dagger\bar{B} = B^\dagger\bar{A}\,. \tag{2.10}$$

Likewise we have

$$C^\dagger C - D^\dagger D = 1\,, \qquad C^\dagger\bar{D} = D^\dagger\bar{C}\,. \tag{2.11}$$

Starting with $(\psi, A\chi)_{\mathcal{H}_1}$ with $\psi \in \mathcal{H}_1$ and $\chi \in \mathcal{H}_2$ we can show

$$A^\dagger = C \tag{2.12}$$

and with $\big(\psi,\bar{B}\chi\big)_{\mathcal{H}_1}$ with $\psi\in\mathcal{H}_1$ and $\chi\in\bar{\mathcal{H}}_2$ we can show

$$\bar{B}^\dagger = -D\,. \tag{2.13}$$

In particular from Eq. (2.8) it follows that A^{-1} must exist. If not then we could find a non-zero vector v that evaluates to zero when operated upon by A. By considering $\big(v,(A^\dagger A - B^\dagger B)v\big)$ we arrive at a contradiction. Likewise C^{-1} must exist. The unitary transformation along with these projection matrices A,B,C,D constitute the so called Bogoliubov transformation.

Using the transformations above we can ask how does the vacuum state in $\mathcal{H}_1$ Hilbert space looks like in the $\mathcal{H}_2$ decomposition. We might as well ask how any generic state in $\mathcal{H}_1$ is represented in $\mathcal{H}_2$, but it's easiest to start with vacuum. We essentially would like to know

$$\Psi = U|0\rangle_1\,,\qquad \Psi\in\mathcal{F}_s(\mathcal{H}_2)\,, \tag{2.14}$$

where $|0\rangle_1 = (1,0,0,\ldots)_1$. We decompose Ψ in terms of its n-particle amplitudes as in Eq. (1.28). Now we apply Eq. (2.4) for a generic complex solution $\psi\in\mathcal{S}^{\mathbb{C}}$ on Ψ:

$$\begin{aligned}\Big[\mathrm{i}a_2\big(\bar{K}_2\psi\big) - \mathrm{i}a_2^\dagger\big(K_2\psi\big)\Big]\Psi &= U\Big[\mathrm{i}a_1\big(\bar{K}_1\psi\big) - \mathrm{i}a_1^\dagger\big(K_1\psi\big)\Big]U^{-1}U|0\rangle \\ &= -\mathrm{i}Ua_1^\dagger\big(K_1\psi)|0\rangle\,.\end{aligned} \tag{2.15}$$

Now, we will choose ψ such that $K_1\psi = 0$. Thus, let $\psi = \bar{\chi}\in\bar{\mathcal{H}}_1$, such that

$$\begin{aligned}0 &= \Big[\mathrm{i}a_2\big(\bar{K}_2\bar{\chi}\big) - \mathrm{i}a_2^\dagger\big(K_2\bar{\chi}\big)\Big]\Psi \\ &= \Big[\mathrm{i}a_2\big(\bar{C}\bar{\chi}\big) - \mathrm{i}a_2^\dagger\big(\bar{D}\bar{\chi}\big)\Big]\Psi\,,\end{aligned} \tag{2.16}$$

Now let $\bar{C}\bar{\chi} = \bar{\xi}\in\bar{\mathcal{H}}_2$, and define

$$\mathcal{E}\equiv\bar{D}\bar{C}^{-1}\,. \tag{2.17}$$

Thus we have the solution for all $\bar{\xi}\in\mathcal{H}_2$:

$$\Big[a_2\big(\bar{\xi}\big) - a_2^\dagger\big(\mathcal{E}\bar{\xi}\big)\Big]\Psi = 0 \tag{2.18}$$

Now we simply compare the action of two operators using Eqs. (1.29) and (1.30) and find

$$\begin{aligned}\bar{\xi}_a\psi^a &= 0\,,\\ \sqrt{2}\bar{\xi}_a\psi^{aa_1} &= \psi\big(\mathcal{E}\bar{\xi}\big)^{a_1}\,,\\ \sqrt{3}\bar{\xi}_a\psi^{(aa_1a_2)} &= \sqrt{2}\big(\mathcal{E}\bar{\xi}\big)^{(a_1}\psi^{a_2)}\,,\\ \sqrt{4}\bar{\xi}_a\psi^{(aa_1a_2a_3)} &= \sqrt{3}\big(\mathcal{E}\xi\big)^{(a_1}\psi^{a_2a_3)}\,,\end{aligned} \tag{2.19}$$

The equations above hold for any $\bar{\xi}\in\bar{\mathcal{H}}_2$. The ψ above is simply the vacuum component (a complex number) of Ψ. Hence, from the first equation we find $\psi^a = 0$, and consequently all the odd particle amplitudes vanish. Hence, we have

$$\Psi = U|0\rangle_1 = \left(\psi,0,\frac{1}{\sqrt{2}}\psi\mathcal{E}^{a_1a_2},0,\sqrt{\frac{3\cdot 1}{4\cdot 2}}\psi\mathcal{E}^{(a_1a_2}\mathcal{E}^{a_3a_4)},0,\ldots\right). \tag{2.20}$$

Thus we see that the vacuum state in one can correspond to multi-particle state in the other. The symmetry property of the solutions are consistent since $\mathcal{E}$ can be shown using second of Eq. (2.11) to be symmetric, $\mathcal{E}^\dagger = \bar{\mathcal{E}}$.

Crucially, we showed that C^{-1} exits, and hence C cannot vanish. The requirement for multiparticle states to be observed from the perspective of $\mathcal{H}_2$ decomposition is that $\mathcal{E} \neq 0$, or in other words $D \neq 0$. From Eq. (2.5) (or from Fig. 2.1) we see that D corresponds to there being non-zero negative frequency components in a purely positive frequency solution in $\mathcal{H}_1$ decomposition. In our previous analysis of scalar field in flat spacetime, we found that all the global inertial observers find the same sign of frequencies, and hence share the same notion of vacuum, single and multi-particle states. To make this clear through an explicit example, in the next section we will make a comparison between an observer who in the global inertial family and another one who *isn't*.

2.2 The Unruh effect

The preceding discussion suggests that even if we restrict to scalar field in Minkowski spacetime, a *non-inertial* observer may not agree with global inertial observers on the particle content of a given state. A simplest example of a non-inertial observer is that of one moving with *uniform proper acceleration.* This example in fact turns out to be useful for the context of Hawking radiation in the vicinity of black hole. Note that here we are explicitly bringing in a “spurious” curved spacetime effect by considering an accelerated observer. In the case of black hole, this effect will be relevant for an inertial, freely falling observer who find themselves in a ‘real’ curved spacetime next to a black hole.

Consider an global inertial coordinate system x^μ and an observer moving with constant proper acceleration a^μ along x direction, where

$$a^\mu = \frac{D^2 x^\mu}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}\,, \qquad a^2 = \alpha^2\,. \tag{2.21}$$

The second equation implies that

$$\left(\frac{\mathrm{d}^2 x(\tau)}{\mathrm{d}\tau^2}\right)^2 - \left(\frac{\mathrm{d}^2 t(\tau)}{\mathrm{d}\tau^2}\right)^2 = \alpha^2\,. \tag{2.22}$$

The hyperbolic nature of the equation above suggests a natural parameterization of the coordinates (t, x) in terms of (η, ξ) defined as

$$t(\eta,\xi) = \frac{1}{a} e^{a\xi}\sinh(a\eta)\,, \qquad x(\eta,\xi) = \frac{1}{a} e^{a\xi}\cosh(a\eta)\,, \qquad -\infty < \eta, \xi < \infty \tag{2.23}$$

Note that these coordinates cannot cover the entirety of the Minkowski spacetime for the above ranges of η and ξ, but only the wedge $x > |t|$. We have left the y, z coordinates the same and will suppress them below. In terms of these coordinates Eq. (2.22) now becomes

$$e^{2a\xi}\Big(a(\dot\eta - \dot\xi)^2 - \ddot\eta + \ddot\xi\Big)\Big(a(\dot\eta + \dot\xi)^2 + \ddot\eta + \ddot\xi\Big) = \alpha^2 \tag{2.24}$$

We have only one equation that constrains $\eta(\tau)$ and $\xi(\tau)$ and their derivatives. Exploiting the freedom, we can consider a simple solution where $\ddot{\eta} = 0$ and $\dot{\xi} = \ddot{\xi} = 0$. Hence,

$$e^{a\xi}\dot{\eta}^2 = \frac{\alpha}{a}\,. \tag{2.25}$$

We further exploit the remaining freedom to set

$$\eta(\tau) = \frac{\alpha}{a}\tau\,, \qquad \xi(\tau) = -\frac{1}{a}\ln\Big(\frac{\alpha}{a}\Big)\,. \tag{2.26}$$

Thus, in these coordinates, the proper time is proportional to η. The metric in these coordinates is given by

$$ds^2 = e^{2a\xi}\big(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2\big) \tag{2.27}$$

The spacetime expressed in these coordinates is called *Rindler space*. We see that the metric is independent of η. Hence, translations along η are isometries, with the Killing vector given by

$$b^a = \partial_\eta^a\,. \tag{2.28}$$

In the (t, x) coordinates, the components are given by

$$\begin{aligned} b^a &= \Big(\frac{\partial t}{\partial \eta}\partial_t^a + \frac{\partial x}{\partial \eta}\partial_x^a\Big) \\ &= a\Big(x\partial_t^a + t\partial_x^a\Big)\,. \end{aligned} \tag{2.29}$$

This Killing field generates one-parameter group of Lorentz boost isometries. The vector is time like in the region $|x| > |t|$ with $b^2 = -e^{2a\xi}$. In other regions (η, ξ) coordinates cannot be employed, but b^a in Eq. (2.29) is nevertheless well defined. In the regions $|x| < |t|$ the vector is space like, and finally on $|x| = |t|$ surfaces it is null. In the region $x < -|t|$ it is time-like but past directed which can be easily seen by considering b^a on the hypersurface Σ defined by $t = 0$. We can describe the $x < -|t|$ region by defining

$$t = -\frac{1}{a}e^{a\xi'}\sinh(a\eta')\,, \qquad x = -\frac{1}{a}e^{a\xi'}\cosh(a\eta')\,, \qquad -\infty < \eta', \xi' < \infty\,. \tag{2.30}$$

The primes on the coordinates indicate that they cannot be used elsewhere. The minus signs ensure that the Killing vector b'^a continues to be given by the same expression in Eq. (2.29). Hence, we see that an observer moving with constant proper acceleration is confined to either left and right *Rindler wedges*.

We now go back to the question posed at the end of the previous section – what does the ordinary Minkowski vacuum state $|0\rangle_M$ look like to our observer with constant proper acceleration. More specifically, we define $\mathcal{H}_1$ in the notation of previous section to be the Hilbert space seen by global inertial observer with the initial data specified on the Cauchy surface Σ at $t = 0$. Accordingly, the solutions in $\mathcal{H}_1$ are positive frequency with respect to the ordinary time t. Let the portions of Σ in the right and left Rinder wedges to be Σ_1 and Σ_2. As remarked earlier, the key property of the Cauchy surface Σ is that any causal curve passing through an arbitrary point in the spacetime manifold must pass through Σ. Hence, we can associate the solutions of the Klein-Gordon equation with the initial data on Σ. Furthermore, the initial data on any closed subset $S \subset \Sigma$ alone determines the solution of any causal curve passing through S. Thus, we can consider an alternative quantum field construction for *all of Minkowski spacetime* with the initial data specified on Σ_1 and Σ_2 hypersurfaces. Thus, in our second decomposition the Hilbert space $\mathcal{H}_2$ is defined as

$$\mathcal{H}_2 = \mathcal{H}_L \oplus \mathcal{H}_R \,, \tag{2.31}$$

where the $\mathcal{H}_R$ consists of solutions with initial data specified on Σ_1 in the right Rindler wedge, and that are positive frequency with respect to the generator b^a. Likewise, $\mathcal{H}_L$ is the Hilbert space of the solutions with initial data specified on Σ_2 in the left Rindler wedge that are positive frequency with respect to $-b^a$ (because on Σ_2, b_a points downwards, towards negative time evolution). We will write the solutions in this decomposition as

$$\psi = \psi_L \oplus \psi_R \,, \quad \psi_L \in \mathcal{H}_L \,, \quad \psi_R \in \mathcal{H}_R \,. \tag{2.32}$$

When a solution vanishes in left or right Rindler wedge we simply write

$$\psi_L = \psi_L \oplus 0 \,, \qquad \psi_R = 0 \oplus \psi_R \,. \tag{2.33}$$

Now, the logical way to proceed would be to solve for the Klein-Gordon equation in these coordinates,

$$e^{2a\xi}\big(-\partial_\eta^2 + \partial_\xi^2\big)\phi - m^2\phi = 0 \,, \tag{2.34}$$

in the left and right Rindler wedges, and directly solve for the Bogoliubov transformation matrices by taking inner products of these solutions on Σ_1 and Σ_2 with plane wave solutions in the global inertial coordinates obtained using the entire Σ hypersurface. However, there's a quicker way to get directly the matrix $\mathcal{E}^{ab}$ defined in Eq. (2.17). It turns out that the solution to the Klein-Gordon equation (in whichever frame) is uniquely determined by its restriction on the intersecting null-planes $\boldsymbol{h}_A$ ($x = t$) and $\boldsymbol{h}_B$ ($x = -t$) [4]. An interesting consequence is that if the solution on $\boldsymbol{h}_A \cup \boldsymbol{h}_B$ is positive frequency with respect to either the global inertial time t or the Rindler time $\eta, -\eta'$ (but not necessarily both), then it will remain so throughout the entire Minkowski space. Thus, we simply need to relate the two sets of solutions, more specifically their Fourier transforms on $\boldsymbol{h}_A \cup \boldsymbol{h}_B$ with respect to appropriate time coordinates, and thus find the Bogoliubov transformation matrices.

Now, since we are limiting ourselves to the solutions on the null planes, it need a new set of coordinates as on the null planes $\boldsymbol{h}_{A,B}$ as ξ, ξ' blow up there. To this end, we define the null coordinates (U, V, y, z):

$$U \equiv t - x \,, \qquad V \equiv t + x \,. \tag{2.35}$$

On $\boldsymbol{h}_A$, we have $U = 0$ and on $\boldsymbol{h}_B$ $V = 0$. Thus, we can trade V for time direction on $\boldsymbol{h}_A$ in the global inertial coordinates, and U for time direction on $\boldsymbol{h}_B$. In other words V and U are Killing parameter times on the null planes in global inertial coordinates. For simplicity we will refer to them as "inertial time" coordinates.

Next, we must specify the Killing parameter time on the null planes in Rindler coordinates. On $\boldsymbol{h}_A \cup \boldsymbol{h}_B$, the boost Killing vector becomes

$$b^a = V\partial_V^a - U\partial_U^a \,. \tag{2.36}$$

On these null planes we only need to specify a single coordinate which we will choose to be the Killing parameter time. We call them v and u, the "accelerating time" coordinates, defined by $b(v) = 1$ on $\boldsymbol{h}_A$ and $b(u) = 1$ on $\boldsymbol{h}_B$. Thus, we have

$$b(v) = V\frac{\partial v}{\partial V} = 1 \,, \tag{2.37}$$

and hence

$$v = \frac{1}{a}\ln|V| \,. \tag{2.38}$$

Since v grows in the direction of b^a, for $V > 0$, v points in increasing time direction and vice versa for $V < 0$. Likewise, the relation $b(u) = 1$ on $\boldsymbol{h}_B$ yields

$$u = -\frac{1}{a}\ln|U| \,. \tag{2.39}$$

Now consider a solution $\psi_{R,\omega}$ that is specified via initial data on Σ_1 in the right Rindler Wedge and vanishes in the left Rindler Wedge. Furthermore, we require this solution to oscillate with a positive frequency $\omega > 0$ with respect to the Rindler time in right Rindler Wedge (η inside the wedge and u, v on $\boldsymbol{h}_{B,A}$ null planes). By the above initial value formulation, it will then remain positive frequency throughout the right wedge. Let $f_{R,\omega}$ be its restriction on $\boldsymbol{h}_A$. Then we have

$$f_{R,\omega}(V, y, z) = \Theta(V)h(y, z)e^{-\mathrm{i}\omega v(V)} \,. \tag{2.40}$$

The Fourier transform with respect to V will allow us to identify the frequency content of this solution in the global inertial $\mathcal{H}_1 \cup \bar{\mathcal{H}}_1$ decomposition of the Klein Gordon solution space. Let us label the frequencies conjugate to t (or V on $\boldsymbol{h}_A$ null plane) in this decomposition by σ. Then, we have

$$\begin{aligned} \hat{f}_{R,\omega}(\sigma, y, z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}V\, e^{+\mathrm{i}\sigma V} f_{R,\omega}(V, y, z) \\ &= \frac{1}{\sqrt{2\pi}} h(y, z)\int_{0}^{\infty} \mathrm{d}V\, e^{+\mathrm{i}\sigma V} \exp\Big(-\frac{\mathrm{i}\omega}{a}\ln V\Big) \,. \end{aligned} \tag{2.41}$$

Next, we can use $\psi_{R,\omega}$ to write a solution in the left wedge by performing the "wedge reflection" isometry $(t, x, y, z) \to (-t, -x, y, z)$. Note that doing so maps solutions in $\mathcal{H}_R$ to $\bar{\mathcal{H}}_L$ due to time reversal. Then the restriction of negative frequency solutions in the left wedge on the same null plane, $\boldsymbol{h}_A$ is given by

$$\bar{f}_{L,\omega} = \Theta(-V)h(y, z)e^{-\mathrm{i}\omega v(V)} = f_{R,\omega}(-V, y, z) \,. \tag{2.42}$$

Note that dispite the similarity with $f_{R,\omega}$, this is a negative frequency solution since it's aligned with b^a (as opposed to $-b^a$) in the left Rindler Wedge. Its Fourier transform with respect to V is given by

$$\begin{aligned}\hat{\bar{f}}_{L,\omega}(\sigma,y,z) &= \hat{f}_{R,\omega}(-\sigma,y,z)\,, \qquad \sigma>0 \\ &= \frac{1}{\sqrt{2\pi}}h(y,z)\int_0^\infty \mathrm{d}V\, e^{-\mathrm{i}\sigma V}\exp\Big(-\frac{\mathrm{i}\omega}{a}\ln V\Big)\,.\end{aligned} \tag{2.43}$$

Next, to proceed further, we simplify the V integral in Eqs. (2.41) and (2.43) by rotating the contours to be along positive and negative imaginary axes (so that the exponential $e^{\mathrm{i}\sigma V}$ converges). To be able to do this, we choose the branch cut of the Logarithm to be along negative real axis. Hence for $V=+\mathrm{i}y$ and $y>0$ in the first integral the substitution

$$\ln V = \frac{\mathrm{i}\pi}{2} + \ln y\,, \tag{2.44}$$

gives

$$\hat{f}_{R,\omega}(\sigma,y,z) = \frac{\mathrm{i}e^{\frac{\pi\omega}{2a}}}{\sqrt{2\pi}}h(y,z)\int_0^\infty \mathrm{d}y\, e^{-\sigma y}\exp\Big(-\frac{\mathrm{i}\omega}{a}\ln y\Big)\,. \tag{2.45}$$

and setting $V=-\mathrm{i}y$ in Eq. (2.43) gives

$$\hat{\bar{f}}_{L,\omega}(\sigma,y,z) = \frac{-\mathrm{i}e^{\frac{-\pi\omega}{2a}}}{\sqrt{2\pi}}h(y,z)\int_0^\infty \mathrm{d}y\, e^{-\sigma y}\exp\Big(-\frac{\mathrm{i}\omega}{a}\ln y\Big)\,. \tag{2.46}$$

We immediately notice that the following linear combination vanishes for $\sigma>0$:

$$e^{-\frac{\pi\omega}{a}}\hat{f}_{R,\omega}(\sigma,y,z) + \hat{\bar{f}}_{L,\omega}(\sigma,y,z) = 0 \tag{2.47}$$

Thus, conversely the function

$$\begin{aligned}\hat{F}_\omega &\equiv e^{-\frac{\pi\omega}{a}}\hat{f}_{R,\omega}(-\sigma,y,z) + \hat{\bar{f}}_{L,\omega}(-\sigma,y,z) \\ &= e^{-\frac{\pi\omega}{a}}\hat{\bar{f}}_{L,\omega}(\sigma,y,z) + \hat{f}_{R,\omega}(\sigma,y,z)\,,\end{aligned} \tag{2.48}$$

vanishes for $\sigma<0$, and is thus a purely positive frequency solution with respect to the inertial time. Thus from this it follows that the solution in the entire spacetime obeys

$$\Psi_i = \psi_{R,\omega_i} + e^{-\frac{\omega_i}{a}}\bar{\psi}_{L,\omega}\,, \qquad \Psi_i\in\mathcal{H}_1\,, \tag{2.49}$$

where $\psi_{R,\omega_i}\in\mathcal{H}_R$ and $\bar{\psi}_{L,\omega_i}\in\mathcal{H}_L$. By repeating these arguments but starting with a negative frequency solution in the right Rindler wedge and combining with the Wedge reflected solution in the left Rindler Wedge, we can derive that the solution

$$\Psi_i' = \psi_{L,\omega_i} + e^{-\frac{\omega_i}{a}}\bar{\psi}_{R,\omega_i}\,, \qquad \Psi_i'\in\mathcal{H}_1\,. \tag{2.50}$$

is also purely positive frequency with respect to the inertial observer. Hence, in terms of the projectors C and D defined above in Eq. (2.5) that project an element in $\mathcal{H}_1$ onto $\mathcal{H}_2\cup\bar{\mathcal{H}}_2$, we have

$$\begin{aligned} C\Psi_i &= \psi_{R,\omega_i}\,, & C\Psi_i' &= \psi_{L,\omega_i}\,, \\ D\Psi_i &= e^{-\frac{\pi\omega_i}{a}}\bar{\psi}_{L,\omega_i}\,, & D\Psi_i' &= e^{-\frac{\pi\omega_i}{a}}\bar{\psi}_{R,\omega_i}\,,\end{aligned} \tag{2.51}$$

Hence, we find that

$$\begin{aligned} DC^{-1}\psi_{R,\omega_i} &= e^{-\frac{\pi\omega_i}{a}}\bar{\psi}_{L,\omega_i} \,, \\ DC^{-1}\psi_{L,\omega_i} &= e^{-\frac{\pi\omega_i}{a}}\bar{\psi}_{R,\omega_i} \,, \end{aligned} \tag{2.52}$$

Since $\{\psi_{R,\omega_i}\}$ and $\{\psi_{L,\omega_i}\}$ form a complete basis of $\mathcal{H}_2 = \mathcal{H}_L \oplus \mathcal{H}_R$, we have thus determined the operator $\bar{\mathcal{E}}_{ab} = DC^{-1}$ defined in Eq. (2.17). Note that the matrix

$$\bar{\mathcal{E}}^{(\omega_i)}_{ab} = e^{-\pi\omega_i/a}\big(\bar{\psi}_{L,\omega_i\,a}\bar{\psi}_{R,\omega_i\,b} + \bar{\psi}_{L,\omega_i\,b}\bar{\psi}_{R,\omega_i\,a}\big) \tag{2.53}$$

acting on ψ^a_{R,ω_i} yields

$$\begin{aligned} \bar{\mathcal{E}}^{(\omega_i)}_{ab}\psi^a_{R,\omega_j} &= e^{-\pi\omega_i/a}\bar{\psi}_{L,\omega_i\,b}\bar{\psi}_{R,\omega_i\,a}\psi^a_{R,\omega_i} \\ &= e^{-\pi\omega_i/a}\bar{\psi}_{L,\omega_i\,b} \,, \end{aligned} \tag{2.54}$$

which is consistent with Eq. (2.52). In the first line we set the inner product of a left and right solution to zero. The corresponding operator $\bar{\mathcal{E}}_{ab}$ is then given by

$$\bar{\mathcal{E}}_{ab} = \prod_i e^{-\pi\omega_i/a}\big(\bar{\psi}_{L,\omega_i\,a}\bar{\psi}_{R,\omega_i\,b} + \bar{\psi}_{R,\omega_i\,a}\bar{\psi}_{L,\omega_i\,b}\big) \tag{2.55}$$

The operator above mixes the left and right frequency components of a given solution in the accelerating frame weighted by the factor $e^{\pi\omega_i/a}$. The corresponding conjugate operator is given by

$$\mathcal{E}^{ab} = \prod_i e^{-\frac{\pi\omega_i}{a}}\, 2\big(\psi_{R,\omega_i}\big)^{(a}\big(\psi_{L,\omega_i}\big)^{b)} \,, \tag{2.56}$$

To proceed further, from Eq. (2.20), we see that

$$U|0\rangle_1 = \prod_i \left(1, 0, e^{-\pi\omega_i/a}\frac{1}{\sqrt{2}}2\psi^{(a}_{R,i}\psi^{b)}_{L,i}, 0, e^{-2\pi\omega_i/a}\sqrt{\frac{3\cdot 1}{4\cdot 2}}4\,\psi^{(a}_{L,i}\psi^{b}_{L,i}\psi^{c}_{R,i}\psi^{d)}_{R,i}, 0, \dots\right). \tag{2.57}$$

On the other hand, from Eq. (1.30) we note that

$$\begin{aligned} \big[a^\dagger(\psi_L)\big]\big[a^\dagger(\psi_R)\big]\big(1,0,\dots\big) &= \big(0,0,\sqrt{2}\psi^{(a}_L\psi^{b)}_R, 0, \dots\big) \\ \frac{1}{\sqrt{2!}}\big[a^\dagger(\psi_L)\big]^2\frac{1}{\sqrt{2!}}\big[a^\dagger(\psi_R)\big]^2\big(1,0,\dots\big) &= \big(0,0,0,0,\frac{1}{2}\sqrt{2\cdot 3\cdot 4}\,\psi^{(a}_L\psi^{b}_L\psi^{c}_R\psi^{d)}_R, 0, \dots\big) \end{aligned} \tag{2.58}$$

Thus, we have a very compact expression of of the action of U on the inertial vacuum state:

$$U|0\rangle_1 = \prod_i\sum_{n=0}^{\infty} e^{-\frac{n\pi\omega_i}{a}}|n_{i,R}\rangle\otimes|n_{i,L}\rangle \tag{2.59}$$

where, following Eq. (1.20), $|n_{i,R}\rangle$ describes a state with n particles with frequency ω_i in the right Rindler wedge, and correspondingly $|n_{i,L}\rangle$ for the left Rindler Wedge. We press on further and calculate the density matrix for restriction of Minkowski vacuum in the right Rindler Wedge. This is simply given by

$$\rho_R = \prod_i \sum_{n=0}^{\infty} e^{-\frac{2n\pi\omega_i}{a}} |n_{i,R}\rangle\langle n_{i,R}| \tag{2.60}$$

We have found something quite remarkable. Note that the n-particle state has energy $n\omega_i$ such that the density matrix in Eq. (2.60) is simply the canonical ensemble of states with temperature

$$T = \frac{a}{2\pi} \,. \tag{2.61}$$

Hence, the the Minkowski vacuum is seen as a thermal state by the accelerated observer! Furthermore, from Eq. (2.59) we see that the state in left and right wedges are highly correlated: An observer in one of the wedges will observe n particles if and only if n particles are observed in the other wedge as well. The state in Eq. (2.59) is termed as the *Thermofield double*. Further interesting details on Unruh effect can be found in Ref. [5].

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