Feynman Integrals from Integrability and Geometry

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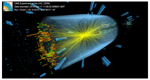
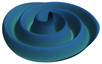
based on Phys.Rev.Lett. 130 4, 2023 + work in progress with
C. Duhr, A. Klemm, C. Nega, F. Porkert

DESY Theory Workshop 2023
Hamburg

Motivation

Feynman integrals needed for phenomenology



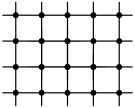


1) in practice: study differential equations
2) express result in terms of special functions (polylogs, elliptic . . .)

This talk: input from two string-inspired areas

1) AdS/CFT integrability → differential equations
2) Calabi-Yau geometry → special functions

Laboratory to study higher loop structure: conformal fishnet integrals



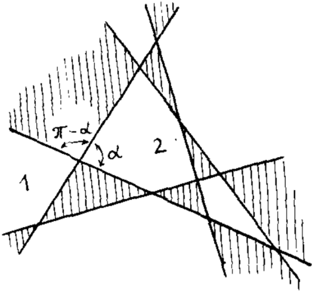
Fishnet Integrals and Integrability

"FISHING-NET" DIAGRAMS AS A COMPLETELY INTEGRABLE SYSTEM

A.B. ZAMOLODCHIKOV

The Academy of Sciences of the USSR, L.D. Landau Institute for Theoretical Physics, Chernogolovka, USSR

Received 29 July 1980



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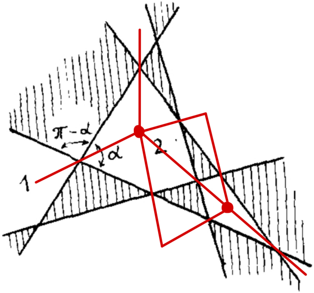
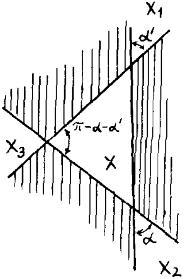


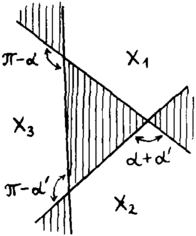
Vertex  : $\int \mathrm{d}^D x$

Propagator : $\frac{1}{x_{jk}^{2\alpha}} \equiv \frac{1}{(x_j - x_k)^{2\alpha}}$ $\qquad (x^2 = x^\mu x_\mu)$

Integrability and Conformal Symmetry



$\simeq$



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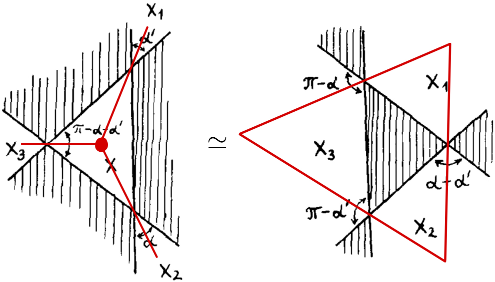
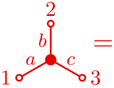
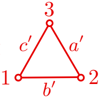


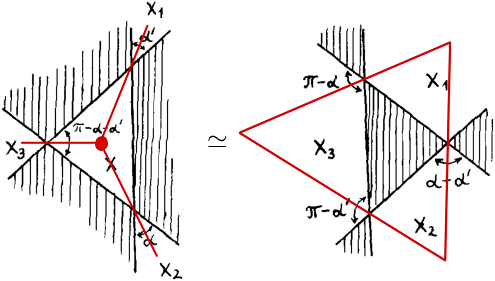


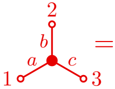
$$= \int \frac{\mathrm{d}^D x_0}{x_{10}^{2a} x_{20}^{2b} x_{30}^{2c}} \overset{a+b+c=D}{=} \frac{X_{abc}}{x_{12}^{2c'} x_{23}^{2a'} x_{31}^{2b'}} \simeq$$



with $X_{abc} = \pi^{\frac{D}{2}} \frac{\Gamma_{a'}\Gamma_{b'}\Gamma_{c'}}{\Gamma_a \Gamma_b \Gamma_c}$ and $a' = \frac{D}{2} - a$

Integrability and Conformal Symmetry





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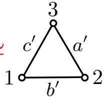

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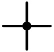
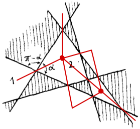
Do these graphs look like fishnets?

Fishnets in 4D

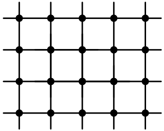
Euclidean integrals made from four-point vertices:

<u>Propagator:</u> j —— k : $\frac{1}{x_{jk}^{2a}} = \frac{1}{(x_j - x_k)^{2a}}$ with $a = 1$

<u>Vertex:</u>  : $\int \mathrm{d}^4 x$ with $\sum_{j=1}^{n} a_j = 4$



$\rightarrow$



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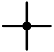
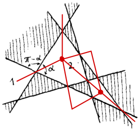
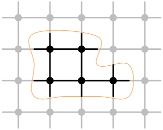
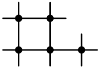
Vertex:  : $\int \mathrm{d}^4 x$ with $\sum_{j=1}^{n} a_j = 4$

 $\rightarrow$ e.g.  → 

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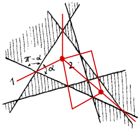
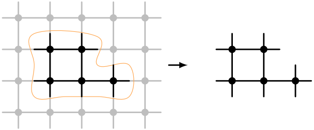
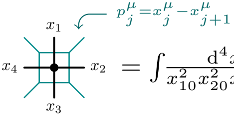


→ e.g.



Simplest example: cross integral [Ussyukina '93, Davydychev]: 

$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$

Fishnets in 4D

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Propagator: j ——— k : $\frac{1}{x_{jk}^{2a}} = \frac{1}{(x_j - x_k)^{2a}}$ with $a = 1$

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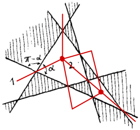

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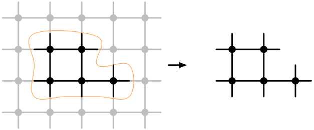

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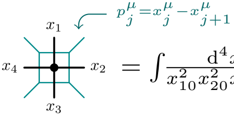

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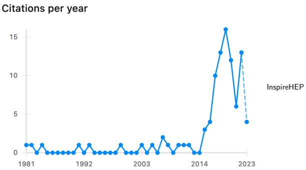
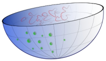
Fishnets and AdS/CFT integrability:



Fishnet Graphs as Correlation Functions

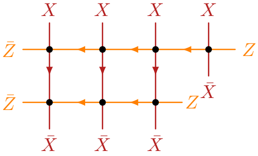
Double-scaling limit in AdS/CFT: 

$\mathcal{N}=4$ SYM		γ-Deformation		Fishnets
$\mathcal{L}_{\mathcal{N}=4}$	$\xrightarrow{XY \to e^{i\gamma_j(\dots)}XY}$	$\mathcal{L}^{\gamma}_{\mathcal{N}=4}$	$\xrightarrow[\xi = g e^{-i\gamma_3/2}\ \text{fix}]{g\to 0,\ \gamma_3 \to i\infty}$	$\mathcal{L}_{\mathsf{F}}$

Resulting bi-scalar fishnet theory: [Gürdoğan Kazakov 2015]

$$\mathcal{L}_{\mathsf{F}} = N_{\mathsf{c}}\, \mathrm{tr}\Big(-\partial_\mu \bar{X}\partial^\mu X - \partial_\mu \bar{Z}\partial^\mu Z + \xi^2 \bar{X}\bar{Z}XZ\Big)$$

- Correlators given by single fishnet Feynman graphs.



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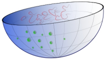

$$\underline{\mathcal{N}=4\ \text{SYM}} \qquad\qquad \underline{\gamma\text{-Deformation}} \qquad\qquad \underline{\text{Fishnets}}$$

$$\mathcal{L}_{\mathcal{N}=4} \xrightarrow{X\,Y \to e^{i\gamma_j(\dots)}XY} \mathcal{L}^{\gamma}_{\mathcal{N}=4} \xrightarrow[\xi = g e^{-i\gamma_3/2}\ \text{fix}]{g\to 0,\ \gamma_3 \to i\infty} \mathcal{L}_{\mathsf{F}}$$

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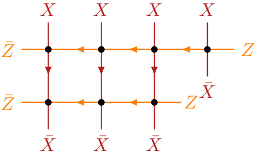
- Correlators given by single fishnet Feynman graphs.
- Fishnet integrals inherit conformal Yangian symmetry $Y[\mathfrak{so}(1,5)]$:

differential operator

$\widehat{\mathrm{J}}^a$ 

$= 0.$

[Chicherin, Kazakov, FL Müller, Zhong 2017]

Integrability and the Yangian

- The Yangian is an infinite dimensional extension of a Lie algebra $\mathfrak{g}$.
- It underlies rational quantum integrable models (rational S-matrix).

Yangian algebra $Y[\mathfrak{g}]$ (first realization): [Drinfeld 1985]

Level 0 : $$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}_k^a$$

Level 1 : $$\widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c \, \mathrm{J}_k^b$$

Serre relations: $$[\widehat{\mathrm{J}}_a, [\widehat{\mathrm{J}}_b, \mathrm{J}_c]] - [\mathrm{J}_a, [\widehat{\mathrm{J}}_b, \widehat{\mathrm{J}}_c]] = \mathcal{O}(\mathrm{J}^3).$$

Examples:

- AdS/CFT: $\mathfrak{g} = \mathfrak{psu}(2,2|4)$
- fishnet integrals: $\mathfrak{g} = \mathfrak{so}(1, D+1)$

Infinite extension of conformal algebra also beyond $D = 2$ dimensions.

Yangian PDEs for Feynman Integrals

Level 0:

$$\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}^a_k \qquad \text{with} \qquad \mathrm{J}^a \in \begin{cases} \mathrm{D} = -ix_\mu \partial^\mu - i\Delta, \\ \mathrm{L}_{\mu\nu} = ix_\mu \partial_\nu - ix_\nu \partial_\mu, \\ \mathrm{P}_\mu = -i\partial_\mu, \\ \mathrm{K}_\mu = ix^2 \partial_\mu - 2ix_\mu x^\nu \partial_\nu - 2i\Delta x_\mu. \end{cases}$$

$\Rightarrow I_n = V_n\, \phi$ with $\phi(z_1, z_2, \dots)$ function of conformal cross ratios.

Level 1: additional non-local generators $\widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{j<k} \mathrm{J}^c_j\, \mathrm{J}^b_k$ e.g.

$$\widehat{\mathrm{P}}^\mu = \sum_{j<k=1}^{n} \left[\left(\mathrm{L}^{\mu\nu}_j + \eta^{\mu\nu} \mathrm{D}_j \right) \mathrm{P}_{k,\nu} - (j \leftrightarrow k) \right] + \sum_{k=1}^{n} s_k \mathrm{P}_k$$

Yangian invariance: $$0 = \widehat{\mathrm{P}}^\mu I_n = V_n \sum_{j<k=1}^{n} \frac{x^\mu_{jk}}{x^2_{jk}}\, \mathrm{PDE}_{jk} \phi$$

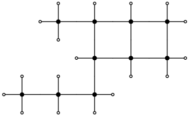
Leads to **system of Yangian PDEs** in the cross ratios: [FL, Müller Münkler 2019]

$$\mathrm{PDE}_{jk}\, \phi = 0, \qquad 1 \le j < k \le n.$$

Fishnet Integrals in Lower Dimensions

Conformal Fishnets in 1D and 2D

Fishnet integrals in lower dimensions with conformal choice of powers in propagators $|x|^{-2a_j}$: [Duhr, Klemm, FL; Nega, Porkert, in progress]

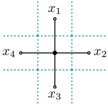


$$1\text{D}: a_j = \tfrac{1}{4}, \qquad 2\text{D}: a_j = \tfrac{1}{2}, \qquad \sum_{j=1}^{4} a_j = D.$$

Integrals are correlators in D-dimensional fishnet theory: [Kazakov, Olivucci 2018]

$$\mathcal{L} = N_c \,\mathrm{tr}\Big[X(-\partial_\mu\partial^\mu)^{\frac{D}{4}}\bar{X} + Z(-\partial_\mu\partial^\mu)^{\frac{D}{4}}\bar{Z} + \xi^2 X Z \bar{X}\bar{Z}\Big]$$

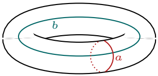
Simplest example: Cross integral in 1D



$$= \int \frac{\mathrm{d}x_0}{x_{10}^{2\frac{1}{4}} x_{20}^{2\frac{1}{4}} x_{30}^{2\frac{1}{4}} x_{40}^{2\frac{1}{4}}} \xrightarrow{\text{conf. transf.}} \int \frac{\mathrm{d}x}{\sqrt{x(x-1)(x-z)}} = \int \frac{dx}{y}$$

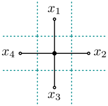
Natural geometry given by Legendre family of elliptic curves

$$y^2 = x(x-1)(x-z) = P_{\text{cross}}(x,z)$$



1D Box from Yangian Bootstrap

Consider Yangian over 1D conformal algebra $Y[\mathfrak{sl}(2,\mathbb{R})]$ on one-loop box:

$$I_4 =$$ 

$$= \frac{1}{\sqrt{x_{13}x_{24}}}\phi(z)$$

Yangian differential equation (= Legendre equation):

$$0 = \phi(z) + 4(2z-1)\phi'(z) + 4(z-1)z\phi''(z), \qquad z = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

Two solutions in terms of elliptic K integral: $K(z) = \int_0^{\pi/2} \mathrm{d}\theta \frac{1}{\sqrt{1-z\sin^2\theta}}$.

1 power series	1 single-log solution
$K(z) = \sum_j c_j z^j$	$K(1-z) = \log(z)\sum_j c_j z^j + \ldots$

For $\vec{\Pi} = (K(z), K(1-z))$ integral must be given by

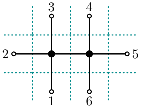
$$\phi(z) = \vec{v}\cdot\vec{\Pi}.$$

Fix linear combination using e.g. numerics to find $\vec{v} = (4,4)$.

1D Double Box from Yangian Bootstrap

Two loops:

$$I_6 =$$



$$= \frac{1}{\sqrt{x_{14}x_{26}x_{35}}}\phi(z_1, z_2, z_3)$$

Full set of PDEs from Yangian and permutation symmetries σ:

$$\widehat{\mathrm{P}} I_6 = 0, \qquad (\sigma \circ \widehat{\mathrm{P}}) I_6 = 0$$

Frobenius Method: Ansatz yields 5-dimensional solution vector $\vec{\Pi}$

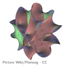
1 power series	3 single-log	1 double-log solution
$\sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$	$\log(z_a) \sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$	$\log(z_a)\log(z_b) \sum_{jkl} c_{jkl} z_1^j z_2^k z_3^l$

Fix linear combination e.g. by using numerics:

$$\phi(z_1, z_2, z_3) = \vec{v} \cdot \vec{\Pi}$$

$\ell = 1$: elliptic curve, $\ell = 2$ geometry? $\to$ need Calabi-Yaus

Mini Calabi-Yau Overview



A Calabi–Yau ℓ-fold is an ℓ-dimensional complex Kähler manifold with vanishing first Chern class.

Uniquely defined by triplet: $\left[\begin{array}{ll} M & \leftarrow \text{complex, } \ell\text{-dimensional manifold} \\ \Omega & \leftarrow (\ell, 0) \text{ form} \\ \omega & \leftarrow \text{Kähler } (1,1) \text{ form} \end{array}\right.$

Integrating Ω over the cycles Γ_j of the CY yields a vector $\vec{\Pi}$ of associated periods $\Pi_j(z) = \int_{\Gamma_j} \Omega$

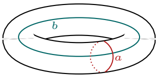
For every family of CYs there is a set of differential operators, the Picard–Fuchs Ideal (PFI), whose solutions are exactly the periods.

Example: 1D Box Integral (CY 1-fold = Elliptic Curve)

Triplet $(\mathcal{E}, \mathrm{d}a = \frac{\mathrm{d}x}{y}, A\,\mathrm{d}a \wedge \mathrm{d}\bar{a})$

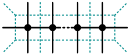
Periods $\vec{\Pi} = (K(z), K(1-z))$

PFI $= \{1 + 4(2z-1)\partial_z + 4z(z-1)\partial_z^2\}$



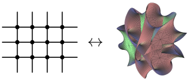
General Fishnets in 1D

Traintracks are Calabi–Yau ℓ-folds. [Duhr, Klemm, FL, Nega, Porkert]



Loops	1	2	3	...
Geometry	Elliptic Curve	K3 surface	CY 3-fold	...
Periods	1|1	1|3|1	1|5|5|1	...

Generic fishnets satisfy Calabi-Yau condition:



Picture Wiki/Floriang - CC

- integrand $\frac{1}{P_G^{(c-1)/c}}$ with polynomial P_G of degree $n = \frac{2c}{c-1}$
- Four-point fishnets have $c = 2$ and $n = 4$.

Conjecture

Yangian with permutations generates Picard–Fuchs ideal of differential operators with Calabi–Yau periods as solutions!

From 1 to 2 Dimensions

Double Copy in 2D

Split 2D Yangian into holomorphic and anti-holomorphic part:

$$Y[\mathfrak{sl}(2,\mathbb{R})] \oplus \overline{Y[\mathfrak{sl}(2,\mathbb{R})]}$$

Double Copy Structure: Same Yangian invariants $\vec{\Pi}$ as in 1D:

$$\phi(z) = \vec{\Pi}^\dagger \cdot \Sigma \cdot \vec{\Pi}$$

Indeed: Box integral in 2D given by linear combination of two factorized Yangian invariants [Derkachov, Kazakov Olivucci 2018] [Corcoran, FL Miczajka 2021]:

$$\begin{aligned}\phi(z,\bar{z}) &= 4\big[K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})\big] \\ &= 4i\begin{pmatrix} K(z) & iK(1-z)\end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} K(\bar{z}) \\ -iK(1-\bar{z})\end{pmatrix}\end{aligned}$$

What is the role of the matrix Σ?

Intersection Matrix, Kähler Potential, Volume

[Duhr, Klemm, FL Nega, Porkert 2022]

- The intersection matrix Σ of the Calabi–Yau defines a natural bilinear pairing of the periods. We observe

$$\phi(z) = \vec{\Pi}^\dagger \cdot \Sigma \cdot \vec{\Pi} = e^{-V}$$

Here V denotes the Kähler potential.

- Identify fishnet graph G with Calabi-Yau volume (or quantum volume of mirror Calabi-Yau), cf. [Greene Kanter 1996] [Gross, Huybrechts Joyce 2001] [Lee,Lerche Weigand 2019]

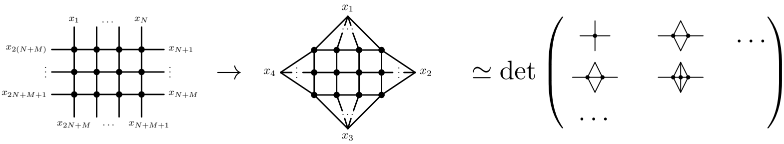
calibration

$$\mathrm{Vol}(M_G) = \int_{M_G} \frac{\omega_G^\ell}{\ell!} = c \int_{M_G} \Omega_G \wedge \bar{\Omega}_G$$

Calabi-Yaus and Basso-Dixon Formula

Four-Point Limits of Fishnet Integrals

In 4D: Basso–Dixon (BD) found determinant representation for fishnet integrals in four-point coincidence limit of $M \times N$ lattice [Basso Dixon 2017]



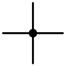
In 2D: generalization of [Derkachov, Kazakov Olivucci 2018] agrees with above structure

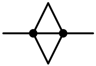
$$\phi_{MN} \simeq \det_{1\leq j,k\leq M}\left[(z\partial_z)^{j-1}(\bar{z}\partial_{\bar{z}})^{k-1}\partial_\varepsilon^{M+N-1}\big|{}_{M+N+1}F_{M+N}(\varepsilon,z)\big|^2\right]_{\varepsilon=0}$$

How does this fit into the Calabi-Yau picture?

Four-Point Graphs in 2D

Compact 1-/2-loop results: [Derkachov, Kazakov Olivucci 2018] [Corcoran, FL Miczajka 2021] [Duhr, Klemm, FL Nega, Porkert 2022]

Elliptic:  $= \frac{4}{\pi}\left(K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})\right) = \vec{\Pi}^\dagger \cdot \Sigma \cdot \vec{\Pi}$

K3:  $= \frac{8}{\pi^2}\left(K_+\overline{K}_- + K_-\overline{K}_+\right)^2 = \vec{\Pi}^\dagger \cdot \Sigma \cdot \vec{\Pi},$

with $K_\pm = K\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1-z}\right)$.

Implications of four-point limit:

- 2D $M \times N$ Basso–Dixon integrals depend on <u>single</u> variable $z \in \mathbb{C}$
- associated Picard–Fuchs ideal is generated by <u>single</u> differential operator $L_{M,N}$ that annihilates CY periods

Such differential operators are well studied in Calabi-Yau community. . .

Symmetric Power of CY Operators

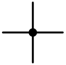
[Duhr, Klemm, FL
Nega, Porkert, to appear]

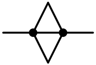
Symmetric power:

$$\mathrm{Sym}^p L := L_{\mathrm{Sym}^p(\mathrm{Sol}(L))}$$

with $\mathrm{Sol}(L) =$ invariants of L

Example:
Differential operator relations, e.g. $L_2 = \mathrm{Sym}^2(L_1)$ [Doran 2000][M.Bogner 2013], are inherited by their invariants, e.g.

$\phi_1 =$  $= \frac{4}{\pi}(K(z)K(1-\bar{z}) + K(1-z)K(\bar{z}))$

$\phi_2 =$  $= \frac{8}{\pi^2}\left(K_+\overline{K}_- + K_-\overline{K}_+\right)^2$

Exterior Power and Basso–Dixon

Exterior power: $\wedge^p L$ denotes operator of minimal degree that annihilates determinants of the form (with y_i invariants of L)

$$\begin{vmatrix} y_{i_1} & \cdots & y_{i_p} \\ \theta_z y_{i_1} & & \theta_z y_{i_p} \\ \vdots & \ddots & \vdots \\ \theta_z^{p-1} y_{i_1} & \cdots & \theta_z^{p-1} y_{i_p} \end{vmatrix}, \qquad \text{with } \theta_z = z\partial_z .$$

Example:
2D BD formula equivalent to $L_{M,N} = \wedge^M L_{M+N-1}$ [Duhr, Klemm, FL Nega, Porkert, to appear]

$W = M+N-1$

$$\phi_{M,N}(z) \simeq \det\left[\theta_{\bar{z}}^{i-1}\theta_z^{j-1}\phi_W(z)\right]_{1\leq i,j\leq M} \simeq D_I^{(W)}(\bar{z})\left(\Sigma_{M,N}\right)_{IJ} D_J^{(W)}(z)$$

$$\text{with} \quad \begin{aligned} I &= (i_1,\ldots,i_M) \\ J &= (j_1,\ldots,j_M) \end{aligned} \quad \text{and} \quad D_I^{(W)} = \begin{vmatrix} \Pi_{W,i_1} & \cdots & \Pi_{W,i_M} \\ \theta_z \Pi_{W,i_1} & & \theta_z \Pi_{W,i_M} \\ \vdots & \ddots & \vdots \\ \theta_z^{M-1}\Pi_{W,i_1} & \cdots & \theta_z^{M-1}\Pi_{W,i_M} \end{vmatrix}$$

Basso-Dixon Formula for Calabi-Yau Volume

Observations for fishnet integrals:

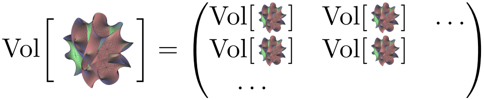
- obey Basso-Dixon formula
- compute volume of Calabi-Yau

which implies

Basso-Dixon formula for Calabi-Yau volume:

$$\mathrm{Vol}(\mathcal{M}_{M,N}) = \det\left[\bar{\vartheta}^{i-1}\vartheta^{j-1}\mathrm{Vol}(\mathcal{M}_{1,M+N-1})\right]_{0\le i,j<M}$$

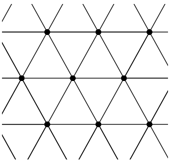
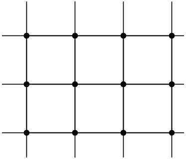
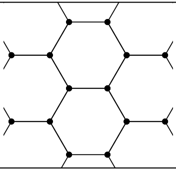
with a derivation ϑ, cf. [Duhr, Klemm, FL, Nega, Porkert, to appear]



Outlook: Beyond Square Fishnets

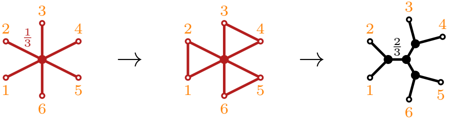
Isotropic Fishnets in 2D

Yangian symmetry also for other graph structures/propagator powers:
[Chicherin, Kazakov, FL Müller, Zhong 2017] [FL, Müller Münkler 2019] [Kazakov, Levkovich-Maslyuk Mishnyakov 2023]

Propagator	$\lvert x_{ij}\rvert^{-2\,\frac{1}{3}}$	$\lvert x_{ij}\rvert^{-2\,\frac{1}{2}}$	$\lvert x_{ij}\rvert^{-2\,\frac{2}{3}}$
Isotropic Fishnet			

Conformal: $\sum_{j\in\text{vertex}} a_j = D = 2$

Star-triangle identity gives:



Form of Feynman graph not unique.

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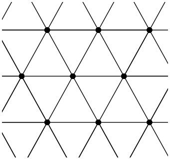
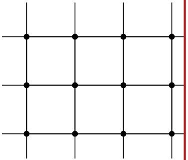
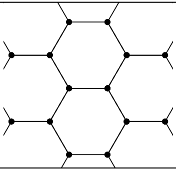
Isotropic Fishnet			

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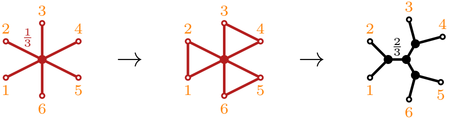
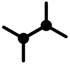
Star-triangle identity gives: 

Form of Feynman graph not unique.

Geometry and Graph Representation

Example: Two-loop three-point graph in 1D: [Duhr, Klemm, FL Nega, Porkert, in progress]

$$\int \frac{\mathrm{d}x_0 \mathrm{d}x_{\bar{0}}}{x_{10}^{\frac{2}{3}} x_{20}^{\frac{2}{3}} x_{0\bar{0}}^{\frac{2}{3}} x_{3\bar{0}}^{\frac{2}{3}} x_{4\bar{0}}^{\frac{2}{3}}} =$$



$\simeq$

$$= \int \frac{\mathrm{d}x_0}{x_{10}^{\frac{2}{3}} x_{20}^{\frac{1}{3}} x_{30}^{\frac{2}{3}} x_{40}^{\frac{1}{3}}}$$

CY condition: integrand $\frac{1}{P_G^{(c-1)/c}}$ with polynomial of degree $n = \frac{2c}{c-1}$

LHS: Natural geometry is singular K3 with $c = n = 3$

$$y^3 = (x_1 - x_0)(x_2 - x_0)(x_0 - x_{\bar{0}})(x_3 - x_{\bar{0}})(x_4 - x_{\bar{0}})$$

RHS: Natural geometry is Picard-curve:

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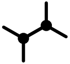

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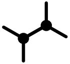

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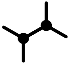

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$\Rightarrow$ emphasizes importance of differential equations

Conclusions

Fishnet integrals are rich topic relating

- AdS/CFT integrability
- Feynman graphs
- Calabi-Yau geometry

Many directions to explore:

- more loops, legs, masses, dimensions
- curved space: Witten diagrams, cf. [Rigatos Zhou '22]

Beyond 2 dimensions:

- we loose holomorphic factorization, but ...
- conformal Yangian still yields building blocks for integrals [FL, Müller Münkler '19]
- underlying geometry still Calabi–Yau [...][e.g. Snowmass review Bourjaily et al. '21] [...]