

Entanglement entropy and proton's structure



The Henryk Niewodniczański
Institute of Nuclear Physics
Polish Academy of Sciences

Krzysztof Kutak



NCN



Motivation

bounds and properties of EE may provide some new insight on behavior of pdfs

links to other areas (thermodynamics, gravity, quantum information, conformal field theory)

Interesting in context of parton saturation and thermalization problem of Quark Gluon Plasma

Various approaches to entropy in the low x limit: entropy of gluon density, thermodynamic entropy, momentum space entanglement, coordinate space entanglement, Wehrl entropy,...

K. Kutak '11, R. Peschanski '13,

A. Kovner, M. Lublinsky '15, A. Kovner, M. Lublinsky, M. Serino '18,

Hagiwara, Y, Hatta, B. Xiao, Yuan'18, N. Armesto, F. Dominguez, A. Kovner, M. Lublinsky, V. Skokov'19

Z. Tu, D. Kharzeev, T. Ulrich '20, C. Akkaya, H. Duan , A. Kovner, V. Skokov '20

K. Zhang, K. Hao, D. Kharzeev, V. Korepin' 21, E. Levin, D. Kharzeev '21,

D. Kharzeev '21, M. Hentschinski, K. Kutak '21,

Dvali, Venugopalan' 21 Liu, Nowak, Zahed, 21;

A. Dumitriu, Kolbusz '22, H. Duan , A. Kovner, V. Skokov '22

Boltzman and von Neuman entropy formulas – reminder

The entropy S of macrostate is given by the log of number W of distinct microstates that compose it

$$S = - \sum_{i=1}^W p(i) \ln p(i) \quad \text{Gibbs entropy}$$

For uniform distribution $p(i) = \frac{1}{W}$ the entropy is maximal Boltzmann entropy
 $S = \ln W$

Since partons are introduced as the microscopic constituents that compose the macroscopic state of the proton, it is natural to evaluate the corresponding entropy or entropy corresponding to parton density.

K. Kutak '11, Peschanski'12
 A. Kovner, M. Lublinsky '15
 D. Kharzeev, E. Levin '17,...

But proton as a whole is a pure state and the von Neuman entropy is 0. Can one get any nontrivial result?

For pure state (one state) density matrix is: For mixed state i.e. classical statistical mixture

$$\rho = |\psi\rangle\langle\psi|$$

$$S_{VN} = -Tr[\rho \ln \rho] = -1 \ln 1 = 0$$

$$\rho = \sum p(i) |\psi_i\rangle\langle\psi_i|$$

$$S_{VN} \neq 0$$

Kharzeev, Levin '17

Entanglement entropy in DIS

The composite system is described by

$$|\Psi_{AB}\rangle \text{ in } A \cap B$$

entangled

if the product can not be expressed as separable product state

$$|\Psi_{AB}\rangle = \sum_{i,j} c_{ij} |\varphi_i^A\rangle \otimes |\varphi_j^B\rangle$$

Schmidt decomposition

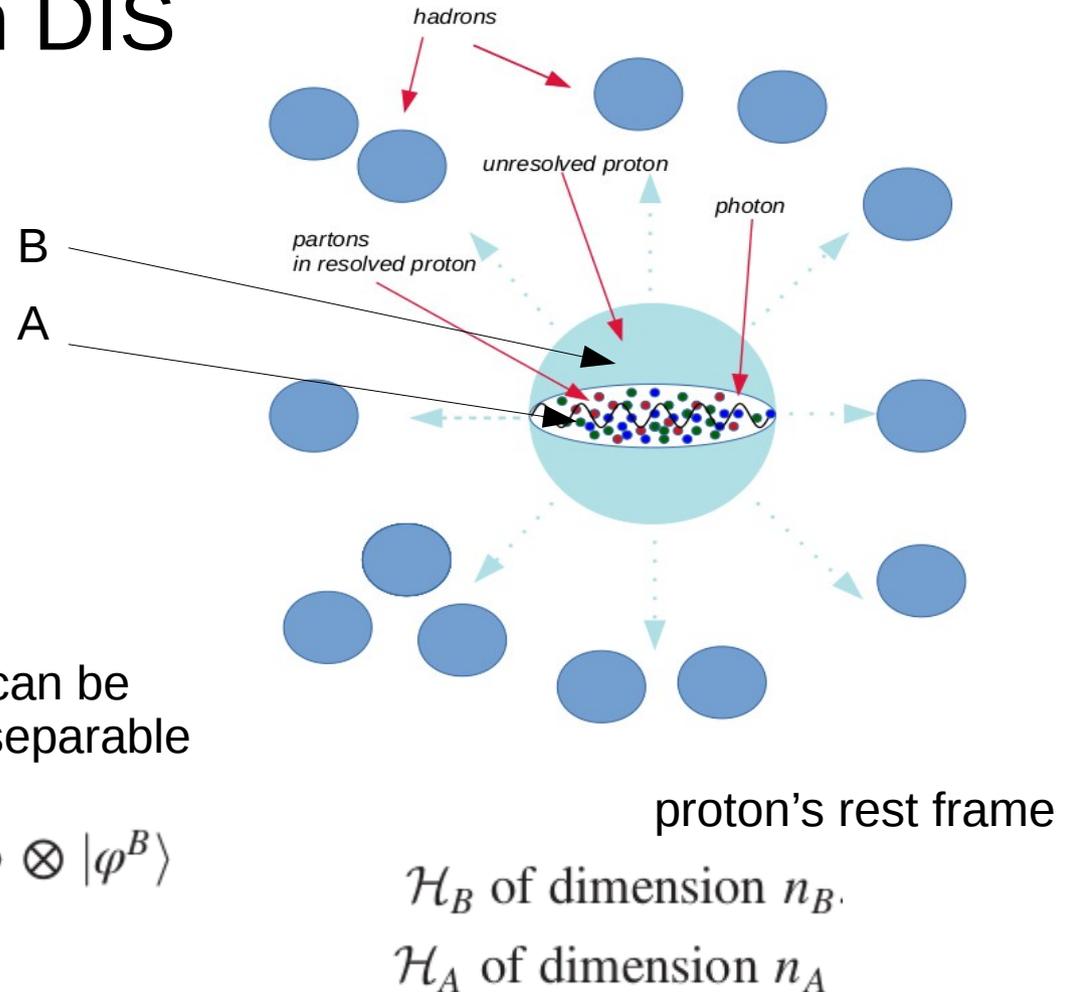
$$|\Psi_{AB}\rangle = \sum \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

← orthonormal states belonging to A
 ← orthonormal states belonging to B
 ← related to matrix C

separable

if the product can be expressed as separable product state

$$|\Psi_{AB}\rangle = |\varphi^A\rangle \otimes |\varphi^B\rangle$$



Khazzeev, Levin '17

Entanglement entropy in DIS

$$|\Psi_{AB}\rangle = \sum_n \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

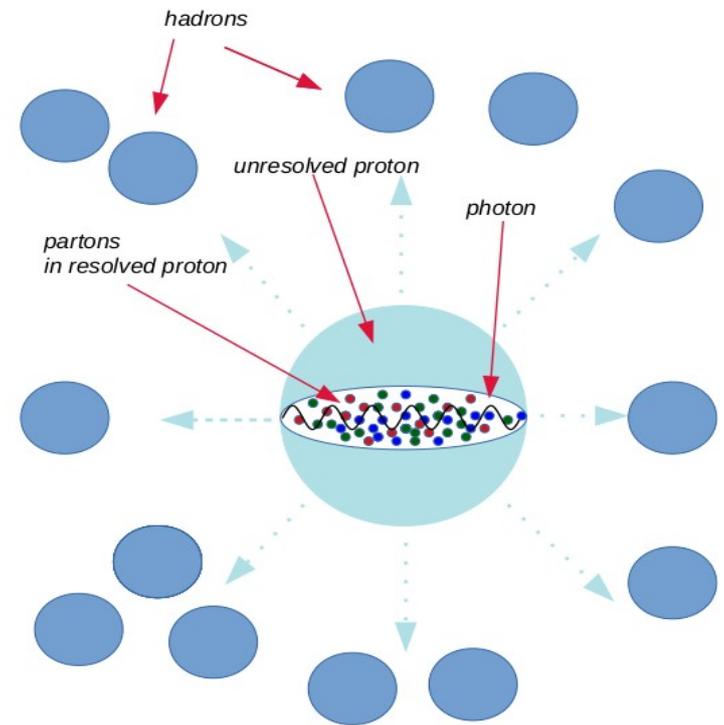
$$\rho_{AB} = |\Psi_{AB}\rangle \langle \Psi_{AB}|$$

$$\rho_A = \text{tr}_B \rho_{AB} = \sum_n \alpha_n^2 |\Psi_n^A\rangle \langle \Psi_n^A|$$

$\alpha_n^2 \equiv p_n$ probability of state with n partons

$$S = - \sum_n p_n \ln p_n$$

entropy results from the entanglement between the regions A and B, and can thus be interpreted as the entanglement entropy. Entropy of region A is the same as entropy in region B.



The density matrix of the mixed state probed in region A

Khazzev, Levin '17

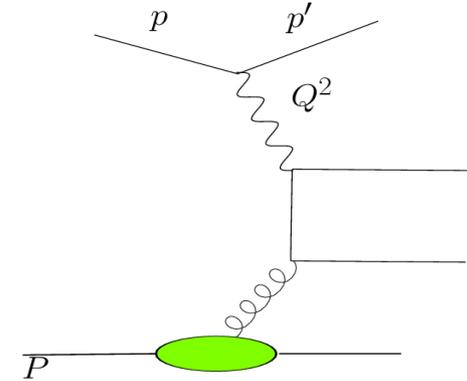
Proton structure function and dipole cross section

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2} \alpha_s \sum_q e_q^2 \int d^2k \mathcal{F}(x, k^2) (S_L(k^2, Q^2, m_q^2) + S_T(k^2, Q^2, m_q^2))$$

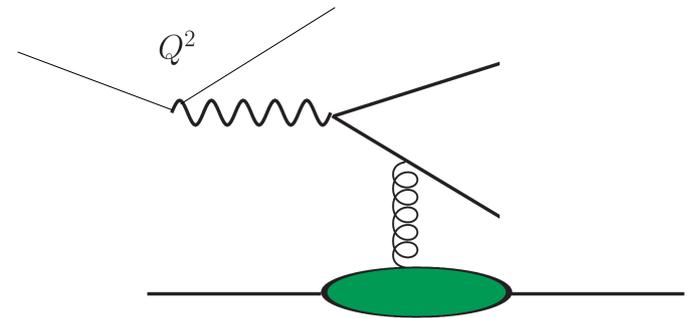
dipole gluon density

impact factors ~ hard coefficients

In the kt factorization



In the dipole formalism



$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2 \alpha_{em}} \int d^2b \int_0^1 dz \int d^2r (|\psi_L(z, r)|^2 + |\psi_T(z, r)|^2) N(x, r, b)$$

wave function

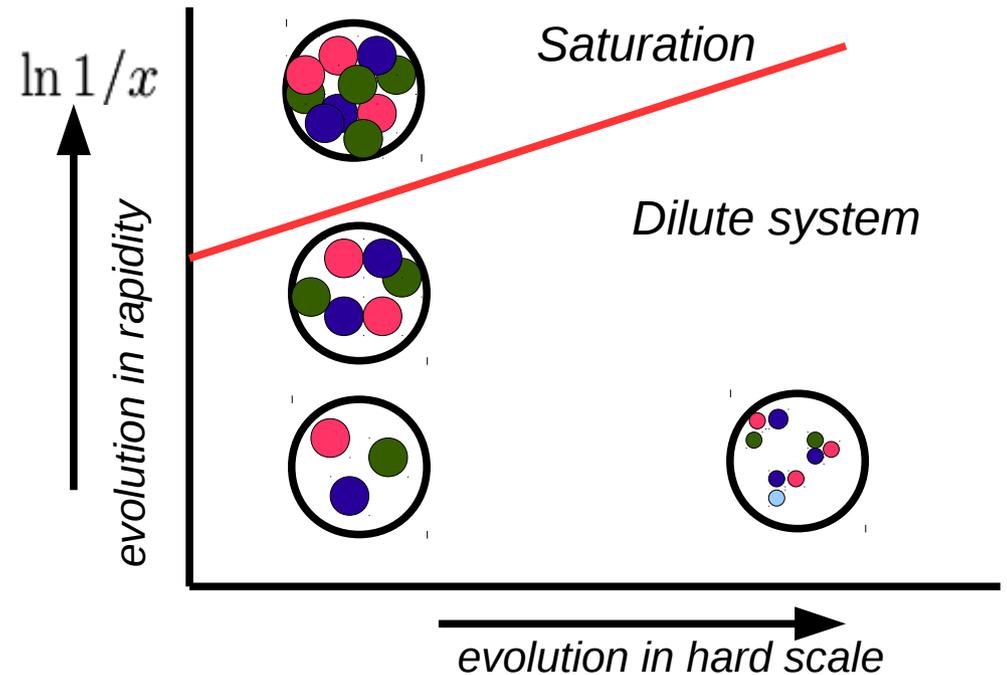
dipole amplitude

Gluons at high energies

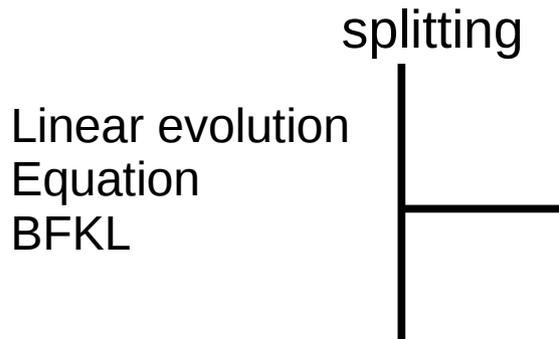
Saturation – state where number of gluons stops growing due to high occupation number. Way to fulfill unitarity requirements in high energy limit of QCD.

L.V. Gribov, E.M. Levin, M.G. Ryskin
 Phys.Rept. 100 (1983) 1-150

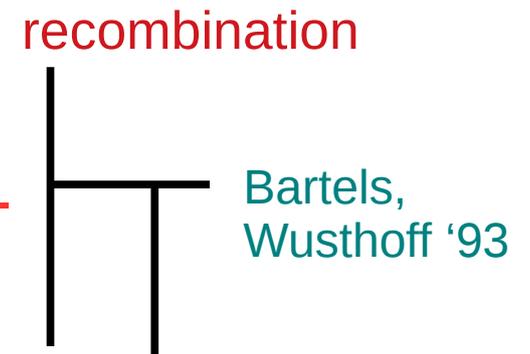
Larry D. McLerran, Raju Venugopalan
 Phys.Rev. D49 (1994) 3352-3355



On microscopic level it means that
 gluon apart splitting recombine



Jailian-Marian, Iancu
 McLerran, Weigert, Leonidov, Kovner

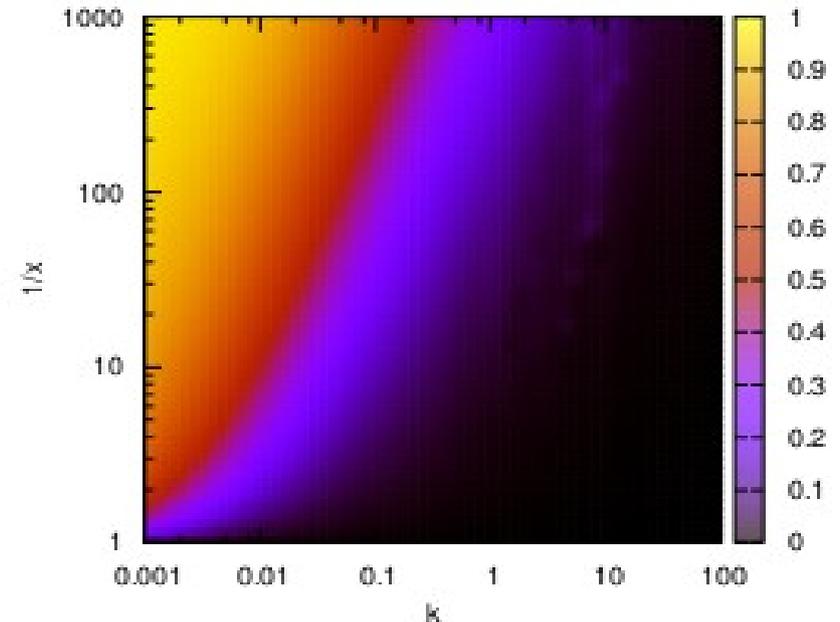


Gluons at high energies

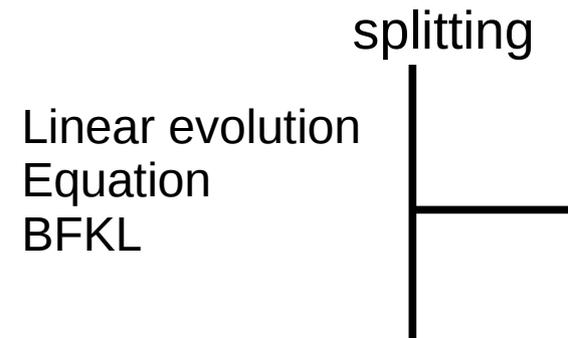
Saturation – state where number of gluons stops growing due to high occupation number. Way to fulfill unitarity requirements in high energy limit of QCD.

L.V. Gribov, E.M. Levin, M.G. Ryskin
Phys.Rept. 100 (1983) 1-150

Larry D. McLerran, Raju Venugopalan
Phys.Rev. D49 (1994) 3352-3355

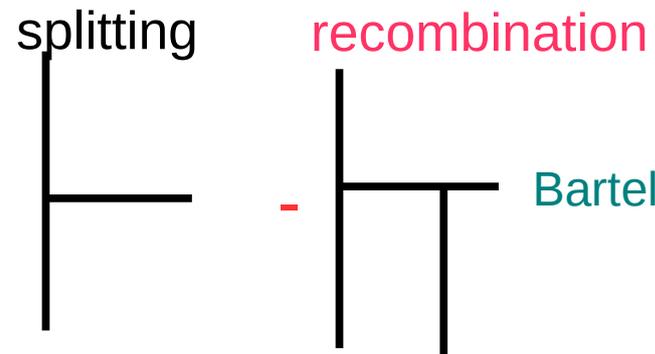


On microscopic level it means that
gluon apart splitting recombine



**Nonlinear evolution
equations**
BK, JIMWLK
Balitsky-Kovchegov,

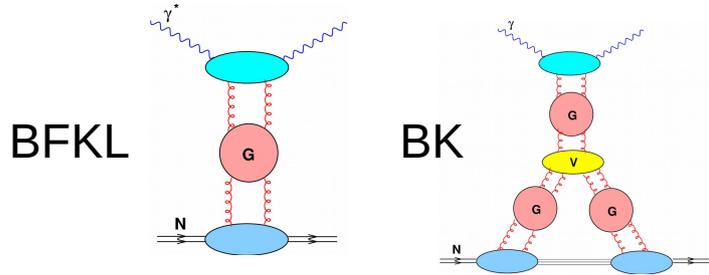
Jailian-Marian, Iancu
McLerran, Weigert, Leonidov, Kovner



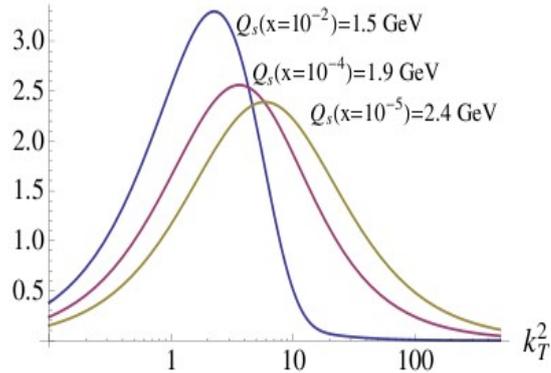
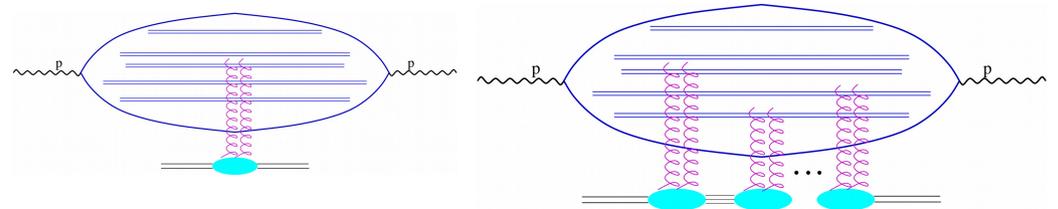
Bartels, Wusthoff'99

Momentum space vs coordinate space

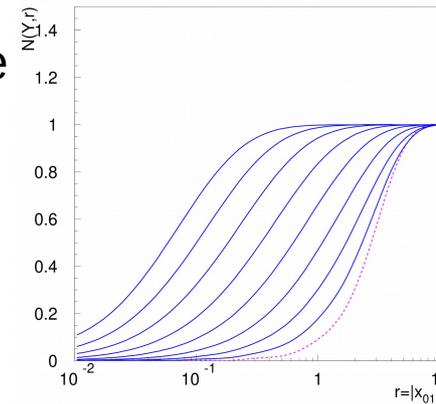
momentum space - Bjorken frame



position space - Mueller frame



gluon ~ color dipole



from A. Stasto
Acta Phys.Polon.
B35 (2004) 3069-3102

$$\mathcal{F}(x, k) = \mathcal{F} + K_{ms} \otimes \mathcal{F}(x, k) - \frac{1}{R^2} TPV \otimes \mathcal{F}(x, k)^2 \quad N(x, r, b) = N_0 + K_{ps} \otimes (N(x, r, b) - N(x, r, b)^2)$$

dipole unintegrated gluon density

related by Fourier transform

Evolved with BK dipole amplitude – expectation value of product of Wilson lines in fundamental representation

The dipole cross section and integrated gluon

$$\sigma(x, r) = \frac{4\pi^2}{N_c} \int \frac{dk^2}{k^2} (1 - J_0(kr)) \mathcal{F}(x, k^2)$$

$$\sigma(x, r) \approx \frac{4\pi^2}{N_c} \int \frac{dk^2}{k^2} \left(1 - \left(1 - \frac{k^2 r^2}{4} \right) \right) \mathcal{F}(x, k^2)$$

$$\sigma(x, r) \approx \frac{\pi^2}{N_c} r^2 xg(x, 1/r^2)$$

$$\sigma(x, r) = \sigma_0 N(x, r)$$

$$N(x, r) \approx xg(x, 1/r^2)$$

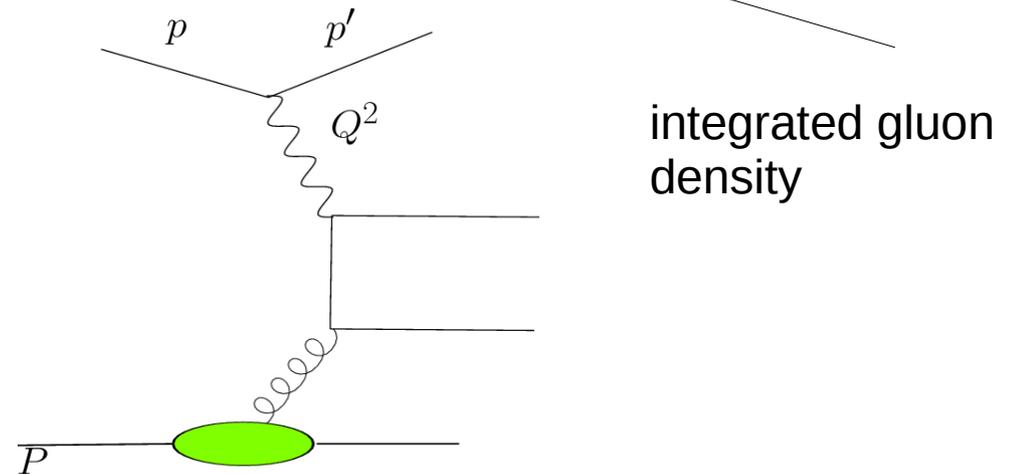
For fixed dipole size one has.

$$N(x) = 1 - Z(x)$$

$$Z(x) \propto \sum_n P_n$$

generating function for dipoles

$$xg(x, Q^2) \equiv \int^{Q^2} dk^2 \mathcal{F}(x, k^2)$$



In the context of the scale dependent GBW model this approximation is viewed as linear approximation

Partonic, dipole cascade

$$p_n = P_n$$

$$\frac{dP_n(Y)}{dY} = -\lambda n P_n(Y) + (n-1)\lambda P_{n-1}(Y)$$

set of partons is described by set of dipoles with fixed sizes, Y is rapidity and is related to energy
 Mueller 95, Lublinsky, Levin '03

$$P_n(Y) = e^{-\lambda Y} (1 - e^{-\lambda Y})^{n-1}$$

depletion of the probability to find n dipoles due to the splitting into $(n+1)$ dipoles.

$$S = -\sum_n p_n \ln p_n$$

the growth due to the splitting of $(n-1)$ dipoles into n dipoles.

$$S(Y) \approx \lambda Y \quad \text{where} \quad Y = \ln 1/x$$

model of BFKL dipole cascade

$$\langle n \rangle = \sum_n n P_n(Y) = \left(\frac{1}{x} \right)^\lambda$$

BFKL intercept = $4 \ln 2 \bar{\alpha}_S$

Assumption $\langle n \rangle \equiv xg(x)$

Kharzeev, Levin '17

$$S(x) = \ln(xg(x))$$

Comments

$$\frac{dP_n(Y)}{dY} = -\lambda n P_n(Y) + (n-1)\lambda P_{n-1}(Y)$$

This equation was also studied in context of the operator growth in the SYK model
[Roberts, Stanford, Streicher '18](#)

The model can be generalized within 3+1 BK and one can argue how to account for hard scale dependence dependence.

$$S(x, Q) = \ln(xg(x, Q))$$

Entropy formula - interpretation

$$P_n(Y) = e^{-\lambda Y} (1 - e^{-\lambda Y})^{n-1}$$

At low x partonic microstates have equal probabilities

In this equipartitioned state the entropy is maximal – the partonic state at small x is maximally entangled.

In terms of information theory as Shannon entropy:

- equipartitioning in the maximally entangled state means that all “signals” with different number of partons are equally likely
- it is impossible to predict how many partons will be detected in a give event.
- structure function at small x should become universal for all hadrons.

From strict bounds on entanglement entropy (from conformal field theory) one can obtain that at low x (in conformal regime) one has

$$xg(x) \leq \text{const } x^{-1/3}$$

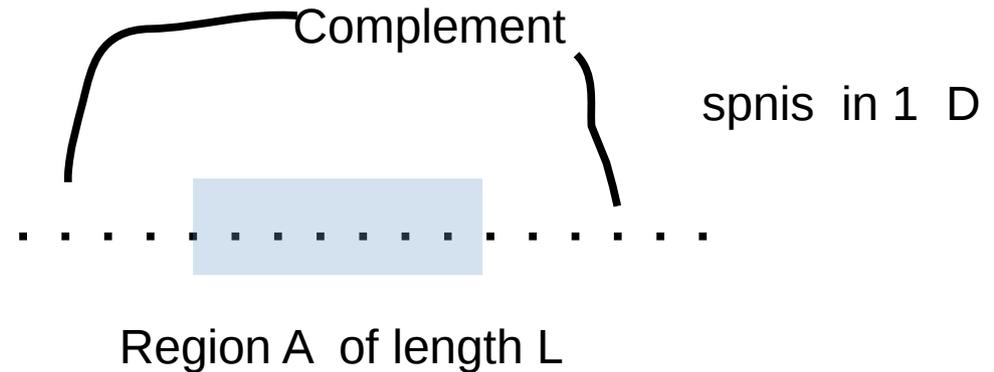
Khazzeev, Levin '17

Furthermore entropy of the final state hadrons can not be smaller than entropy of partons.

Comments

CFT result for EE

$$S = \frac{c}{3} \ln \frac{L}{\epsilon}$$



Entanglement entropy obtained from CFT calculations as well as from gravity using Ryu-Takayanagi formula

Relation to Kharzeev-Levin formula

$$L = (mx)^{-1}$$

$$\epsilon \equiv 1/m$$

Length of tube probed in DIS

Proton's Compton wave length

$$S = \ln \left(\frac{1}{x} \right)^{1/3}$$

$$S(x) = \ln(xg(x))$$

Studied also in the context of 2 D QCD

See also
Callan, Wilczek '94
Calabrese, Cardy '04

and lectures by
Headrick

Liu, Nowak, Zahed, '22

Casini, Huerta, Hosco '05

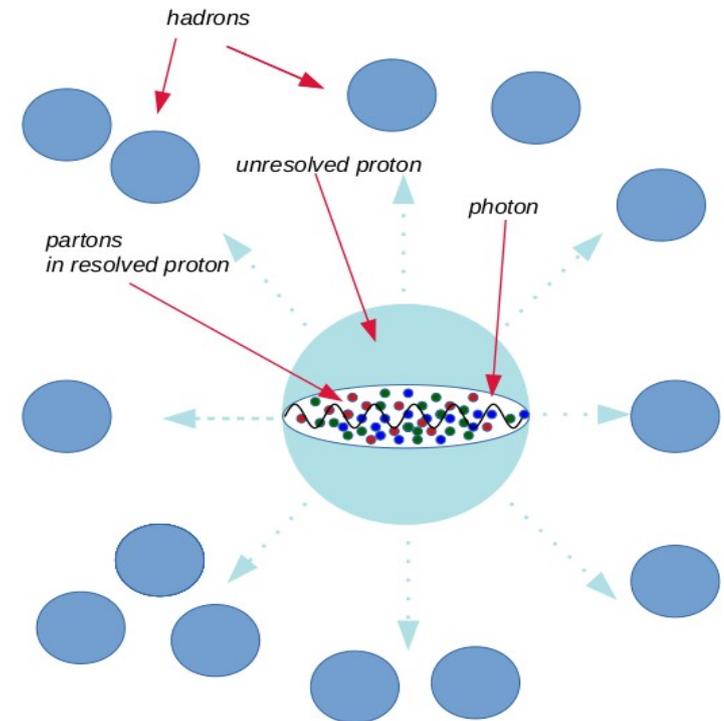
Entanglement entropy – calculation and measurement

For DIS at high energies, this entanglement entropy can be calculated using

$$S(x, Q^2) = \ln \left\langle n \left(\ln \frac{1}{x}, Q \right) \right\rangle$$

$$S_{hadron} = \sum P(N) \ln P(N)$$

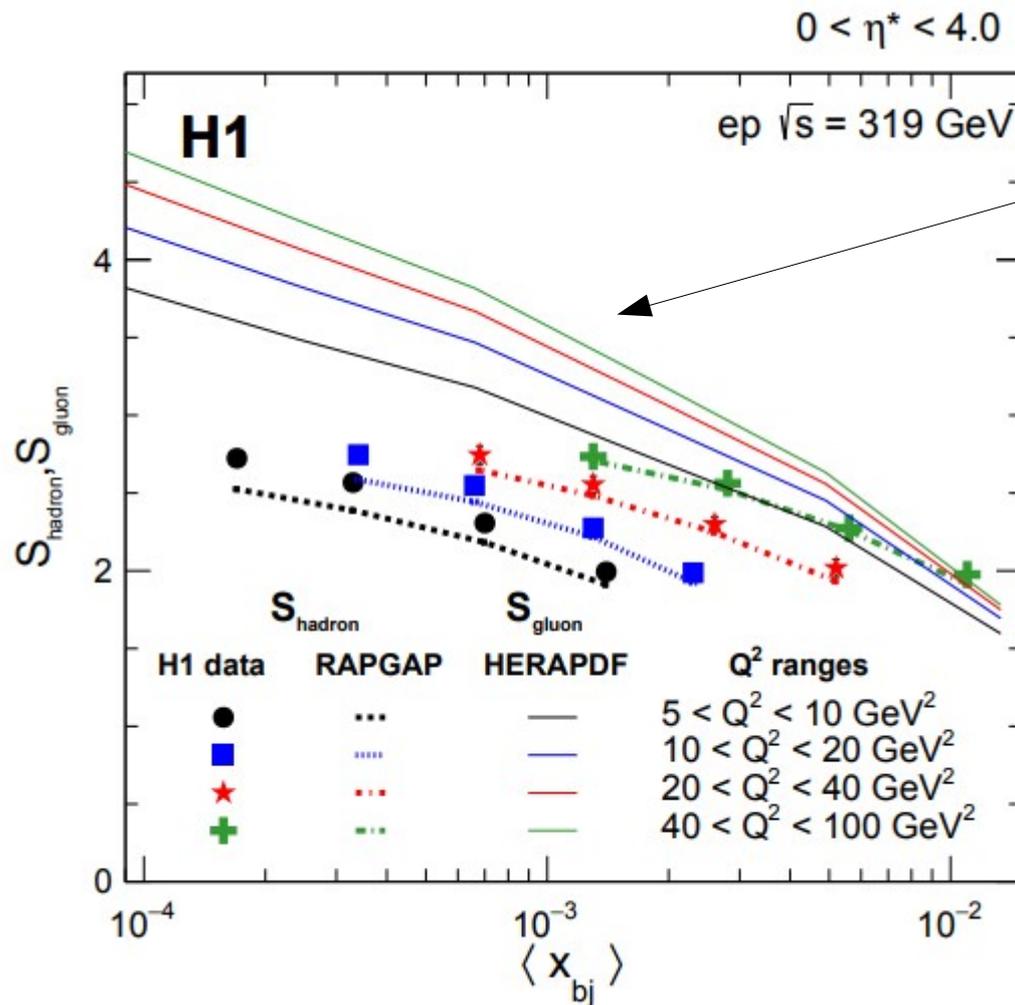
N number of measured hadrons



The charged particle multiplicity distribution measured in either the current fragmentation region or the target fragmentation region.

Fraction of events with charged hadron

Data and EE



HERA pdf used

$$S(x, Q) = \ln(xg(x, Q))$$

Also attempt by Kharzeev and Levin to use quarks instead of gluons
[Phys. Rev. D 104, 031503 \(2021\)](#)

H1

[Eur.Phys.J.C 81 \(2021\) 3, 212](#)

See also [Z. Tu, D. Kharzeev, T. Ulrich '20](#)
 for calculations of EE in p-p.

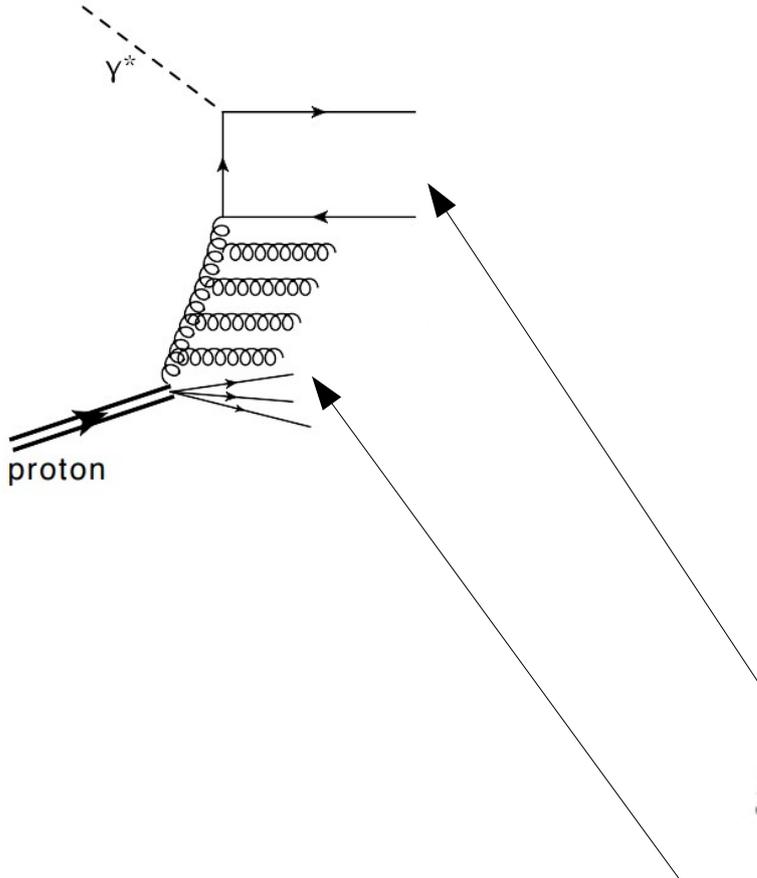
Extension of KL entropy formula

Hentschinski, Kutak '21

$$\left\langle n \left(\ln \frac{1}{x}, Q \right) \right\rangle = xg(x, Q) + x\Sigma(x, Q)$$

To get the entropy of system of partons one needs to account for both quarks and gluons. One can view this as a higher order correction to KL formula. Furthermore it is impossible to isolate quarks from gluons therefore the complete entropy formula should receive contributions from quarks and gluons

Gluon and quark distribution



In the linear regime obeys BFKL equation. In our calculations we use NLO BFKL with kinematical improvements and running coupling. The gluon density has been fitted to F_2 data (exact kinematics was used)

Hentschinski, Sabio-Vera, Salas.
 Phys.Rev.D 87 (2013) 7, 076005
 Phys.Rev.Lett. 110 (2013) 4, 041601

We calculate the sea quarks distribution using

$$x\Sigma(x, Q) = P_{qg}(Q, \mathbf{k}) \otimes \mathcal{F}(x, \mathbf{k}^2)$$

$$xg(x, Q) = \int_0^{Q^2} d\mathbf{k}^2 \mathcal{F}(x, \mathbf{k}^2)$$

Transverse momentum dependent splitting function
 Catani, Hautmann
 Nucl.Phys. B427 (1994) 475-524

Other methods for resummation:
 KMS (Kwiecinski, Martin, Stasto);
 CCSS (Colferai, Ciafaloni, Staśto, Salam)

Gluon distribution

NLO BFKL with collinear resummation

$$\mathcal{F}(x, \mathbf{k}^2, Q) = \frac{1}{\mathbf{k}^2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi i} \hat{g}\left(x, \frac{Q^2}{Q_0^2}, \gamma\right) \left(\frac{\mathbf{k}^2}{Q_0^2}\right)^\gamma$$

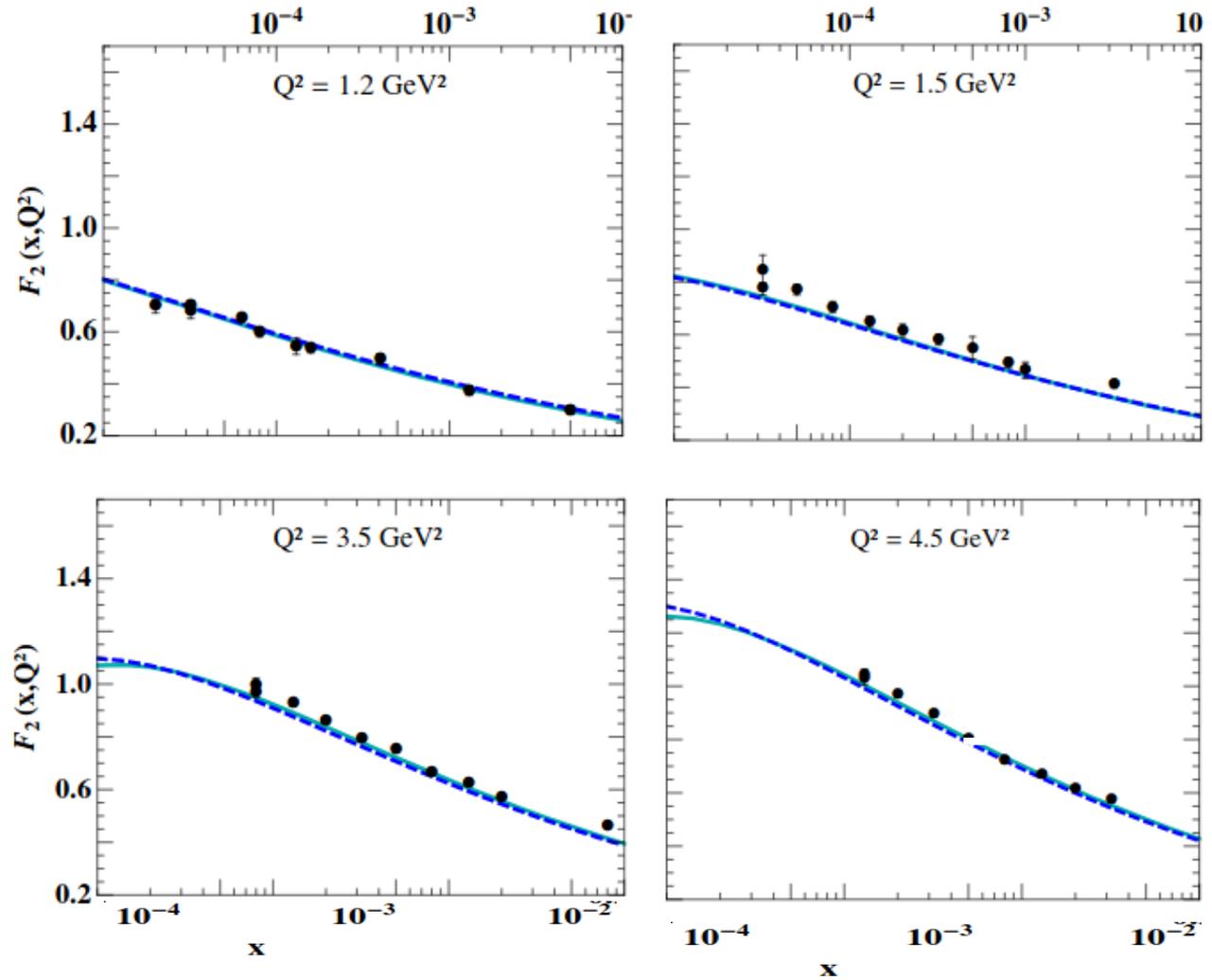
$$\hat{g}\left(x, \frac{Q^2}{Q_0^2}, \gamma\right) = \frac{C \cdot \Gamma(\delta - \gamma)}{\pi \Gamma(\delta)} \cdot \left(\frac{1}{x}\right)^{\chi(\gamma, Q, Q)} \left\{ 1 + \frac{\bar{\alpha}_s^2 \beta_0 \chi_0(\gamma)}{8N_c} \log\left(\frac{1}{x}\right) \left[-\psi(\delta - \gamma) + \log\frac{Q^2}{Q_0^2} - \partial_\gamma \right] \right\}$$

the low x growth

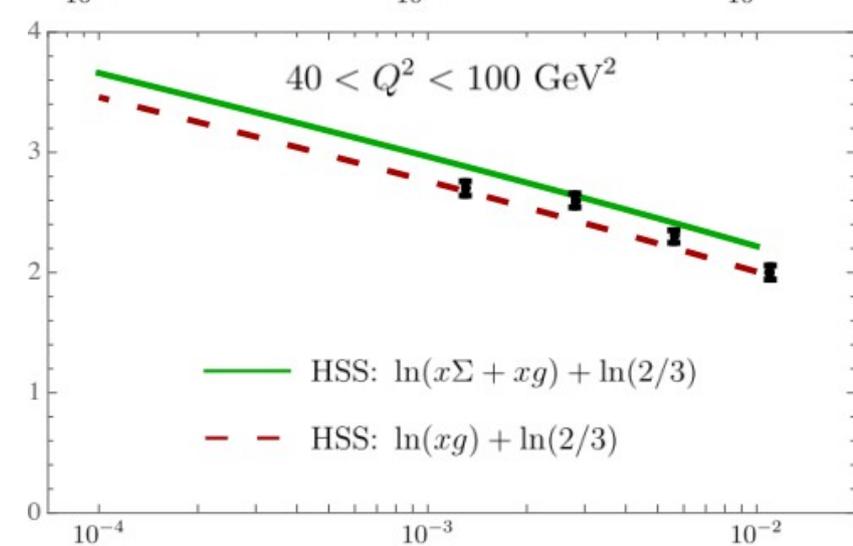
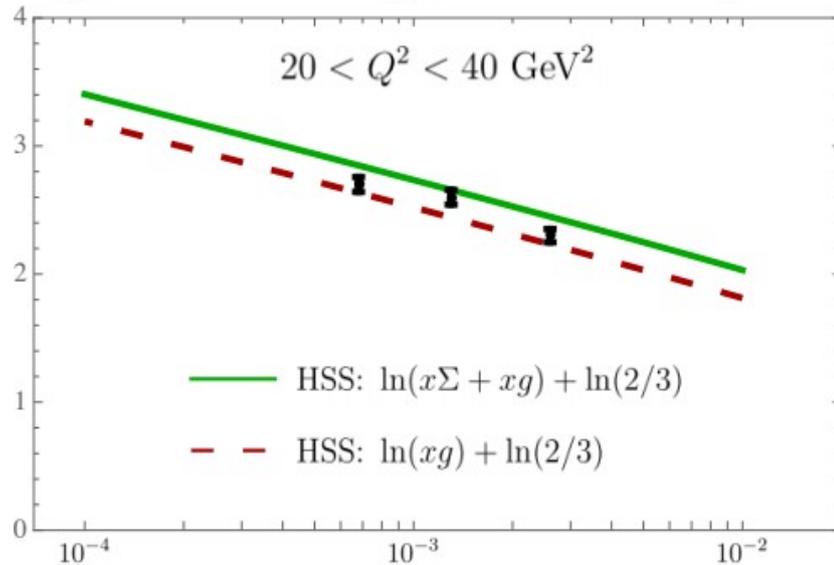
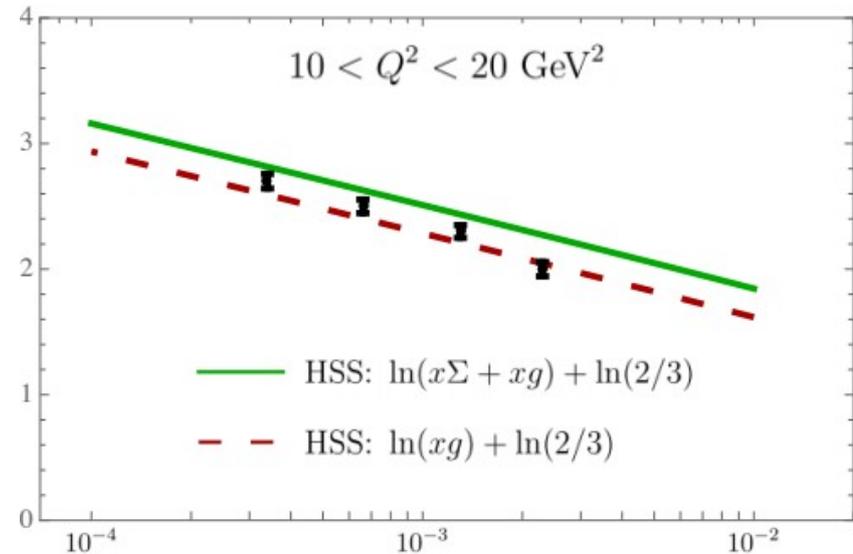
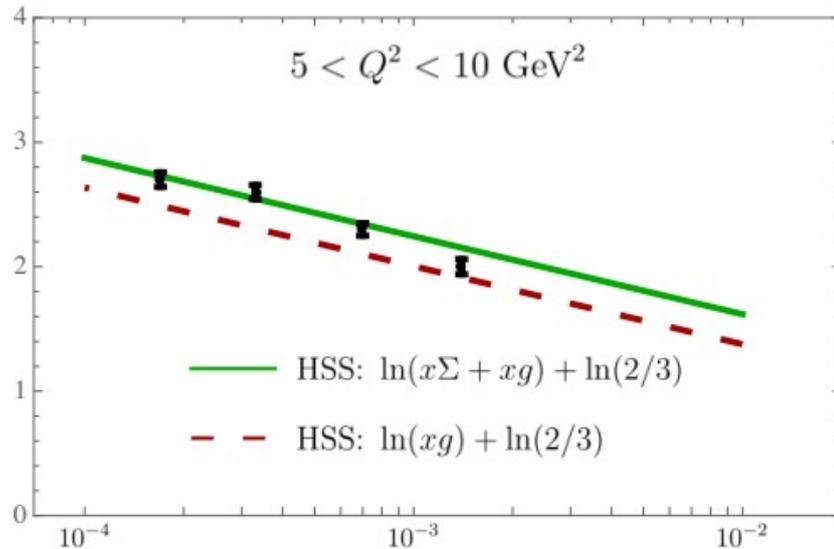
Hentschinski, Sabio-Vera, Salas.
 Phys.Rev.D 87 (2013) 7, 076005
 Phys.Rev.Lett. 110 (2013) 4, 041601

Proton structure function from HSS fit

F_2 data description

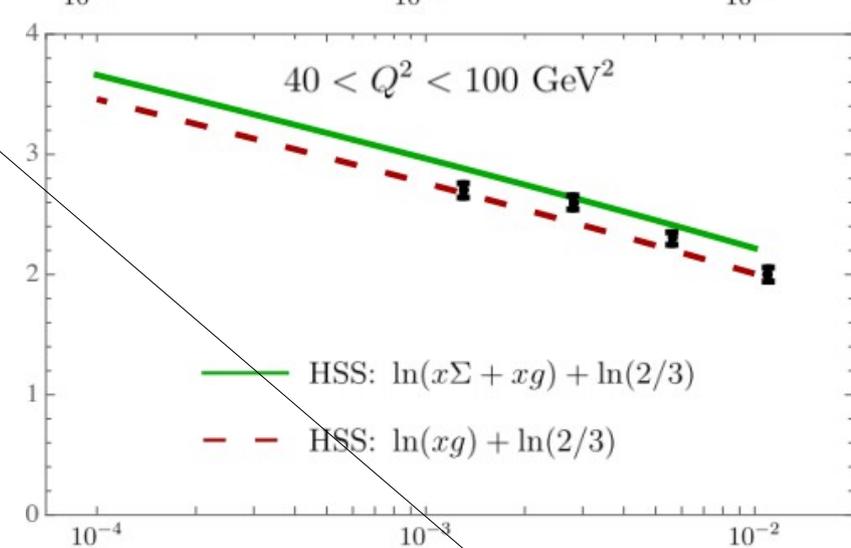
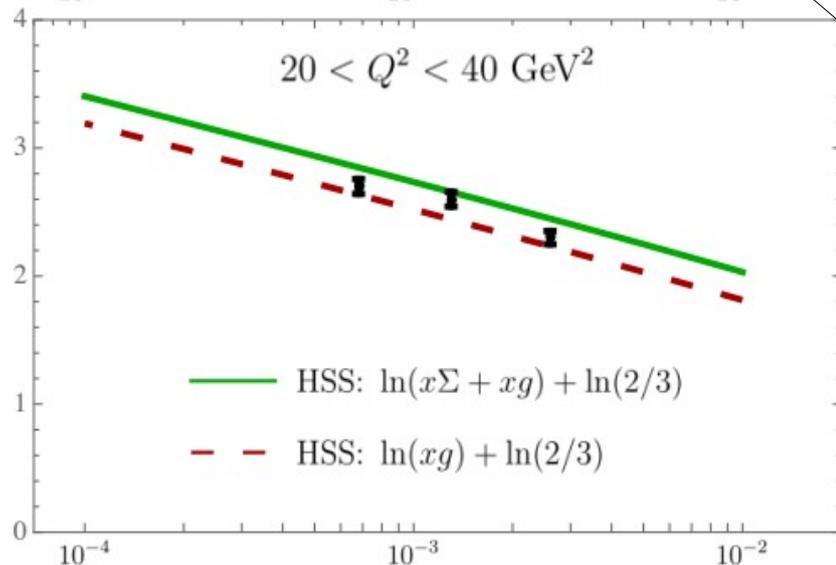
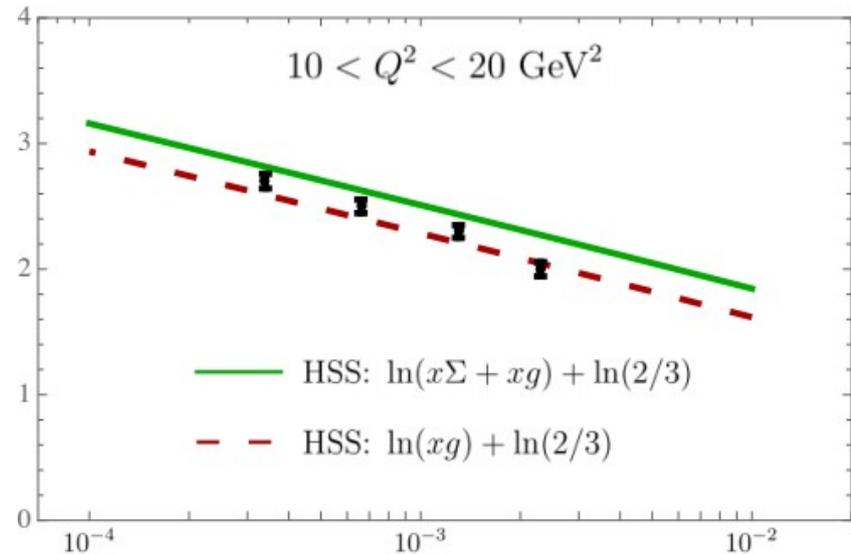
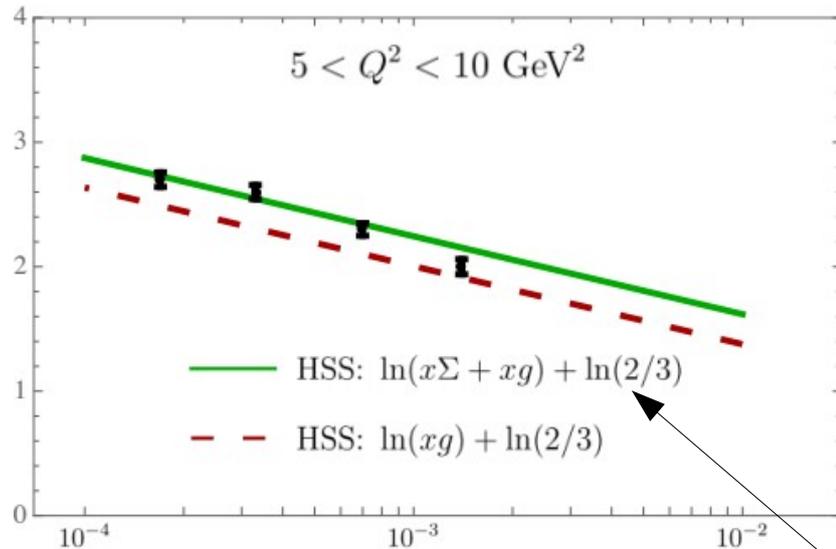


Results



Hint that the general idea works. Gluon dominates over quarks.
 One has to also take into account that only charged hadrons were measured.

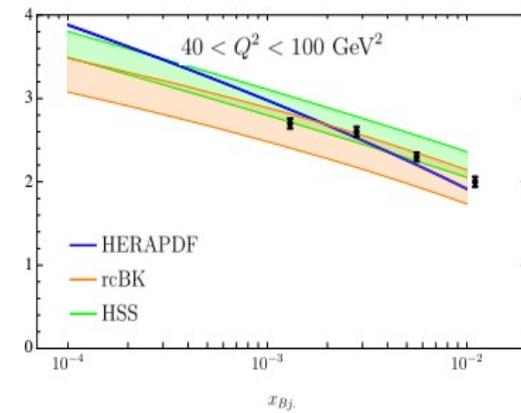
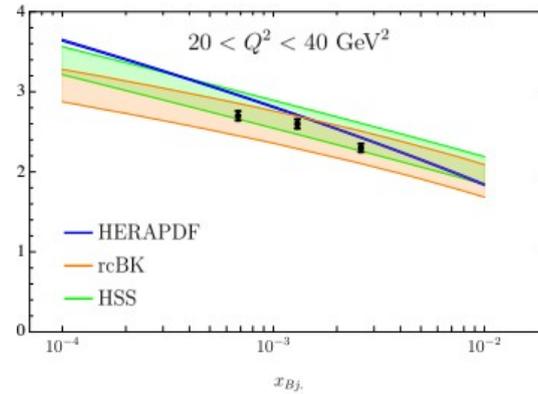
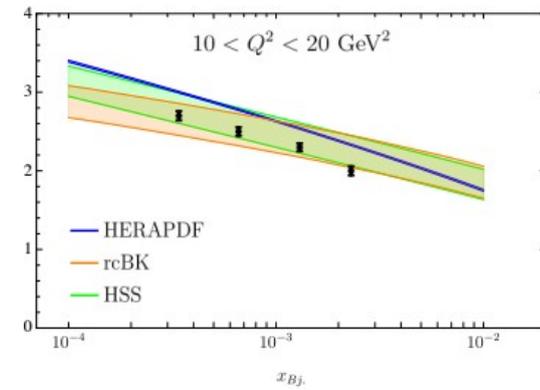
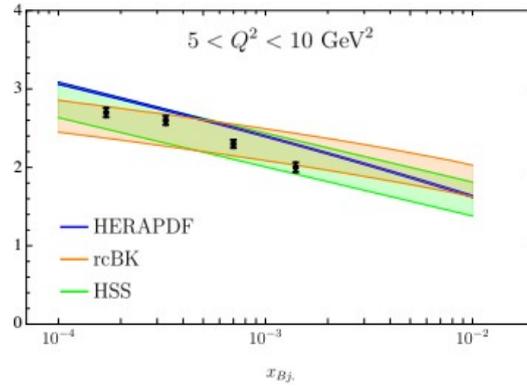
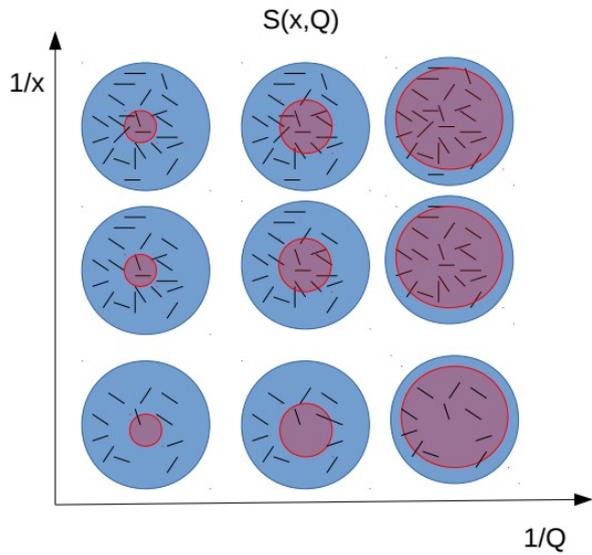
Results



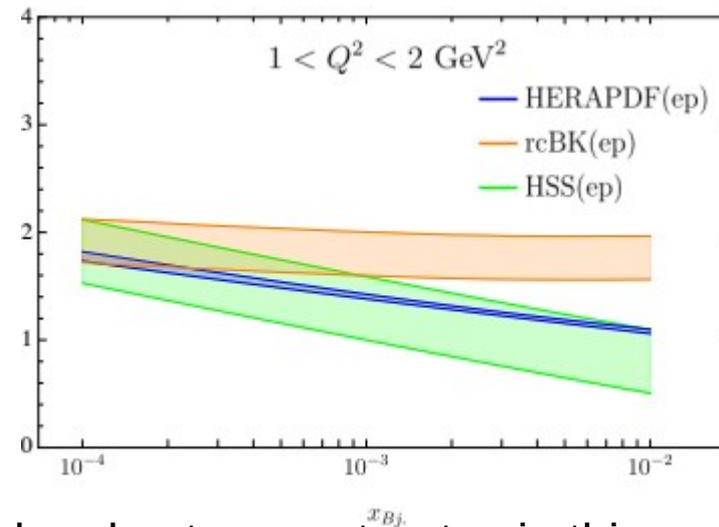
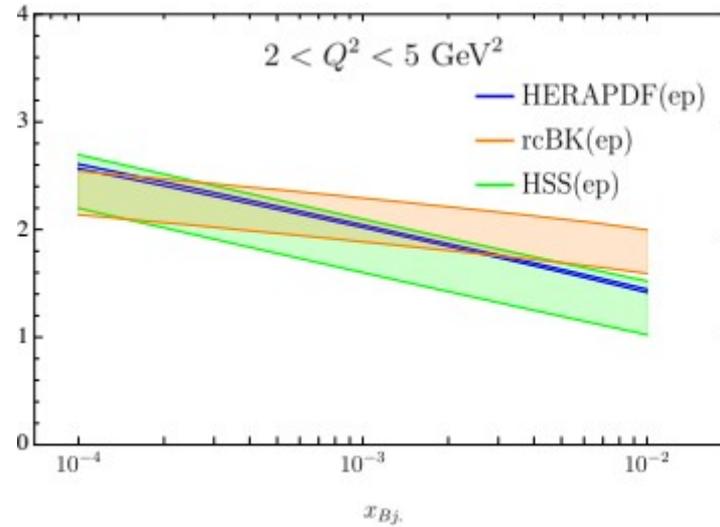
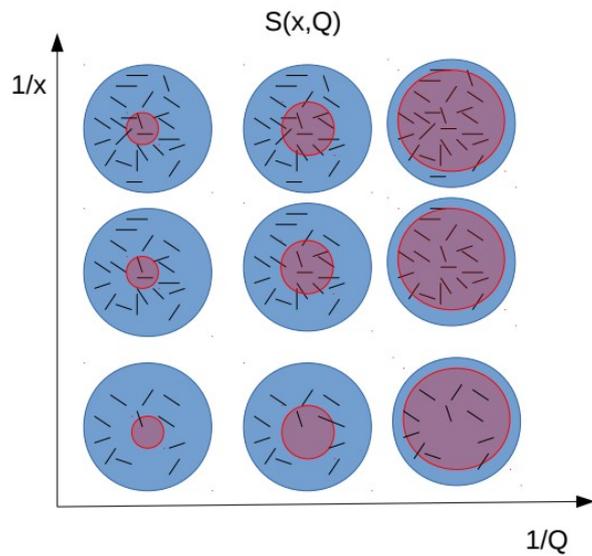
Hint that the general idea works. Gluon dominates over quarks.
One has to also take into account that only charged hadrons were measured i.e $2/3$ of partons contribute

Large scales - description

Martin Hentschinski, KK, Robert Straka '23



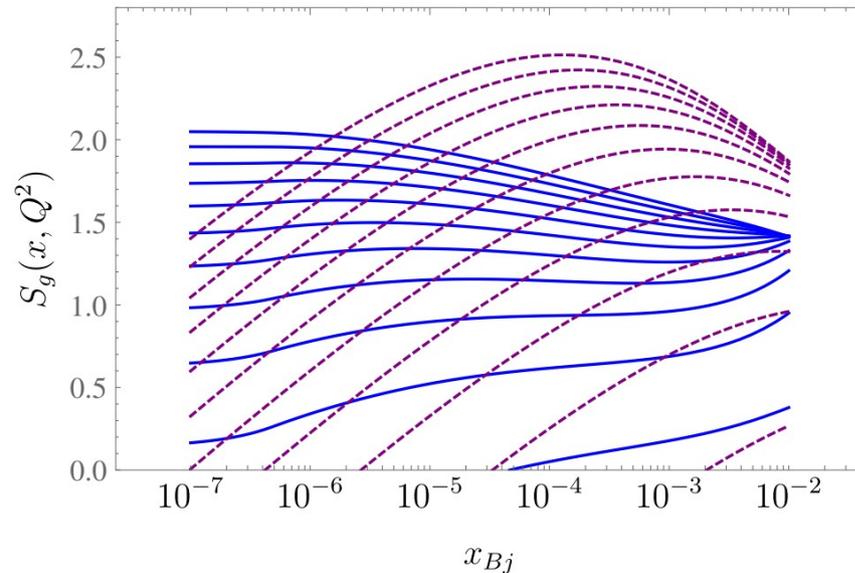
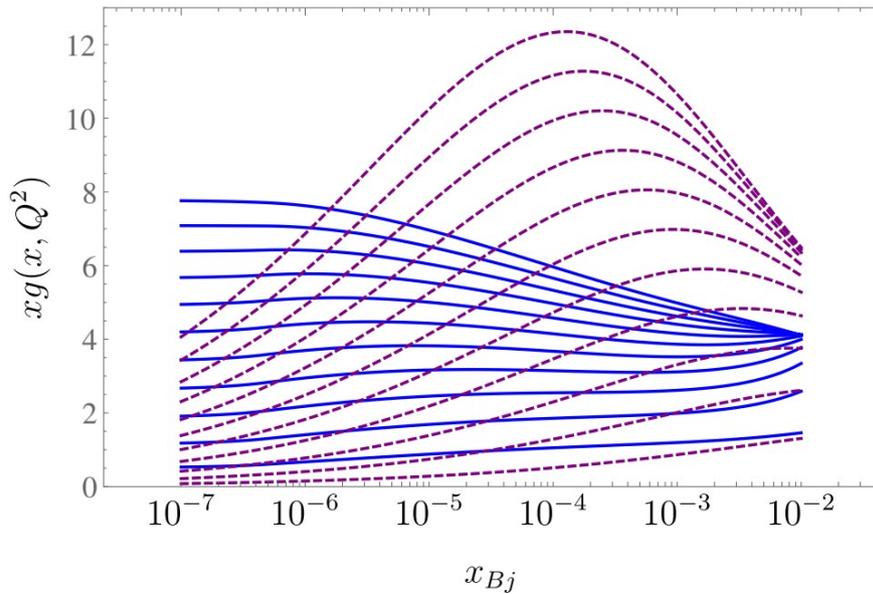
Small scales - prediction



See also [Hagivara, Hatta, Xiao '18](#)

The generalized KL model is used and entropy saturates in this approach and [Nowak, Liu, Zahed '22](#)

Integrated gluon and entropy



Photon can not resolve proton anymore therefore the EE vanishes.

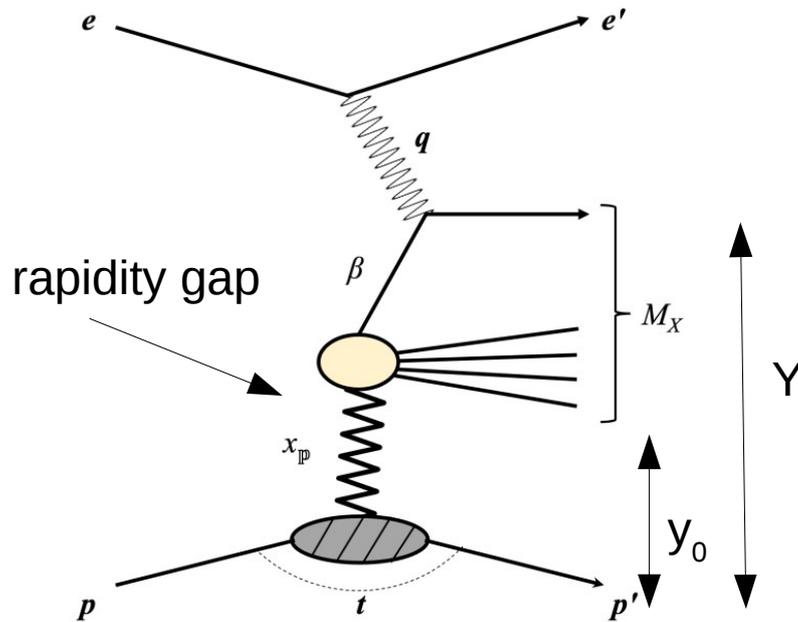
But it might be that the formalism breaks down for low scales.

There might be another source of entropy that keep the total entropy not vanishing → **generalized second law Bekenstein**

$$\lim_{Q^2 \gg Q_s^2} S(x, Q^2) = \ln(S_{\perp} Q_s^2(x)) + \ln \frac{N_c}{8\alpha_s \pi^2} = \lambda \ln \frac{1}{x} + \text{const}$$

$$\lim_{Q^2 \ll Q_s^2} S(x, Q^2) = \ln \left(\frac{S_{\perp} Q^4}{Q_s^2(x)} \right) + \ln \frac{N_c}{16\alpha_s \pi^2}$$

EE in Diffractive Deep Inelastic Scattering



$x_{\mathbb{P}}$ proton's momentum fraction carried by the Pomeron

β denotes the Pomeron's momentum fraction carried by the quark interacting with the virtual photon

$$x = \beta \cdot x_{\mathbb{P}} \quad \text{Bjorken } x$$

$$y_0 \simeq \ln 1/x_{\mathbb{P}} \quad \text{size of rapidity gap}$$

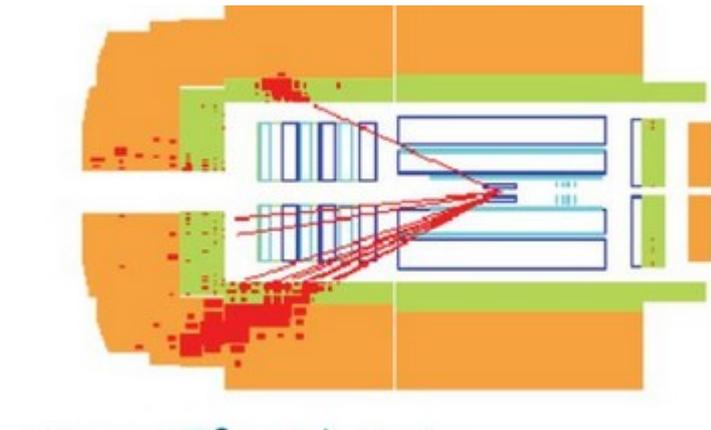
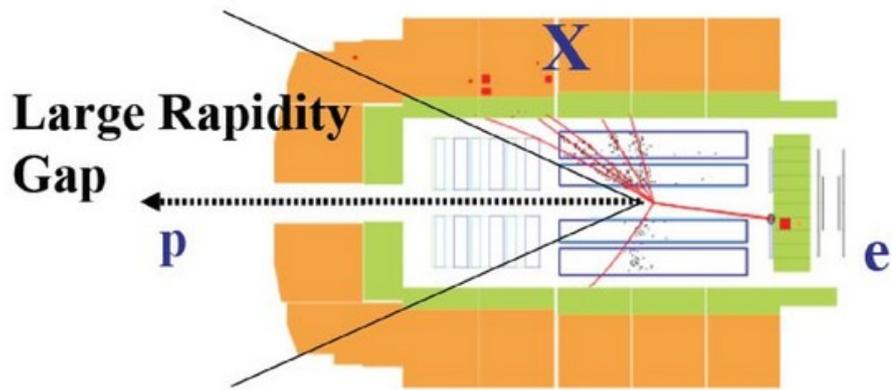
$$Y = \ln 1/x$$

$$y_X = Y - y_0 \simeq \ln 1/\beta$$

Analogous evolution equation as for non-diffractive case but Initial conditions are different and there is delay because of rapidity gap.

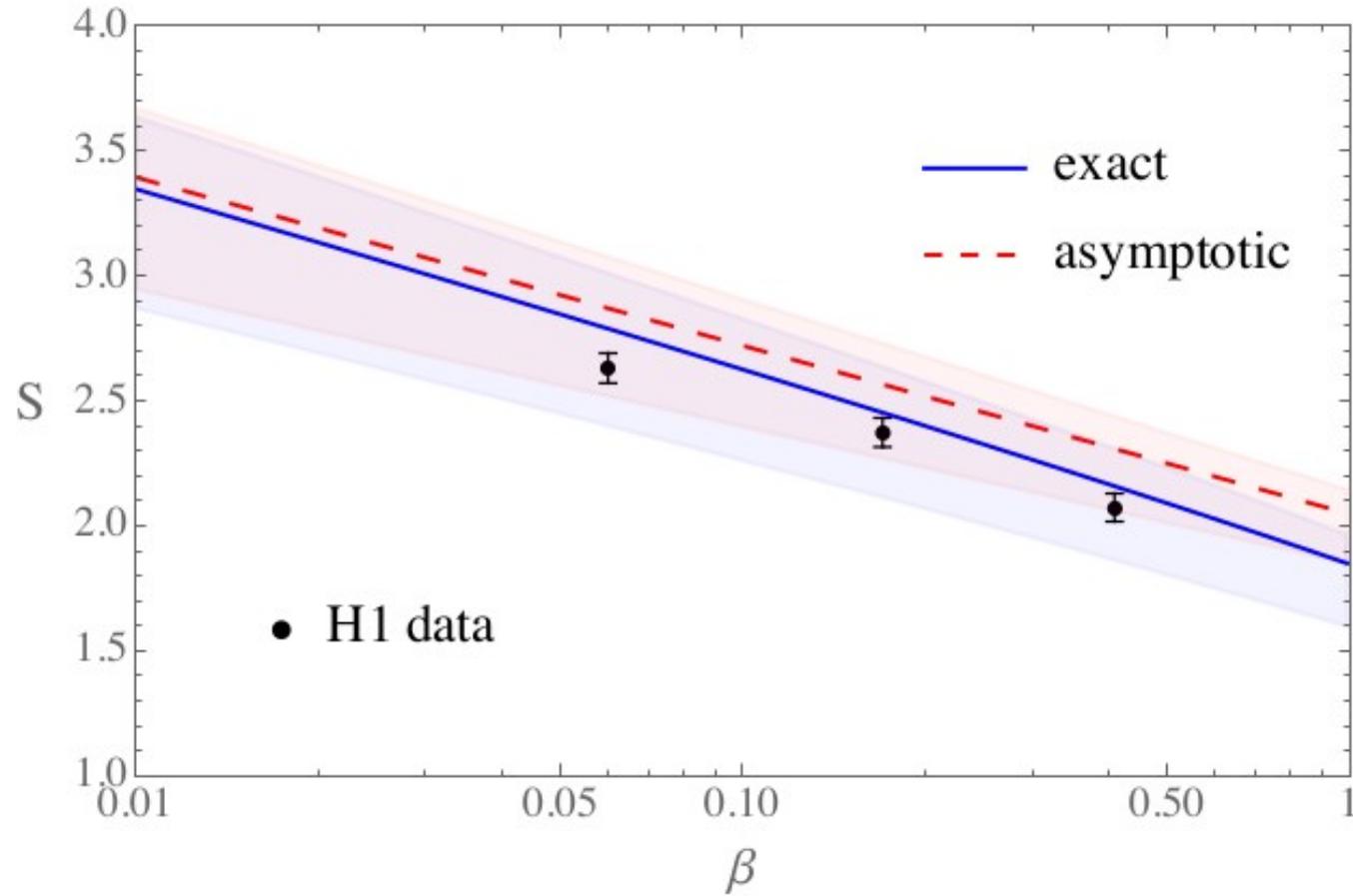
Munier, Mueller Phys. Rev. D 98, 034021 (2018)

Diffraction vs. nondiffraction

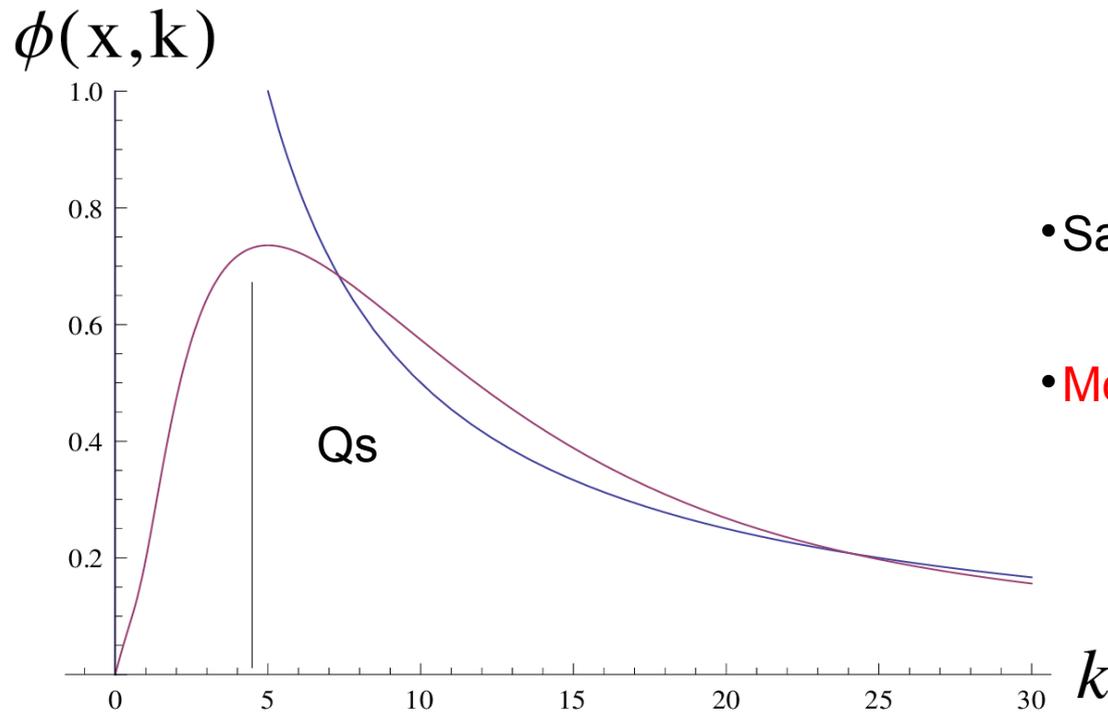


H1 detector

EE in DDIS

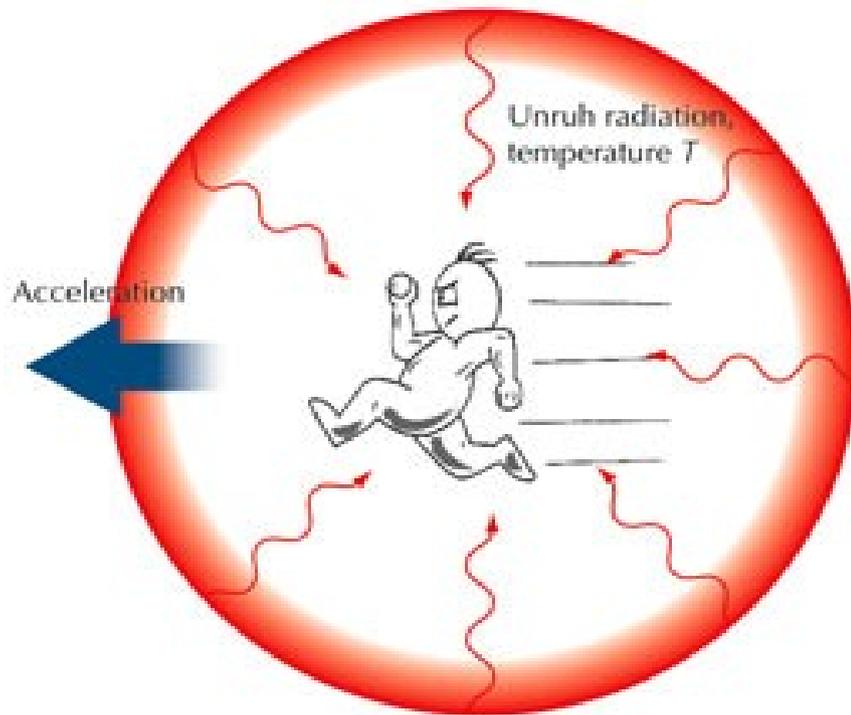


Saturation and gluon density



- Saturation scale regulates the divergence
- **Most of gluons** have momentum of the order of Q_s

Unruh effect



Accelerated observer in its rest frame feels thermal radiation or Bose-Einstein distribution with temperature

$$T = \frac{|a|}{2\pi}$$

Saturation and entropy



The relation $T = \frac{Q_s(x)}{2\pi}$ Can be understood in a generalized sense i.e. that **saturation** scale defines some **temperature**.

Equilibrium thermodynamics relations \longrightarrow Lower bound on produced entropy

It can be shown that the saturation line has an interpretation of a characteristics i.e. line along which the gluon density has a constant value.

Kutak 2011,
arxiv v1 and v2

$$dE = TdS$$

$$dM = TdS$$

$$\frac{dQ_s(x)}{Q_s(x)} = \frac{dS}{2\pi}$$

$$dE = dM$$

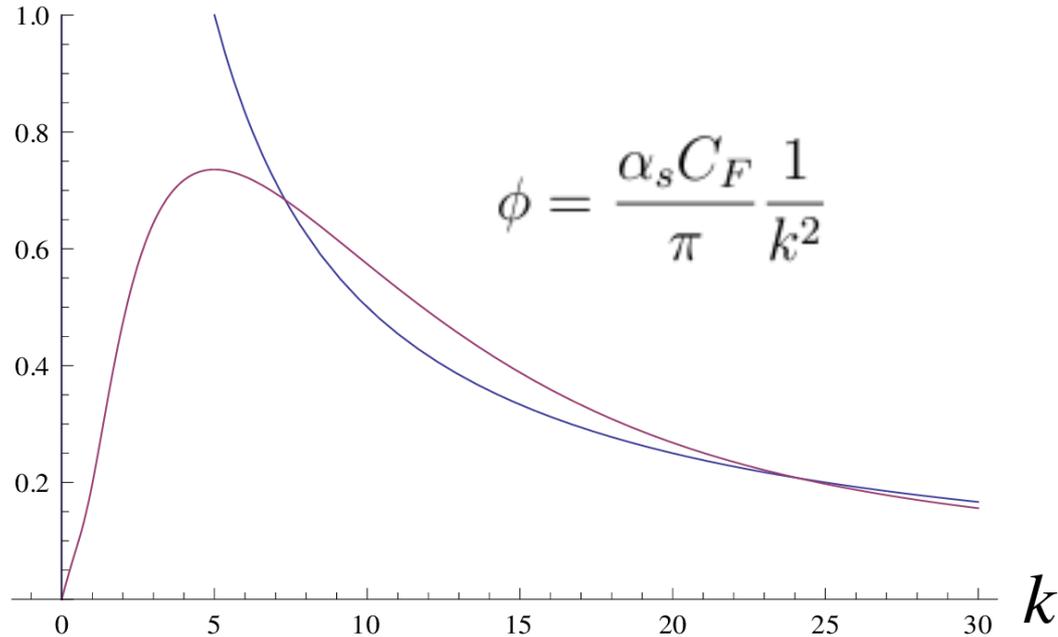
$$dM = dQ_s(x)$$

mass of system
of gluons

Gluon density and entropy

Kutak '11

$\phi(x, k)$



$$\Delta S = \pi \lambda \Delta y$$

$$\Delta y = \ln(x_0/x)$$

$$Q_s^2 = Q_0^2 (x_0/x)^\lambda$$

Many-body interactions



Medium generated mass of gluon.
Framework of Hard Thermal Loops.

Similarly in QED. Cut on photon's kt
Is equivalent to introducing mass.

In presented approach **mass is not fixed it is x dependent**

Conclusions and outlook

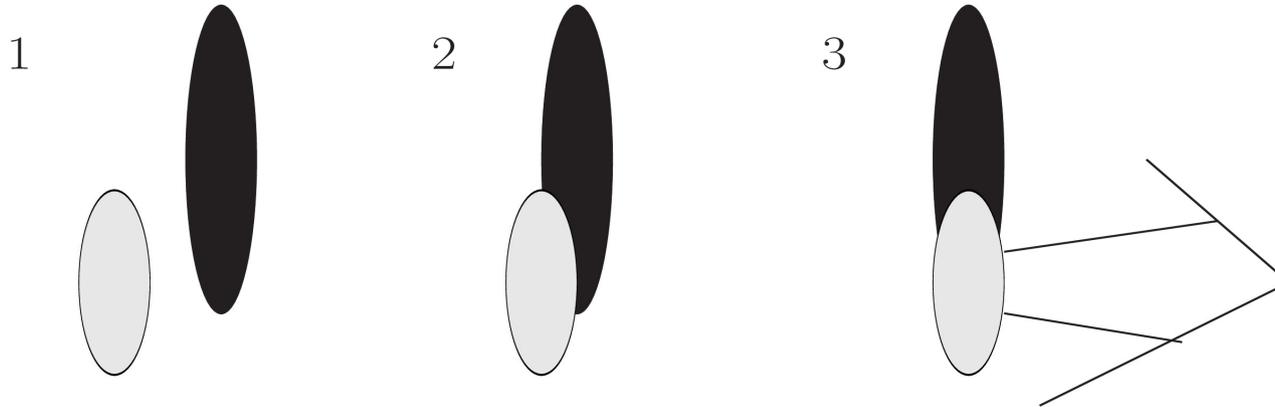
- We show evidences for the proposal for low x maximal entanglement entropy of proton constituents .
- It can be systematically improved (quark contributions, NLO BFKL, rc BK) and can describe successfully H1 data.
- We obtain saturation of entropy at small resolution scales.
- We demonstrate that the proposal works for DDIS and that it can be used to study onset of maximal entanglement
- The thermodynamic based approach agrees with KL approach

Backup

Colliding hadrons and Unruh effect

Stages of collision

Kharzeev, Tuchin '05



$$P(M \leftarrow m) = 2\pi |\mathcal{T}(M \leftarrow m)|^2 \rho(M),$$

$$\int dM P(M \leftarrow m)$$

Probability for transition to final state

should be finite

density of states determined by typical momentum. **Qs emerges**

$$|\mathcal{T}(M \leftarrow m)|^2 \sim \exp(-2\pi M/a)$$

$$\frac{a}{2\pi} \equiv T \leq \frac{\sqrt{6}}{4\pi} \frac{1}{\sqrt{b}} \equiv T_{Hag}$$

$$T = \frac{Q_s(x)}{2\pi}$$

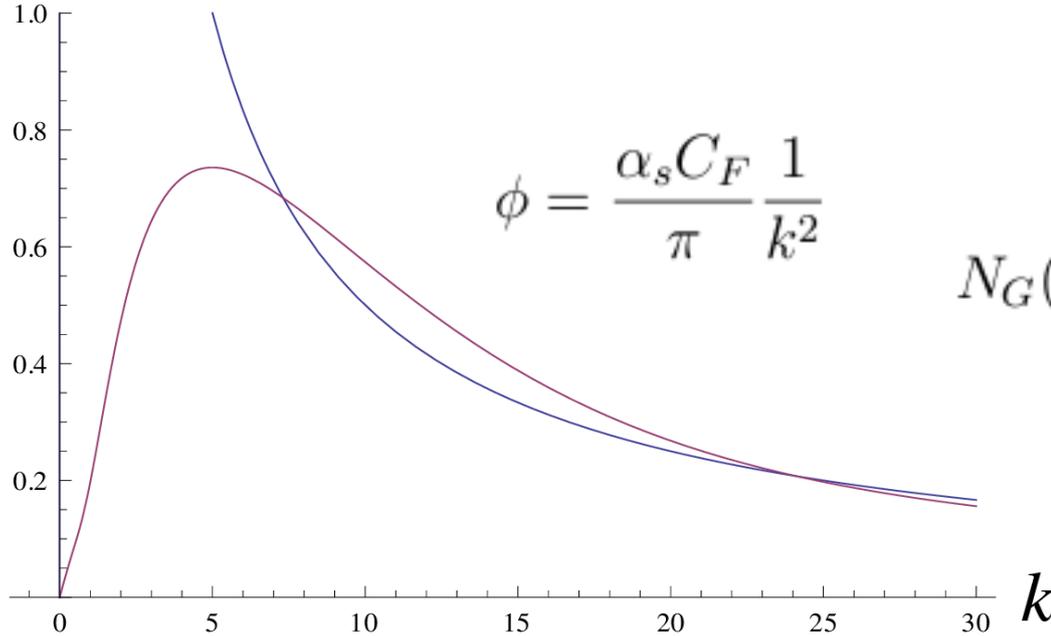
transition amplitude

Gluon production and entropy – another assumptions

Kutak '11

Bialas; Janik; Fialkowski, Wit; Iancu, Blaizot, Peschanski,...

$\phi(x, k)$



$$\phi = \frac{\alpha_s C_F}{\pi} \frac{1}{k^2}$$

$$M_G(x) = Q_s(x)$$

energy dependent
gluon's mass

$$M(x) = N_G(x) M_G(x)$$

mass of system
of gluons

$$N_G(x) \equiv \frac{dN}{dy} = \frac{1}{S_{\perp}} \frac{d\sigma}{dy}$$

number of gluons

$$dE = T dS$$



$$dM = T dS$$

$$d[N_G(x) M_G(x)] = \frac{Q_s(x)}{2\pi} dS$$

Many-body interactions



Medium generated mass of gluon.
Framework of Hard Thermal Loops.

Entropy due to less
dense hadron



$$S = \frac{6 C_F A_{\perp}}{\pi \alpha_s} Q_s^2(x) + S_0$$

Similarly in QED. Cut on photon's kt
Is equivalent to introducing mass.

$$S = 3\pi [N_G(x) + N_{G0}]$$

In presented approach mass is not fixed it is x dependent