Diamonds of integrable deformations

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Based on work
with **Lewis Cole**, **Ryan Cullinan**, **Joaquin Liniado**
and **Dan Thompson**

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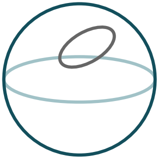
Introduction

- Integrable systems appear in many guises in mathematical and theoretical physics:
  - exact results in QFT,
  - quantum groups and algebras,
  - statistical mechanics,
  - string and gauge theory,
  - gauge/gravity duality.



Introduction

- The landscape of 2d integrable models continues to grow:
  - direct construction (challenging),
  - $\mathcal{E}$-models (2d Hamiltonian formulation), [Klimčík, Ševera, ...]
  - affine Gaudin models (integrable systems), [Feigin, Frenkel, Delduc, Lacroix, Magro, Vicedo, ...]
  - higher-dimensional gauge theories, [Ward, Mason, Woodhouse, Costello, Yamazaki, Witten, ...]
  - dualities and deformations (more general guiding principles). [...]
- These frameworks and approaches are closely connected.

Introduction

- Higher-dimensional gauge theories:

⋆ 4d Chern-Simons (4d CS) [Costello, Yamazaki, Witten, . . .]

$$\boxed{\frac{1}{2\pi i}\int_{\Sigma\times\mathbb{CP}^1}\omega\wedge(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)}$$

– localises on order and disorder defects on $\mathbb{CP}^1$ to give 2d IFTs.

⋆ 4d anti-self-dual Yang-Mills (4d IFT) [Ward, Mason, Woodhouse, . . .]

$$\boxed{F_{AA'BB'}=\epsilon_{AB}\Phi_{A'B'}+\epsilon_{A'B'}\tilde{\Phi}_{AB}\qquad\Phi_{A'B'}=0}$$

– symmetry reduces to 2d IFTs.

Introduction

Conjecture (Costello):

$$
\begin{tikzcd}
& \textbf{6d hCS} \arrow[dl, "\text{reduction}"'] \arrow[dr, "\text{localisation}"] & \\
\textbf{4d CS} \arrow[dr, "\text{localisation}"'] & & \textbf{4d IFT} \arrow[dl, "\text{reduction}"] \\
& \textbf{2d IFT} &
\end{tikzcd}
$$

Introduction

- Higher-dimensional gauge theories:

⋆ 6d holomorphic Chern-Simons (6d hCS)

$$\boxed{\frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge (\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \frac{2}{3}\,\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}})}$$

  - 6d theory on twistor space,
  - localises on defects to give 4d IFTs,
  - symmetry reduces to 4d CS.

$$\begin{tikzcd} & \textbf{6d hCS} \arrow[dl] \arrow[dr] & \\ \textbf{4d CS} \arrow[dr] & & \textbf{4d IFT} \arrow[dl] \\ & \textbf{2d IFT} & \end{tikzcd}$$

Introduction

- What can we learn about:
  - the unification of classifications?
  - the landscape of integrable models?
  - the relation between integrable models?
  - properties of integrable models?

$$\begin{tikzcd}
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Introduction

What was known

- The diamond has been constructed for the PCM + WZ term: [Bittleston, Skinner]
  - Dirichlet (Cauchy) boundary conditions work as expected.
- Early attempts for deformed models were: [Chen, He, Tian]
  - unclear how to lift boundary conditions from 4d CS,
  - unclear what gauge symmetries to expect in the 2d and 4d IFTs.

What is known now

[Cullinan, Cole, BH, Liniado, Thompson]

- The diamond has been successfully constructed for the current-current deformation of the WZW CFT.
- There is a systematic picture of how to implement boundary conditions and what gauge symmetries to expect.

$$\begin{tikzcd} & \text{6d hCS} \arrow[dl] \arrow[dr] & \\ \text{4d CS} \arrow[dr] & & \text{4d IFT} \arrow[dl] \\ & \text{2d IFT} & \end{tikzcd}$$

Outline

1. Introduction
2. The 2d IFTs
3. From 6d hCS to 4d CS to 2d IFT
4. From 6d hCS to 4d IFT to 2d IFT
5. Summary and future directions

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PCM plus WZ term

- The action of the PCM plus WZ term for the simple Lie group G is [Novikov, Witten]

$$\boxed{\mathcal{S} = -\frac{k}{8\pi h}\int_{\Sigma} \mathrm{vol}_2\, \mathrm{tr}\left(J_+ J_-\right) - \frac{k}{8\pi}\int_{\Sigma\times[0,1]} \mathrm{vol}_2 \wedge d\xi\, \mathrm{tr}\left(J_\xi [J_+, J_-]\right)}$$

  - $J = g^{-1}dg \in \mathrm{Lie(G)} = \mathfrak{g}$ where $g \in \mathrm{G}$,
  - $\sigma^{\pm}$ are coordinates on the 2d space-time Σ, ξ is coordinate on $[0,1]$,
  - $J_{\pm}$ and J_ξ are the pull-backs of J to Σ and $\Sigma \times [0,1]$,
  - h is the sigma model coupling,
  - $k \in \mathbb{Z}$ is the level of the Wess-Zumino term,
  - tr is the normalised Killing form.

$$\begin{tikzcd} & \text{6d hCS} \arrow[dl] \arrow[dr] & \\ \text{4d CS} \arrow[dr] & & \text{4d IFT} \arrow[dl] \\ & \textbf{2d IFT} & \end{tikzcd}$$

PCM plus WZ term – classical integrability

- The e.o.m. of the PCM plus WZ term is

$$\boxed{(1+h)\partial_+ J_- + (1-h)\partial_- J_+ = 0}$$

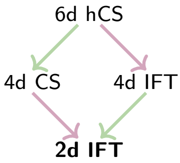
- There is a current that is both conserved and flat on-shell

$$\boxed{(1 \mp h) J_\pm}$$

- Therefore, a Lax connection exists

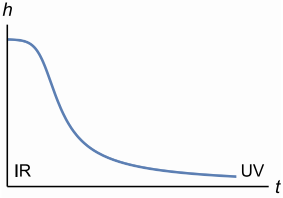
$$\boxed{\mathcal{L}_\pm = \frac{1 \mp h}{1 \mp \zeta} J_\pm \qquad \zeta \in \mathbb{C}}$$

- The Poisson bracket of the Lax matrix is of Maillet form and we have conserved charges in involution.



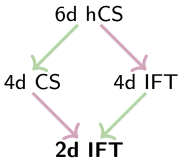
PCM plus WZ term – RG flow

- The RG flow of the PCM plus WZ term is



  - k is an RG invariant, h runs with the RG time t,
  - asymptotically free in the UV,
  - flows to the G_k WZW CFT, $h = 1$, in the IR

$$\boxed{\mathcal{S} = -\frac{k}{8\pi}\int_\Sigma \mathrm{vol}_2\,\mathrm{tr}\left(J_+ J_-\right) - \frac{k}{8\pi}\,\mathcal{S}_{\mathrm{WZ}}(g)}$$



Current-current deformation

- The action of the G_k WZW CFT perturbed by the current-current bilinear is [Sfetsos]

$$\boxed{\mathcal{S} = -\frac{k}{8\pi}\int_\Sigma \mathrm{vol}_2\, \mathrm{tr}\left(J_+ \frac{1+\lambda\,\mathrm{Ad}_g^{-1}}{1-\lambda\,\mathrm{Ad}_g^{-1}} J_-\right) - \frac{k}{8\pi}\mathcal{S}_{\mathrm{WZ}}(g)}$$

  - λ is the deformation parameter,
  - the leading term is a current-current bilinear

$$\boxed{\mathcal{S} = \cdots - \frac{k\lambda}{4\pi}\int \mathrm{vol}_2\, \mathrm{tr}\left(\partial_+ g g^{-1} g^{-1}\partial_- g\right) + \ldots}$$

- Commonly referred to as the λ-model in the literature.

$$\begin{tikzcd}
& \text{6d hCS} \arrow[dl] \arrow[dr] & \\
\text{4d CS} \arrow[dr] & & \text{4d IFT} \arrow[dl] \\
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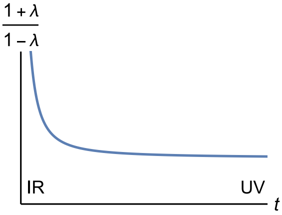
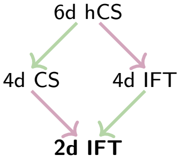
\end{tikzcd}$$

Current-current deformation – RG flow

- The current-current perturbation is marginally relevant



  - k is an RG invariant, λ runs with the RG time t,
  - flows from the G_k WZW CFT in the UV,
  - strongly coupled in the IR.



Current-current deformation - classical integrability

- The action of the deformed model can be written as

$$\boxed{\begin{aligned}\mathcal{S} = &-\frac{k}{8\pi}\int_\Sigma \mathrm{vol}_2\,\mathrm{tr}\left(J_+J_-\right) - \frac{k}{8\pi}\,\mathcal{S}_{\mathrm{WZ}}(g) \\ &+\frac{k}{4\pi}\int_\Sigma \mathrm{vol}_2\left(A_+J_- - A_-\,\mathrm{Ad}_g\,J_+ + A_-\,\mathrm{Ad}_g\,A_+ - \lambda^{-1}A_+A_-\right)\end{aligned}}$$

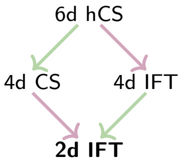
- In terms of the on-shell values of the auxiliary fields $A_\pm$, the e.o.m. can be written

$$\boxed{\partial_\pm A_\mp - \lambda\partial_\mp A_\pm + [A_\pm, A_\mp] = 0}$$

- We can construct a flat conserved current

$$\boxed{\frac{2}{1+\lambda}A_\pm}$$

and the model is classically integrable.



Four-parameter model

- This can be embedded inside a four-parameter 2-field model [Georgiou, Sfetsos]

$$
\begin{aligned}
\mathcal{S} = &- \frac{k}{8\pi} \int_\Sigma \mathrm{vol}_2 \, \mathrm{tr}\left(J_+ J_-\right) - \frac{k}{8\pi} \mathcal{S}_{\mathrm{WZ}}(g) - \frac{\tilde{k}}{8\pi} \int_\Sigma \mathrm{vol}_2 \, \mathrm{tr}\left(\tilde{J}_+ \tilde{J}_-\right) - \frac{\tilde{k}}{8\pi} \mathcal{S}_{\mathrm{WZ}}(\tilde{g}) \\
&+ \frac{k}{4\pi} \int_\Sigma \mathrm{vol}_2 \left(A_+ J_- - B_- \, \mathrm{Ad}_g \, J_+ + B_- \, \mathrm{Ad}_g \, A_+\right) \\
&+ \frac{\tilde{k}}{4\pi} \int_\Sigma \mathrm{vol}_2 \left(B_+ \tilde{J}_- - A_- \, \mathrm{Ad}_{\tilde{g}} \, \tilde{J}_+ + A_- \, \mathrm{Ad}_{\tilde{g}} \, B_+\right) \\
&- \frac{\sqrt{k\tilde{k}}}{4\pi} \int_\Sigma \mathrm{vol}_2 \left(\lambda^{-1} A_+ A_- + \tilde{\lambda}^{-1} B_+ B_-\right)
\end{aligned}
$$

- If we formally take $\tilde{\lambda} \to 1$ and set $k = \tilde{k}$, there is an emergent G gauge symmetry.
- We can set $\tilde{g} = 1$ and solve for $B_\pm$ to recover the current-current deformation of the G_k WZW CFT.

$$
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\end{tikzcd}
$$

Four-parameter model

- This model has many nice features:
  - marginally relevant deformation of the $G_k \times G_{\tilde{k}}$ WZW CFT.

$$\boxed{\mathcal{S} = \dots - \frac{\sqrt{k\tilde{k}}}{4\pi} \int \mathrm{vol}_2 \, \mathrm{tr} \left(\tilde{\lambda} \partial_+ g g^{-1} \tilde{g}^{-1} \partial_- \tilde{g} + \lambda \partial_+ \tilde{g} \tilde{g}^{-1} g^{-1} \partial_- g \right) + \dots}$$

  - flows to non-trivial fixed points in the IR.
  - classically integrable with flat conserved currents ($k_0 = \sqrt{\frac{\tilde{k}}{k}}$)

$$\boxed{\frac{2(1 - k_0^{\mp 1}\lambda)}{1 - \lambda^2} A_\pm \qquad \frac{2(1 - k_0^{\pm 1}\tilde{\lambda})}{1 - \tilde{\lambda}^2} B_\pm}$$

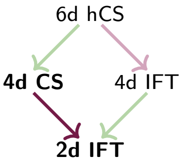
$$\begin{tikzcd} & \text{6d hCS} \arrow[dl] \arrow[dr] & \\ \text{4d CS} \arrow[dr] & & \text{4d IFT} \arrow[dl] \\ & \textbf{2d IFT} & \end{tikzcd}$$

Origin from 4d CS

- The action of 4d CS is [Costello, Yamazaki, Witten, ...]

$$\boxed{\frac{1}{2\pi i}\int_{\Sigma\times\mathbb{CP}^1}\omega\wedge\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)}$$

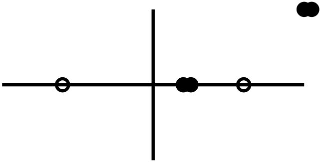
  - $(\sigma^{\pm},\zeta)$ are coordinates on $\Sigma\times\mathbb{CP}^1$,
  - ω is a specified meromorphic 1-form on $\mathbb{CP}^1$,
  - A is a $\mathfrak{g}$-valued 1-form on $\Sigma\times\mathbb{CP}^1$.
- To define the theory, we specify ω and boundary conditions, which ensure that the boundary variation vanishes.
- More mathematically rigorous treatments are available!



4d CS origin of the PCM plus WZ term

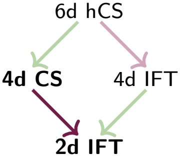
- For the PCM plus WZ term

$$\boxed{\omega \sim \frac{\zeta^2 - 1}{(\zeta - h)^2}\, d\zeta}$$



- The positions of the zeroes and the double pole at infinity are fixed by $SL(2,\mathbb{C})$.
- The remaining freedom is the position of the double pole on the finite plane.
- The boundary conditions are

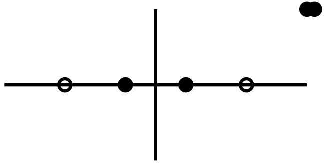
$$\boxed{A_\pm|_{\zeta=h} = 0 \qquad A_\pm|_{\zeta\to\infty} = 0}$$



4d CS origin of the current-current deformation

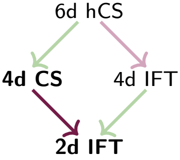
- For the current-current deformation of the G_k WZW CFT

$$\boxed{\omega \sim \frac{\zeta^2 - 1}{\zeta^2 - \alpha^2}\, d\zeta \qquad \alpha = \frac{1-\lambda}{1+\lambda}}$$



- The positions of the zeroes and the double pole at infinity are fixed by $SL(2, \mathbb{C})$.
- The remaining freedom is the position of the simple poles on the finite plane.
- However, note that $\text{res}_{\zeta\to\infty} = 0$.
- The boundary conditions are

$$\boxed{A_\pm|_{\zeta=\alpha} = A_\pm|_{\zeta=-\alpha} \qquad A_\pm|_{\zeta\to\infty} = 0}$$



Origin from 4d CS

- In both cases the boundary conditions can be understood systematically.

  [Delduc,Magro,Lacroix,Vicedo,...]

- From the gauge field and its derivatives evaluated at the poles we construct

$$\boxed{\mathbb{A}_\pm \in \mathfrak{d}}$$

where $\mathfrak{d}$ is a Drinfel'd double with bilinear form $\langle\langle , \rangle\rangle$.

- Setting the boundary variation to vanish implies that

$$\boxed{\langle\langle \mathbb{A}_+, \delta\mathbb{A}_- \rangle\rangle = \langle\langle \mathbb{A}_-, \delta\mathbb{A}_+ \rangle\rangle}$$

- The boundary conditions we've seen so far correspond to taking

$$\boxed{\mathbb{A}_\pm \in \mathfrak{l}}$$

where $\mathfrak{l}$ is a **Lagrangian subalgebra** of $\mathfrak{d}$.

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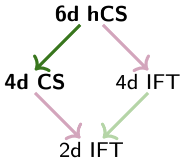
Symmetry reducing 6d hCS to 4d CS

- The action of 6d hCS is

$$\boxed{\frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge \left(\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \frac{2}{3}\,\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right)}$$

  - Euclidean twistor space $\mathbb{PT}$ is diffeomorphic to $\mathbb{R}^4 \times \mathbb{CP}^1$,
  - $(x^{AA'}, \pi_{A'})$ are coordinates on $\mathbb{R}^4 \times \mathbb{CP}^1$.
  - A and A' are left- and right-handed spinor indices: $A \in \{1,2\}$, $A' \in \{1,2\}$,
  - Ω is a specified meromorphic 3-form,
  - $\bar{\mathcal{A}}$ is a $\mathfrak{g}$-valued anti-holomorphic 1-form on $\mathbb{PT}$.
- To define the theory, we specify Ω and boundary conditions, which ensure that the boundary variation vanishes.

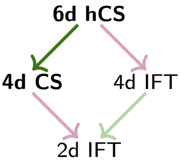


Symmetry reducing 6d hCS to 4d CS

- The relevant meromorphic 3-form is

$$\boxed{\Omega \sim \frac{1}{2}\,\frac{\langle \pi d\pi\rangle \wedge dx^{AA'} \wedge dx^{BB'}\epsilon_{AB}\pi_{A'}\pi_{B'}}{\langle \pi\alpha\rangle\langle \pi\tilde{\alpha}\rangle\langle \pi\beta\rangle^2}}$$

- We have a double pole at $\pi_{A'} = \beta_{A'}$ and two simple poles $\pi_{A'} = \alpha_{A'}$ and $\pi_{A'} = \tilde{\alpha}_{A'}$.
- We introduce bases for left- and right-handed spinors, $\{\kappa_1^A, \kappa_2^A\}$ and $\{\gamma_{1A'}, \gamma_{2A'}\}$:
  - the boundary conditions are specified in terms of $\{\kappa_1^A, \kappa_2^A, q\}$,
  - the symmetry reduction is specified in terms of $\{\gamma_{1A'}, \gamma_{2A'}\}$.
- The parameters of the 6d hCS theory are $\{k, \alpha_{A'}, \tilde{\alpha}_{A'}, \beta_{A'}, \kappa_1^A, \kappa_2^A, q\}$, up to redundancies.
- The symmetry reduction introduces the additional parameters $\{\gamma_{1A'}, \gamma_{2A'}\}$.



Symmetry reducing 6d hCS to 4d CS

- The relevant meromorphic 3-form is

$$\boxed{\Omega \sim \frac{1}{2}\,\frac{\langle \pi d\pi\rangle \wedge dx^{AA'} \wedge dx^{BB'} \epsilon_{AB}\pi_{A'}\pi_{B'}}{\langle \pi\alpha\rangle\langle \pi\tilde{\alpha}\rangle\langle \pi\beta\rangle^2}}$$

- After symmetry reduction the 4d CS meromorphic 1-form is

$$\boxed{\begin{aligned}\omega \sim (V_1 \wedge V_2) \lrcorner\, \Omega &\sim \frac{\langle \pi d\pi\rangle\langle \pi\gamma_1\rangle\langle \pi\gamma_2\rangle}{\langle \pi\alpha\rangle\langle \pi\tilde{\alpha}\rangle\langle \pi\beta\rangle^2} \\ &\sim \frac{(\zeta-\gamma_1)(\zeta-\gamma_2)}{(\zeta-\alpha)(\zeta-\tilde{\alpha})}\, d\zeta\end{aligned}}$$

where $\zeta = \pi_2/\pi_1$, $\gamma_{iA'} = (1,\gamma_i)_{A'}$, $\beta_{A'} = (0,1)_{A'}$, $\alpha_{A'} = (1,\alpha)_{A'}$ and $\tilde{\alpha}_{A'} = (1,\tilde{\alpha})_{A'}$.

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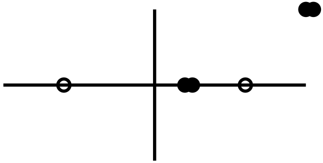
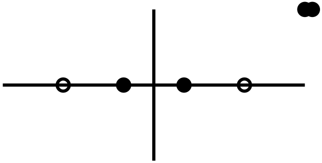
Boundary conditions in 4d CS

PCM plus WZ term

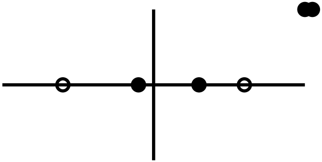


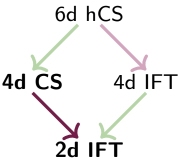
Current-current deformation



- What happens if we introduce the additional parameter in 4d CS and take

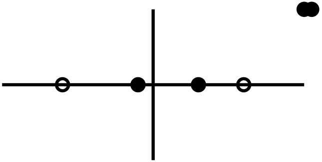
$$\boxed{\omega \sim \frac{\zeta^2 - 1}{(\zeta - \alpha)(\zeta - \tilde{\alpha})}\, d\zeta}$$



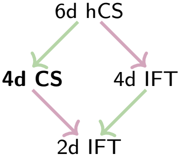


Boundary conditions in 4d CS

- The current-current deformation boundary conditions cannot be further deformed with $\mathbb{A}_{\pm}$ remaining in a Lagrangian subalgebra.
- Therefore, if we are to complete the diamond, we are prompted to consider more general boundary conditions.



- These boundary conditions can be found by symmetry reducing the 6d hCS boundary conditions.
- In the end, we find the four-parameter 2-field model introduced earlier, with the current-current deformation of the G_k WZW CFT as a special point.
- This is in contrast to the PCM plus WZ case, in which we always end up with 1-field model.
- Introducing a non-vanishing residue at infinity corresponds to having a non-trivial IR fixed point.



Boundary conditions in 4d CS

$$\boxed{\langle\!\langle \mathbb{A}_+, \delta \mathbb{A}_- \rangle\!\rangle = \langle\!\langle \mathbb{A}_-, \delta \mathbb{A}_+ \rangle\!\rangle}$$

- **Standard boundary conditions**

⋆ $\mathbb{A}_\pm \in \mathfrak{l}$ where $\mathfrak{l}$ is a Lagrangian subalgebra:
- the boundary conditions preserve gauge symmetry,
- 2d IFT depends on a single field $g \in G$,
- 4d CS gives a classically integrable model by construction,

- **Generalised boundary conditions** [cf. Bassi, Lacroix]

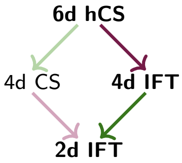
⋆ $\mathbb{A}_\pm \in \mathfrak{l}_\pm$ where $\mathfrak{l}_\pm$ are subspaces of $\mathfrak{d}$ such that $\langle\!\langle \mathfrak{l}_+, \mathfrak{l}_- \rangle\!\rangle = 0$:
- the boundary conditions do not preserve gauge symmetry,
- 2d IFT depends on two fields $g \in G$ and $\tilde{g} \in G$,
- 4d CS gives a Lax connection, but it's not clear if it's enough for classical integrability.

$$\begin{tikzcd}
 & \text{6d hCS} \arrow[dl] \arrow[dr] & \\
\textbf{4d CS} \arrow[dr] & & \text{4d IFT} \arrow[dl] \\
 & \text{2d IFT} &
\end{tikzcd}$$

In the other direction

- First localising from 6d hCS to a 4d IFT, and then symmetry reducing to a 2d IFT works without any further issues.
- The e.o.m. of the 4d IFT can be recast, as a consequence of the construction, as the 4d anti-self-dual Yang-Mills equations.
- This is a new case of Ward's conjecture:

> **"... many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalisations) by reduction."**



The 4d IFT

- The explicit action of the 4d IFT is

$$\boxed{\begin{aligned}\mathcal{S} = -\frac{k}{8\pi}\int_{\mathbb{R}^4} \mathrm{vol}_4\, \mathrm{tr}\,\Big(j_1\,\frac{1+\Lambda^t}{1-\Lambda^t}\,j_2 + \tilde{j}_1\,\frac{1+\Lambda^t}{1-\Lambda^t}\,\tilde{j}_2 \\ - 2q^{-1} j_1\,\frac{1}{1-\Lambda^{-t}}\,\tilde{j}_2 + 2q\tilde{j}_1\,\frac{1}{1-\Lambda^t}\,j_2\Big) \\ -\frac{k}{8\pi}\int_{\mathbb{R}^4\times[0,1]} \mathrm{vol}_4\wedge \mathrm{d}\xi\,\mathrm{tr}\,\big(j_\xi[j_1,j_2] + \tilde{j}_\xi[\tilde{j}_1,\tilde{j}_2]\big)\end{aligned}}$$

where

$$j_i = \kappa_i^A \alpha^{A'} g^{-1}\partial_{AA'} g \qquad \tilde{j}_i = \kappa_i^A \tilde{\alpha}^{A'}\partial_{AA'}\tilde{g}\tilde{g}^{-1}$$

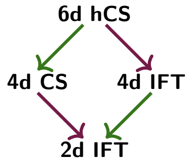
$$\Lambda = q\,\mathrm{Ad}_{\tilde{g}}\,\mathrm{Ad}_g$$

- Generically has a 2d G gauge symmetry. Whether this survives symmetry reduction or not depends on choice of $\gamma_{iA'}$.
- What is this model, what are its properties and what are its limits?

$$\begin{tikzcd} & \text{6d hCS} \arrow[dl] \arrow[dr] & \\ \text{4d CS} \arrow[dr] & & \textbf{4d IFT} \arrow[dl] \\ & \text{2d IFT} & \end{tikzcd}$$

Summary

- The diamond has been constructed for: [Costello]
  - the PCM + WZ term, [Bittleston, Skinner]
  - the current-current deformation of the G_k WZW CFT.
    [Cole, Cullinan, BH, Liniado, Thompson]
- Leads to generalised boundary conditions in 4d CS.
- Equations of motion of deformed integrable sigma models found as the symmetry reduction of 4d anti-self-dual Yang-Mills equations.



Future directions

- Can we generalise further?
  - What happens if Ω has zeroes?
  - What happens if $\mathbb{CP}^1$ is replaced by a more general Riemann surface?

- Can we find new 2d or 4d IFTs from the generalised boundary conditions?

- Can we gain new insights into Ward's conjecture?

$$\begin{tikzcd}
& \textbf{6d hCS} \arrow[dl] \arrow[dr] & \\
\textbf{4d CS} \arrow[dr] & & \textbf{4d IFT} \arrow[dl] \\
& \textbf{2d IFT} &
\end{tikzcd}$$

Thank you!