Correlated spin systems: a tensor network approach

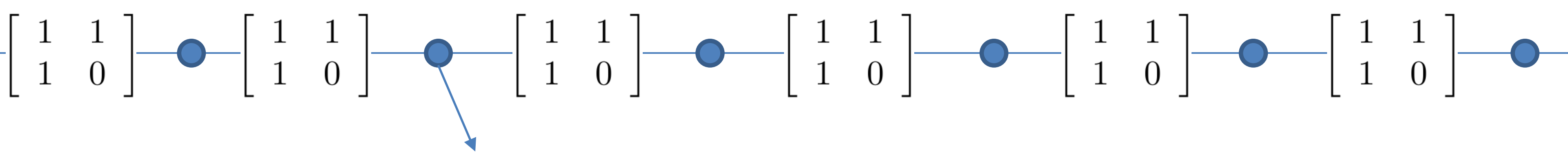
Frank Verstraete

Overview

- Spin systems as tensor networks
  - Partition functions
  - Variational Matrix Product State methods
  - Finite entanglement scaling
- Dualities & (Categorical) symmetries
- Topological phases of matter

Partition functions as counting problems: hard square constants

- 1-dimension:
  - count number of strings of bits such that a 1 is surrounded by 0's:
    - E.g. 0001001010000100
  - Transfer matrix approach: evaluate following tensor network



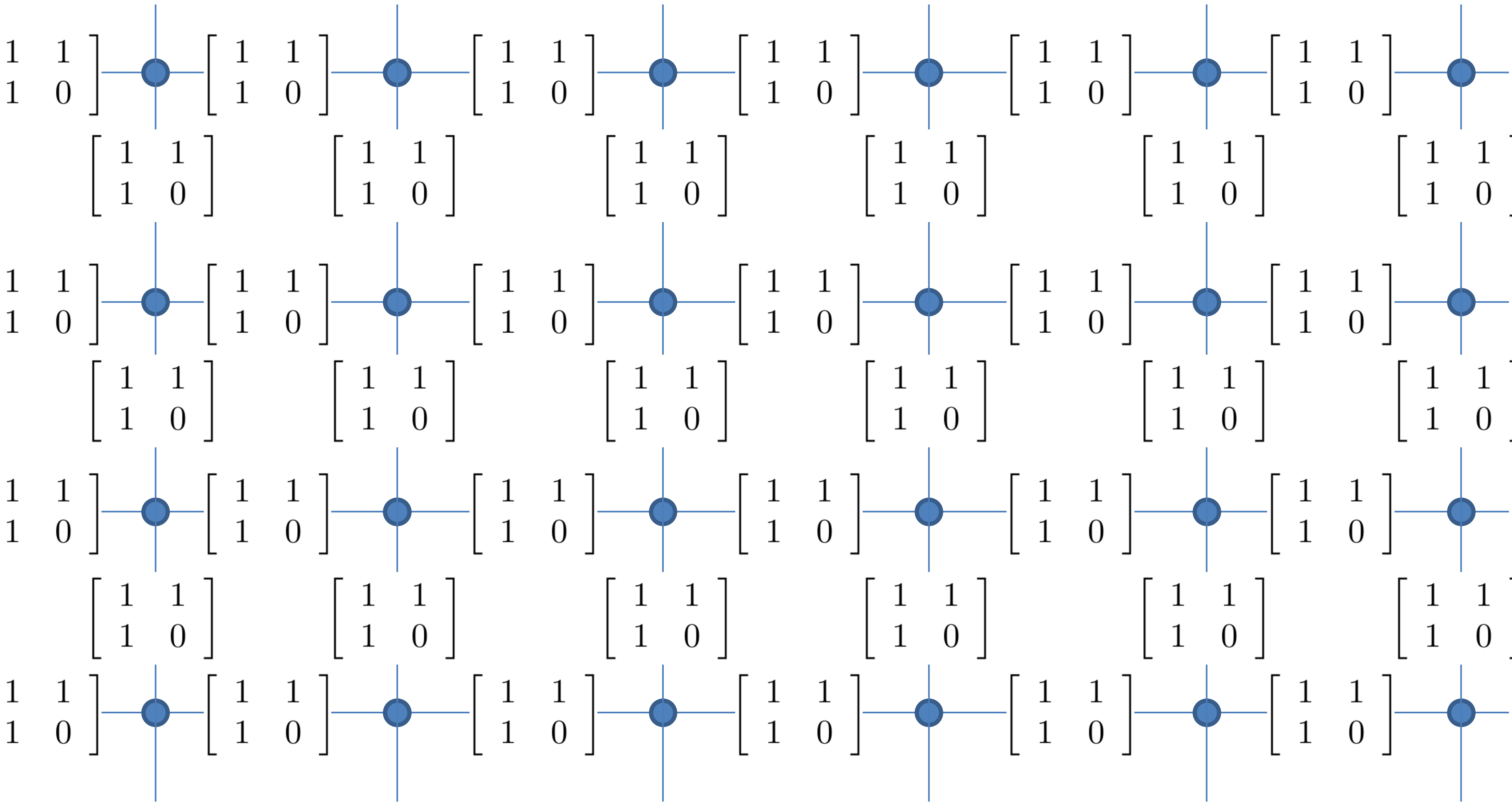
Identity matrix: sum over 0,1

  - Number of configurations is: $\mathrm{Tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^L\right) = \left(\frac{1+\sqrt{5}}{2}\right)^L + \left(\frac{1-\sqrt{5}}{2}\right)^L$

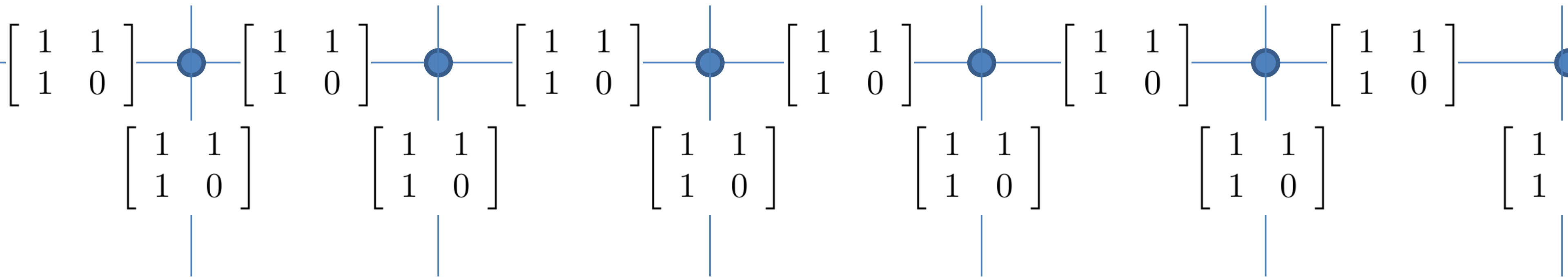
Hard square cst.

- 2 dimensions:

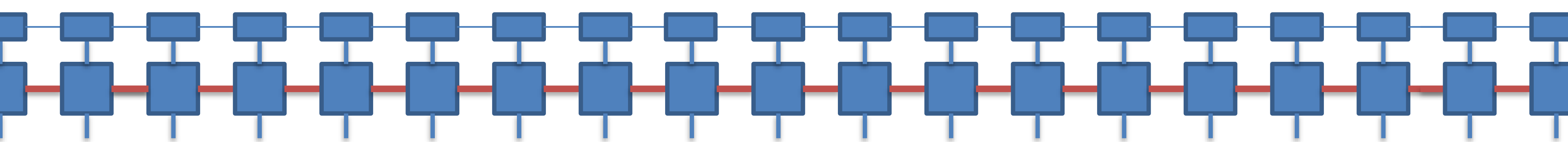
 $$= |0000\rangle + |1111\rangle$$



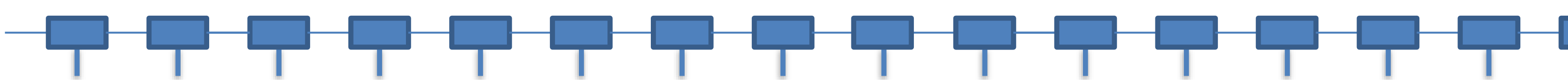
- Problem is equivalent to finding leading eigenvalue of transfer matrix / Matrix Product Operator:



- Is essentially equivalent problem to finding ground state of a local quantum Hamiltonian, for which DMRG / variational MPS methods provide the state of the art: MPS with bond dimension D

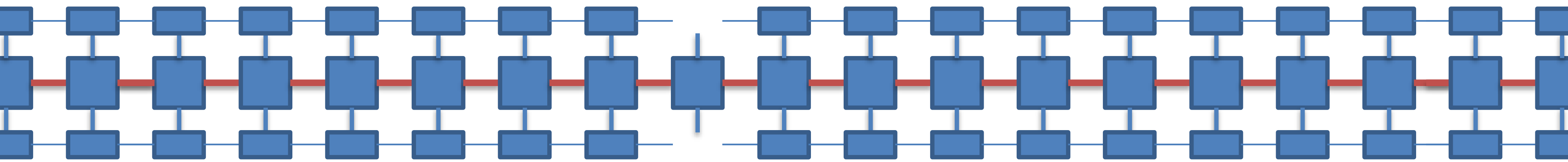


$$= \lambda^N .$$

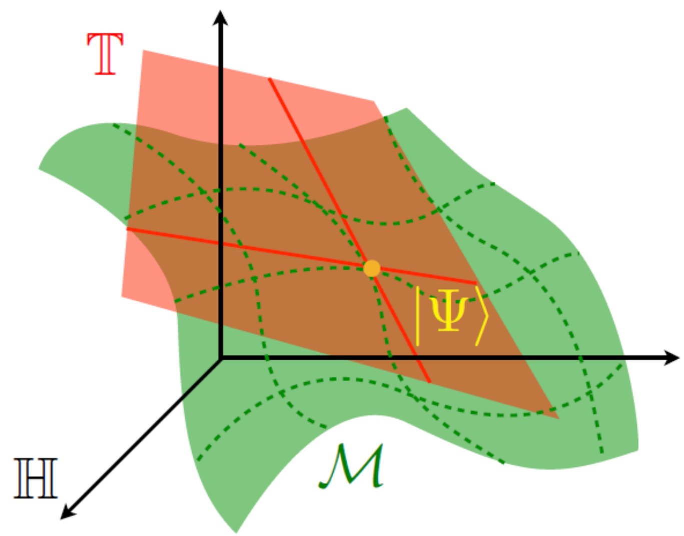


Variational uniform matrix product state algorithm

- Make use of left/right canonical forms to reduce optimization to a sequence of effective eigenvalue problems:



- Essence: enforce that residual of MPO applied to MPS is orthogonal to tangent space of MPS manifold; this leads to a Lanczos-type version of corner transfer matrix (CTM) of Baxter; optimization gives direct access to the free energy and hence of entropy of the stat. mech. Model without need of integration such as in MC



Haegeman, FV, arXiv:1611.08519
Zauner-Stauber et al. ‘18

2	1.5030477
4	1.50304808246
6	1.5030480824753218
8	1.503048082475332642
10	1.5030480824753322643220 58
20	1.503048082475332264322066329475 54
30	1.503048082475332264322066329475553689377
Baxter[128]	1.5030480824753322643220663294755536893857 81
40	1.503048082475332264322066329475553689385781 02
50	1.503048082475332264322066329475553689385781038609
60	1.503048082475332264322066329475553689385781038610303
70	1.503048082475332264322066329475553689385781038610305061
80	1.503048082475332264322066329475553689385781038610305062026556

Table 1: Free energy of the hard squares model; with bond dimension $D = 80$, we get 58 digits of precision.

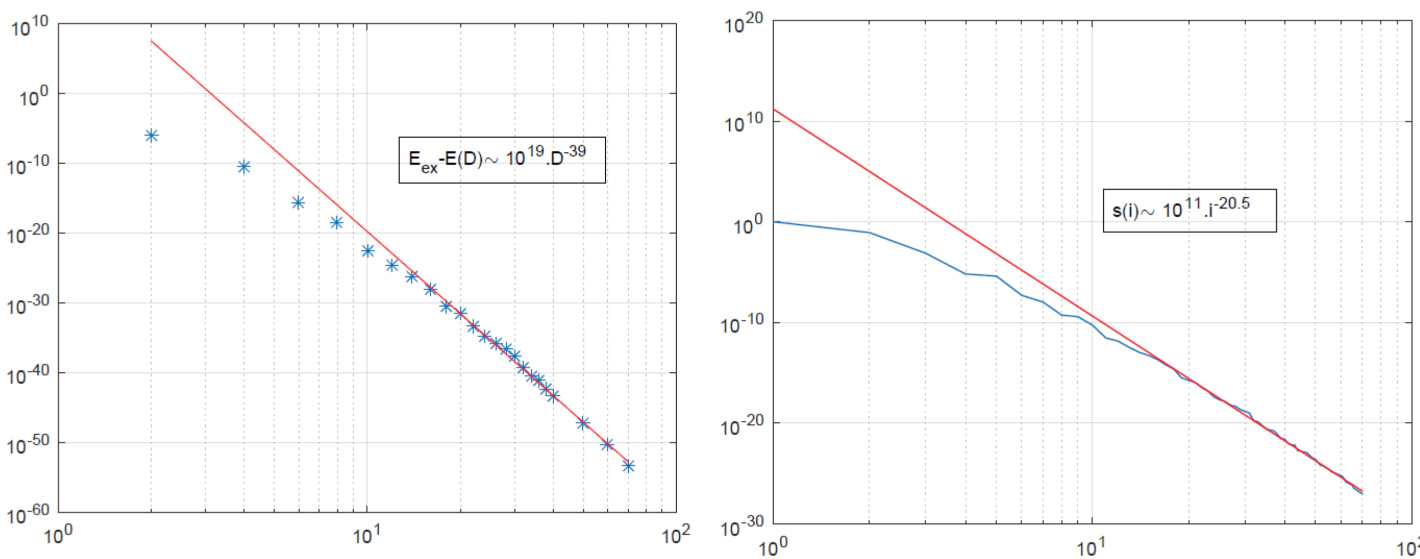
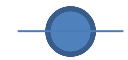


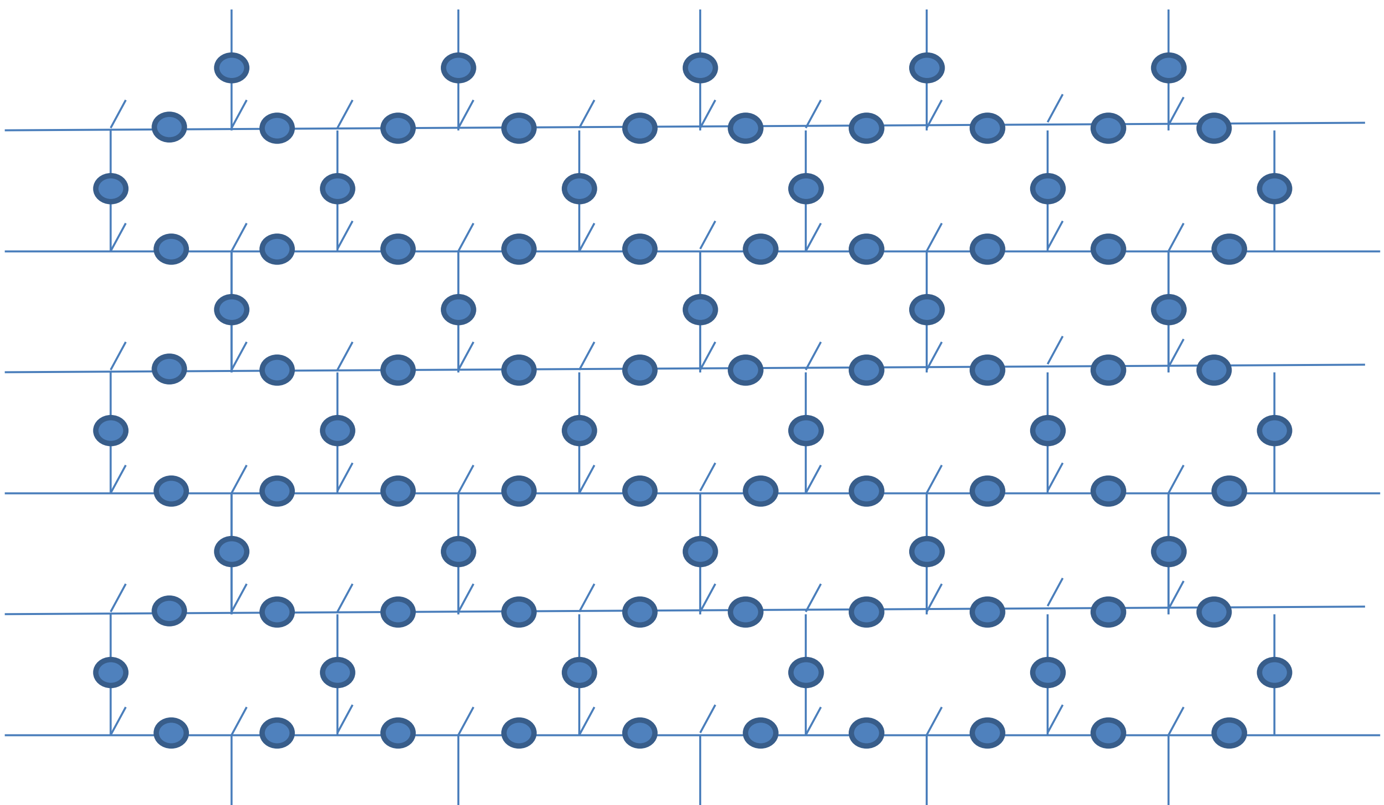
Figure 22: (a) Convergence of free energy versus bond dimension for the hard squares model and (b) decay of Schmidt coefficients for $D = 70$.

Ising model as tensor network

$$= |0000\rangle + |1111\rangle$$



$$= \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$$



Scaling hypothesis for MPS

- When simulating a critical point, a simultaneous scaling in the distance to the critical point and in the bond dimension can be formulated

$$\delta = \sum_{i=1}^{n} c_i \epsilon_i,$$

Rams, Czarnik & Cincio '18

$$\delta^{-\beta/\nu} m(t,\delta) = m(\delta^{-1/\nu} t, 1) = \tilde{m}(\delta^{-1/\nu} t)$$

$$\delta\xi(t,\delta) = \xi(\delta^{-1/\nu} t, 1) = \tilde{\xi}(\delta^{-1/\nu} t)$$

$$\exp\left(\frac{6}{c} S(t,\delta)\right) = s \exp\left(\frac{6}{c} S(s^{-1/\nu} t, s\delta)\right)$$

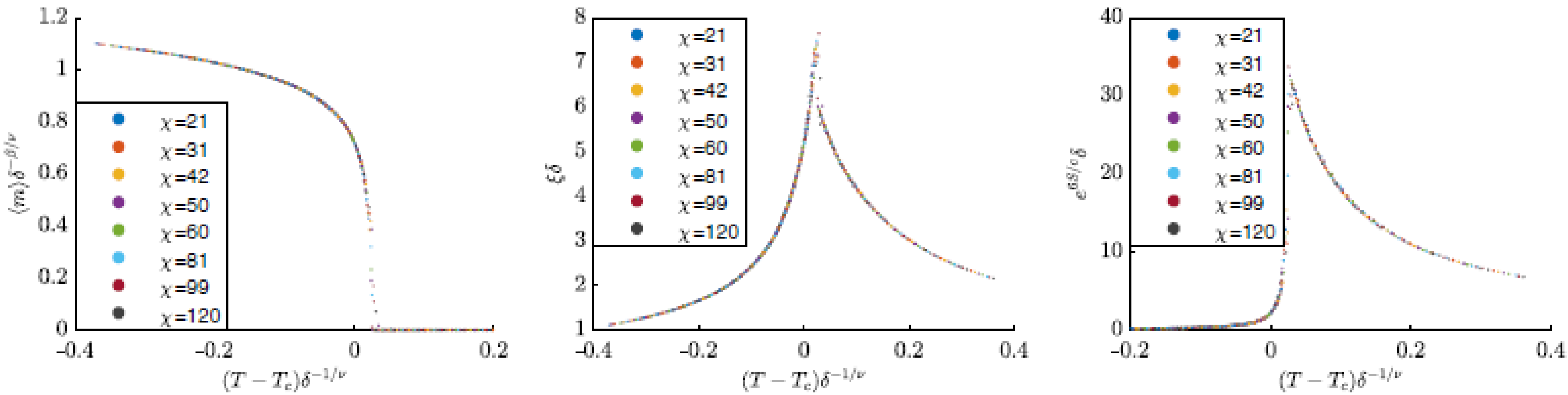


FIG. 2. Collapse plots for the Potts model, calculated with MPS of bond dimension 21,31,42,50,60,81,99, and 120, for 96 different temperatures linearly spaced between $T = 0.9939$ and $T = 0.9954$. Left, magnetization; middle, correlation length; right, bipartite entanglement entropy.

Vanhecke, Haegeman, Van Acoleyen, Vanderstraeten &FV '19

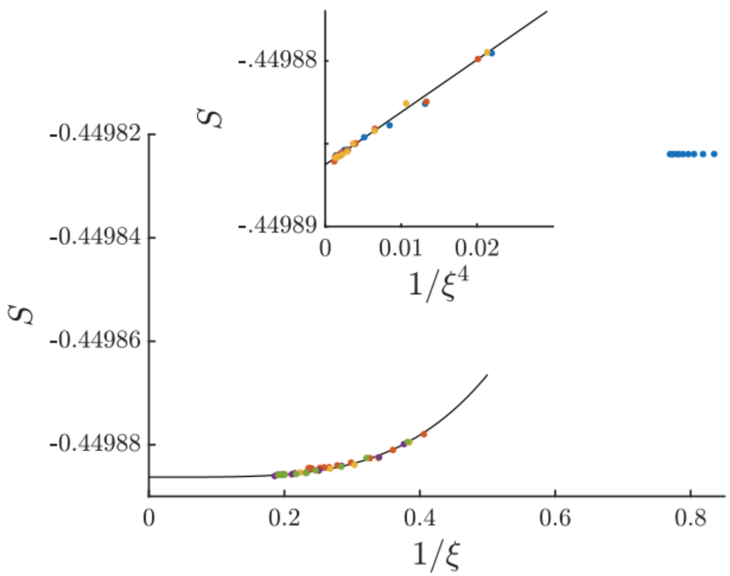


FIG. 1. Residual entropy per site for the dimer covering problem on the cubic lattice. We have optimized PEPS tensors for $D = (2,3,4,5,6)$ and $\chi = (20,26,34,44,58,75,97,126,164,213,276)$; the same MPS bond dimension was used for the contraction of the double- and triple-layer (see supplementary material). The correlation length is extracted from the double-layer boundary MPS. A fit (black) on the $D > 3$ data reveals a clear x^4 power law (see inset) for which the origin is to us an enigma.

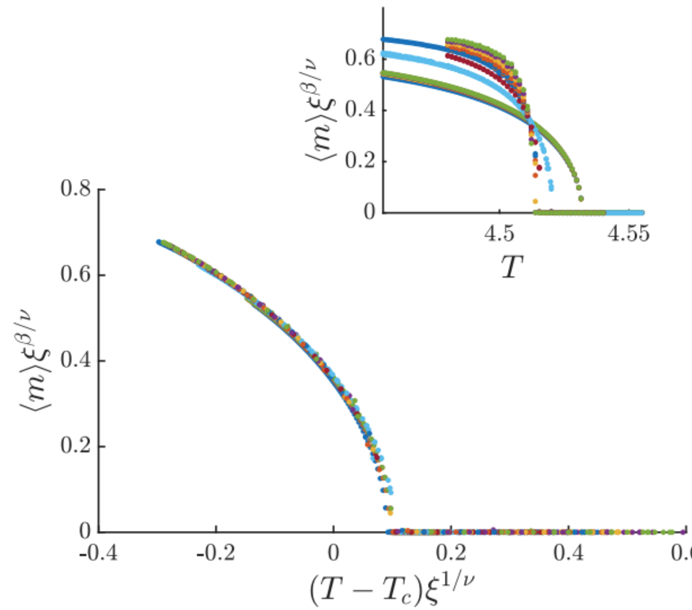


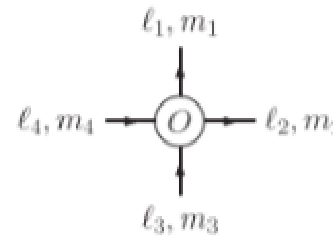
FIG. 2. Magnetization of the classical 3-D Ising model. We have optimized PEPS tensors for $D = (2,3,4)$ and $\chi = (10,15,20,25,30,40,50)$ for 100 equally spaced temperatures around the critical temperature. The same MPS bond dimension was used for the contraction of the double- and triple-layer (see supplementary material). To plot the data, we shift the temperature by $T_c = 4.511528$ (obtained from Monte-Carlo simulations[29]) and rescaled both axes with appropriate powers $\beta = 0.326419$ and $\nu = 0.629971$ of ξ (obtained from the conformal bootstrap method[30]), with ξ the correlation length of the double-layer MPS environment. In the inset we provide a partial data collapse (also showing the abundance of data points), showing the crossing of the rescaled magnetization at the critical temperature.

Vanhecke, Vanderstraeten, FV, PRL ‘22

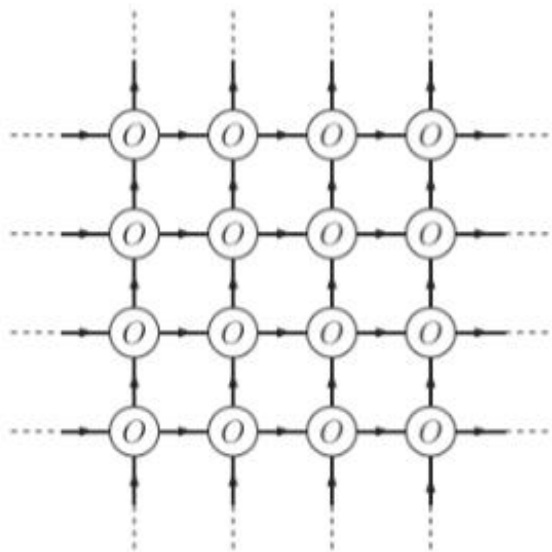
Heisenberg model / RP^2

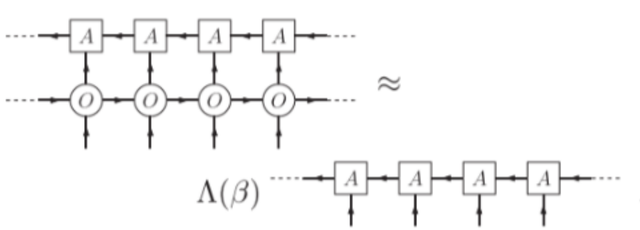
$$\mathcal{Z} = \left(\prod_i \int \frac{d\Omega_i}{4\pi} \right) \left(\prod_{\langle ij \rangle} e^{\beta (\vec{s}_i \cdot \vec{s}_j)^p} \right)$$

$$e^{\beta (\vec{s}_i \cdot \vec{s}_j)^p} = \sum_\ell f_\ell(\beta) \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta_i, \phi_i) Y_{\ell m}(\theta_j, \phi_j).$$



$\mathcal{Z} =$





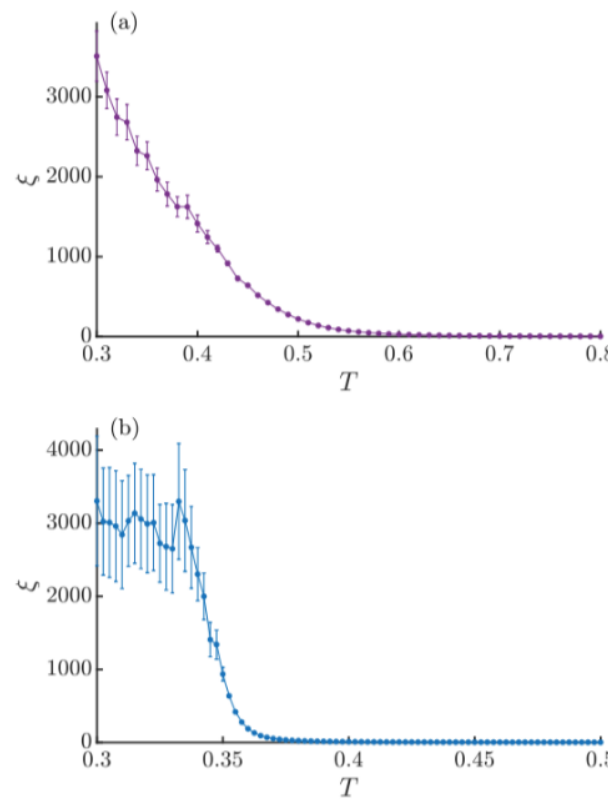


FIG. 4. Extrapolated correlation length as a function of temperature for the (a) Heisenberg and (b) RP^2 models.

What about scaling of MPS for field theories?

gapped spin systems :	$\Lambda_{UV} = 1, \Lambda_{IR} < m$	$\longrightarrow$ Exact
critical spin systems:	$\Lambda_{UV} = 1, \Lambda_{IR} \to 0$	
gapped QFTs:	$\Lambda_{UV} \to \infty, \Lambda_{IR} < m$	$\longrightarrow$ Extrapolation, scaling theory
critical QFTs:	$\Lambda_{UV} \to \infty, \Lambda_{IR} \to 0$	

- Let us look at $\lambda\phi^4$ to see how the two scales manifest themselves in the entanglement degrees of freedom

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu_p^2\phi^2 + \frac{1}{4}\lambda_p\phi^4.$$

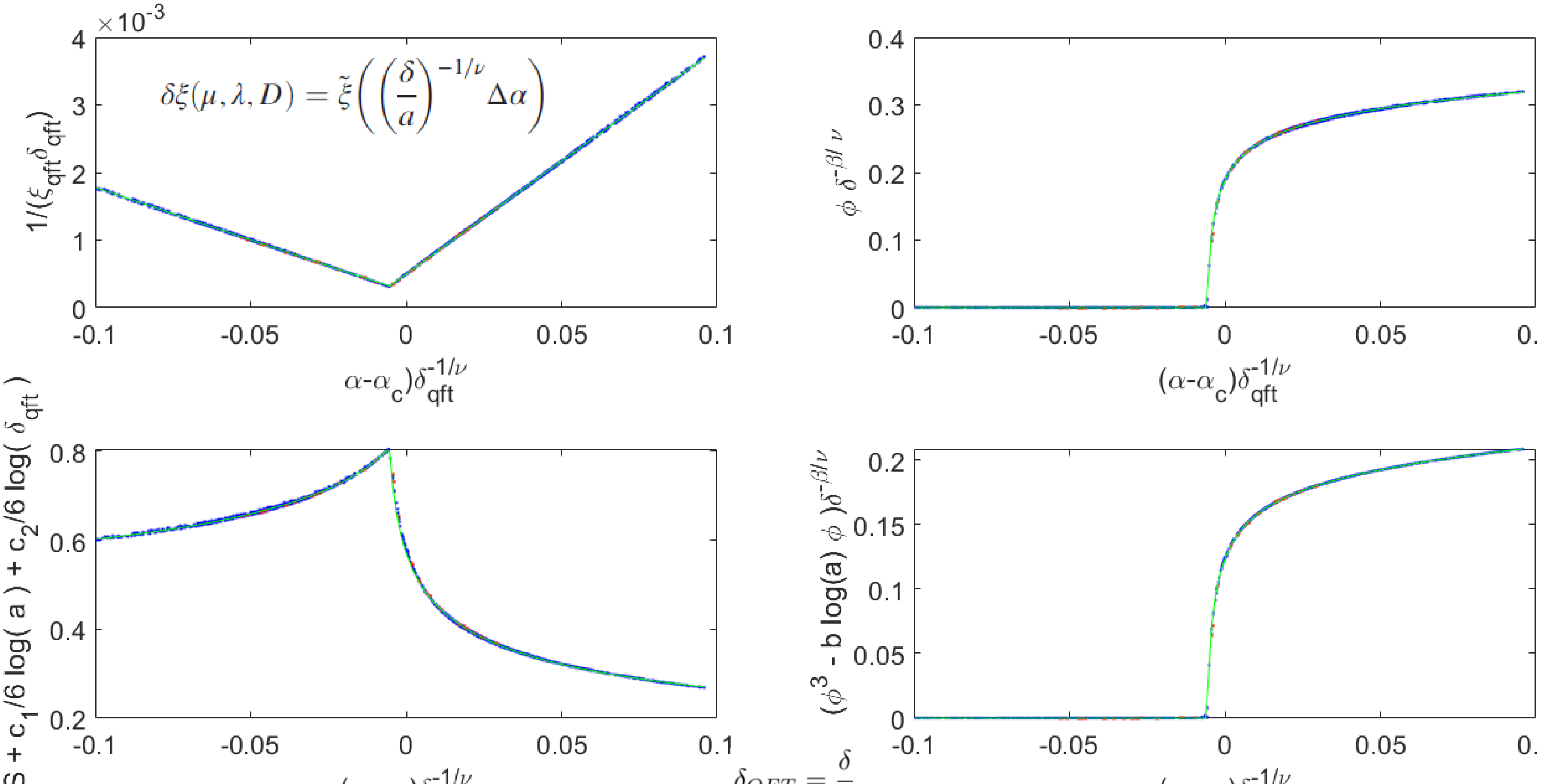
  – Double scaling regime: entanglement scaling + continuum (lattice parameter) should lead to both a c=1 contribution from UV AND a c=1/2 contribution from IR

Vanhecke et al.'19

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu_p^2\phi^2 + \frac{1}{4}\lambda_p\phi^4. \qquad g = \lambda_p/\mu_{Rp}^2$$

$$\alpha = g + \lambda.\left(c_1 \log\lambda + c_2 + c_3\lambda\log\lambda\right)$$

$$a^2 = \lambda.(1 + C(\alpha - \alpha_c)) + \lambda\left(d_1\lambda\log\lambda + d_2\lambda\log^2\lambda\right)$$



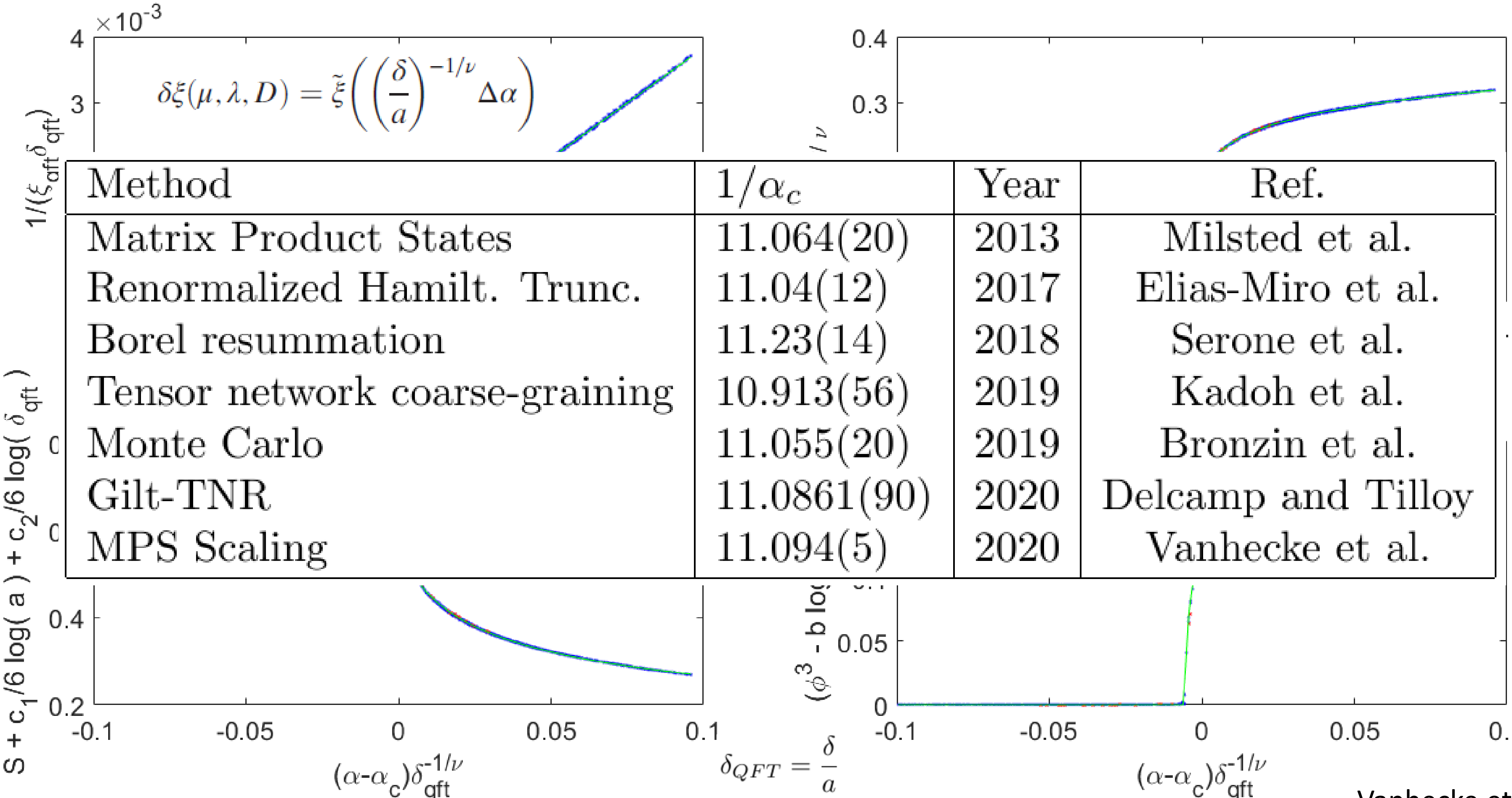
$\delta_{QFT} = \frac{\delta}{a}$

Vanhecke et al.'20

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu_p^2\phi^2 + \frac{1}{4}\lambda_p\phi^4. \qquad g = \lambda_p/\mu_{Rp}^2$$

$$\alpha = g + \lambda.\left(c_1 \log\lambda + c_2 + c_3\lambda\log\lambda\right)$$

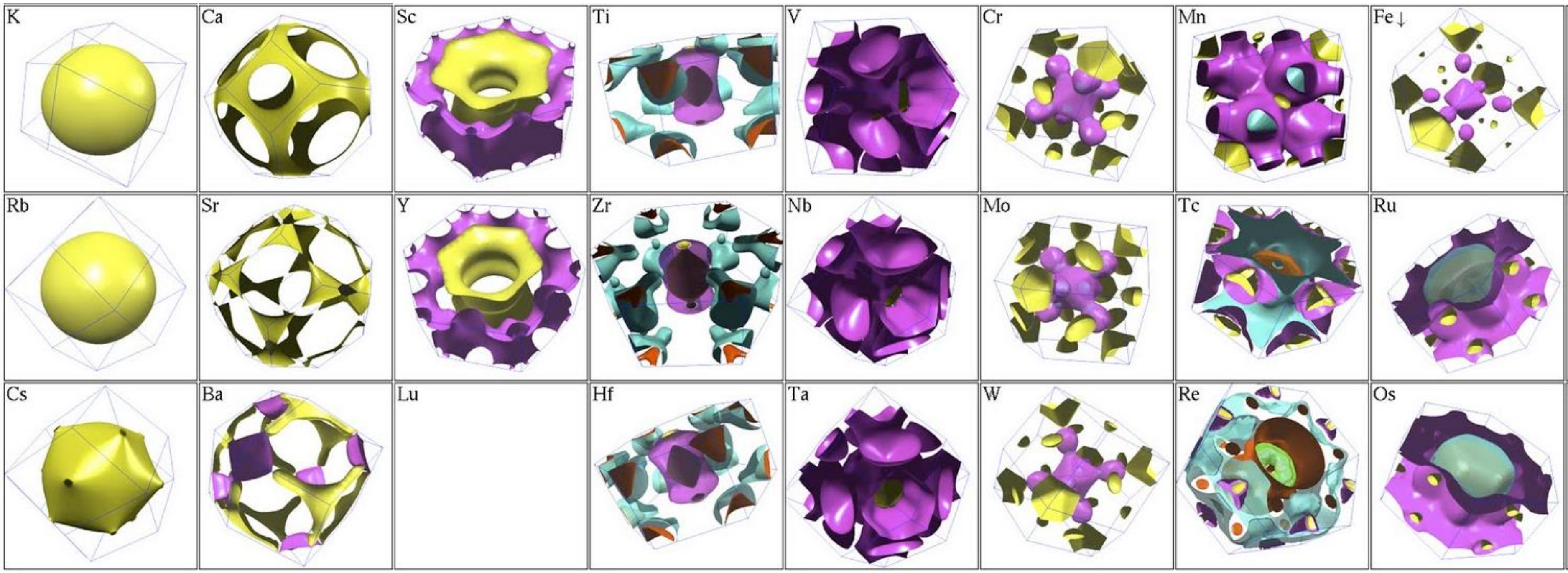
$$a^2 = \lambda.(1 + C(\alpha - \alpha_c)) + \lambda\left(d_1\lambda\log\lambda + d_2\lambda\log^2\lambda\right)$$



Method	$1/\alpha_c$	Year	Ref.
Matrix Product States	11.064(20)	2013	Milsted et al.
Renormalized Hamilt. Trunc.	11.04(12)	2017	Elias-Miro et al.
Borel resummation	11.23(14)	2018	Serone et al.
Tensor network coarse-graining	10.913(56)	2019	Kadoh et al.
Monte Carlo	11.055(20)	2019	Bronzin et al.
Gilt-TNR	11.0861(90)	2020	Delcamp and Tilloy
MPS Scaling	11.094(5)	2020	Vanhecke et al.

Vanhecke et al.'20

What about systems with a Fermi surface?



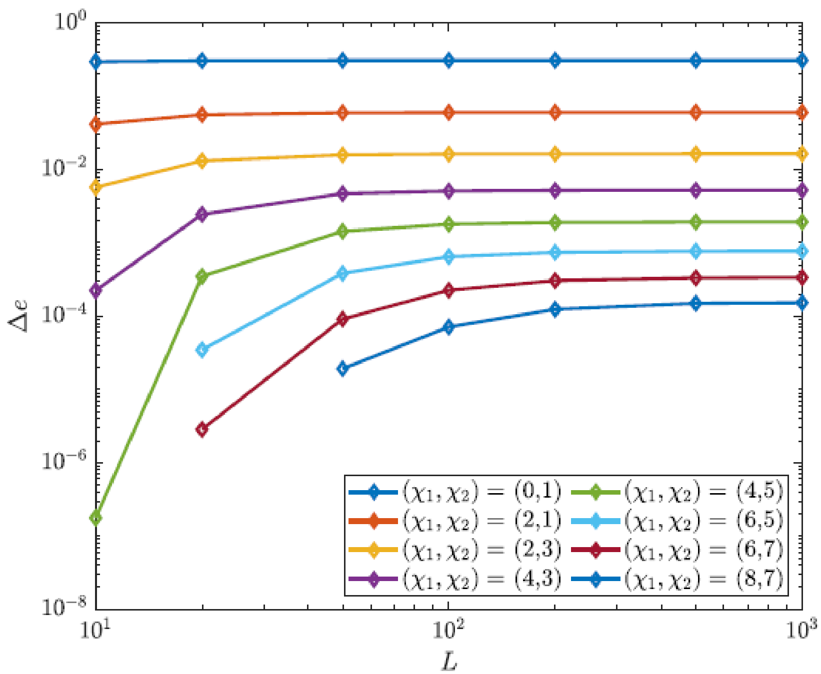
http://www.phys.ufl.edu/fermisurface/

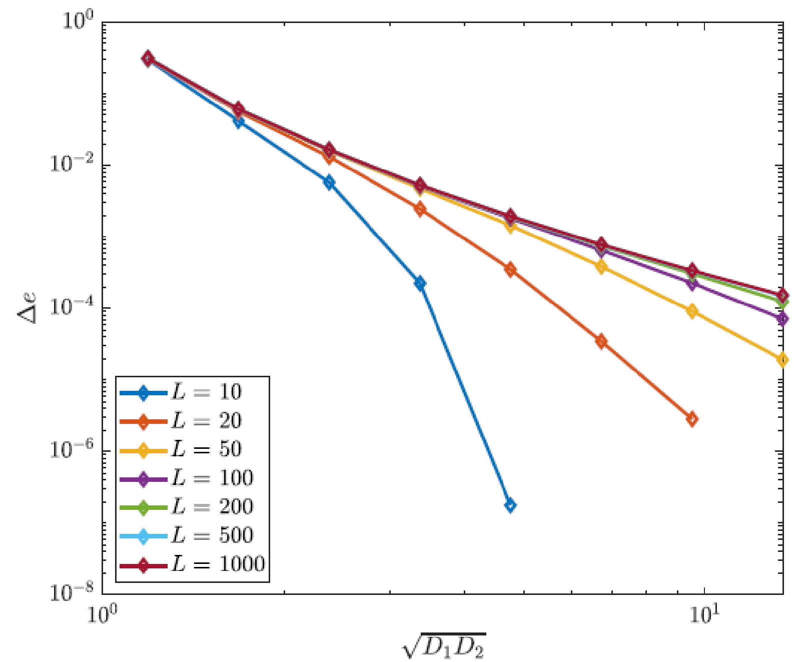
- Violation of area law:

$$S \sim \frac{L^{d-1}\log L}{(2\pi)^{d-1}} \frac{1}{12} \int_{\partial\Omega} \int_{\partial\Gamma} |n_x \cdot n_p| dS_x \mathrm{d}S_p$$

PEPS and Fermi surfaces

$$H_t = -\sum_{\mathbf{n}} \left(t_x \, a_{\mathbf{n}}^\dagger a_{\mathbf{n}_\rightarrow} + t_y \, a_{\mathbf{n}}^\dagger a_{\mathbf{n}_\uparrow} + h.c. \right)$$

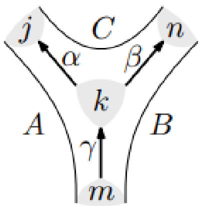




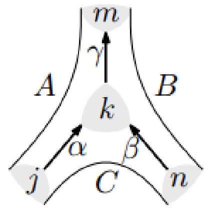
- We conclude: favourable scaling of bond dimension as a function of precision at halve filling => just as in 1D, Fermi surfaces can be captured by using a scaling ansatz for tensor networks
  - Important caveat: it has to be possible to open a gap using a perturbation

Mortier, Schuch, FV, Haegeman ‘22

A critical lattice model for a Haagerup conformal field theory



$$:= \frac{\left({}^{3}F_{B}^{A\alpha\beta}\right)_{C,jn}^{\gamma,km}}{\sqrt{d_C}},$$



$$:= \frac{\left({}_{3}F_{B}^{A\alpha\beta}\right)_{C,jn}^{\gamma,km}}{\sqrt{d_C}},$$

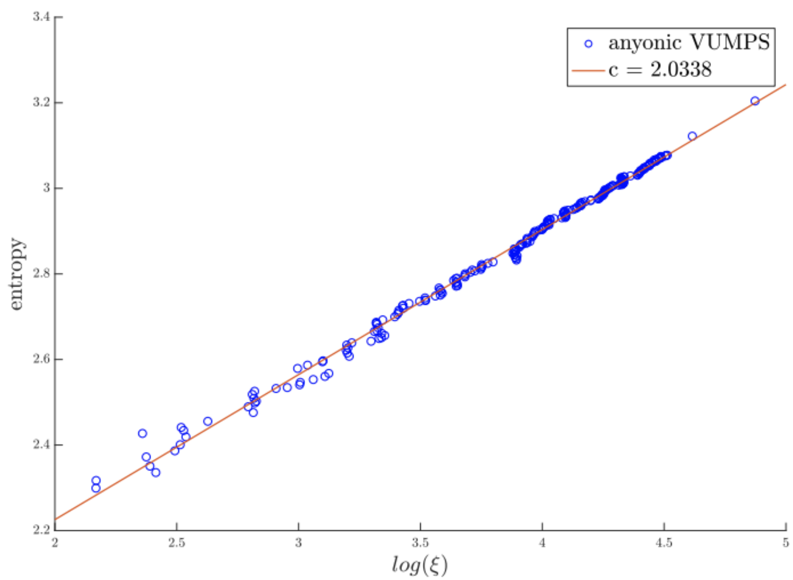


Figure 2. Finite entanglement scaling for the fixed point MPS of the transfer matrix calculated using VUMPS with explicit $\mathcal{H}_3$ anyonic symmetry. The results are consistent with a central charge close to $c = 2$.

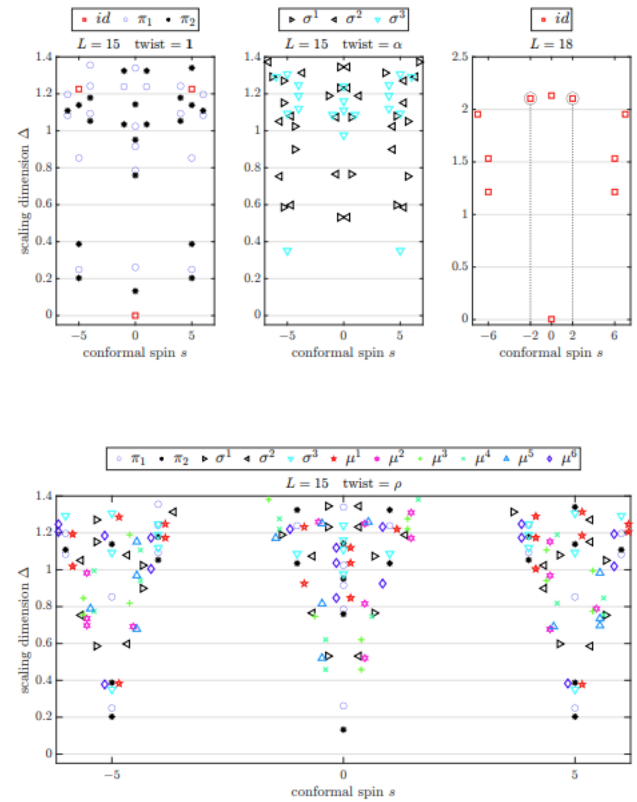
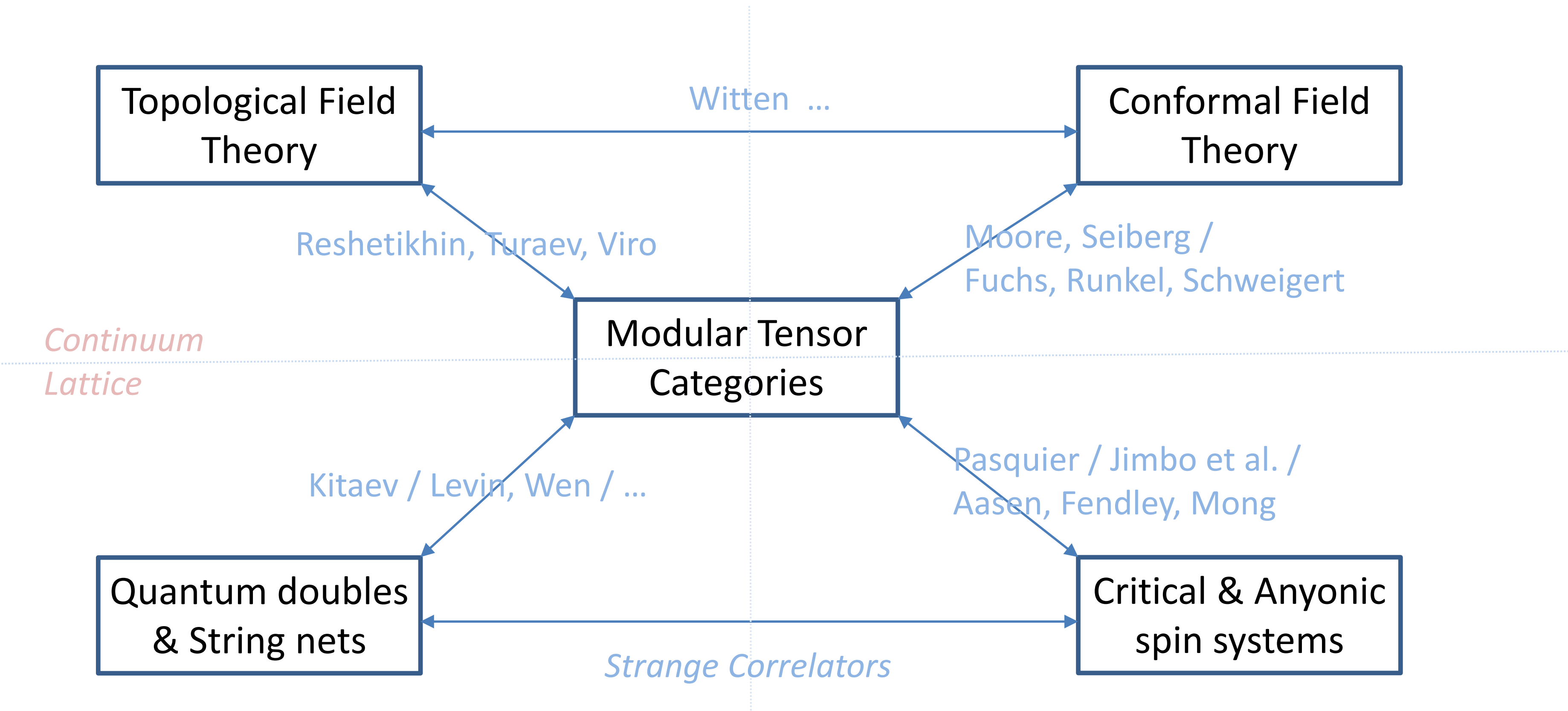


Figure 4. Spectra for the transfer matrix, twisted with topological defects $\mathbf{1}$ (upper left), α (upper middle) and ρ (bottom), numerically obtained with anyonic symmetry-preserving exact diagonalization on $L = 15$ sites. The eigenvalues are labeled by their corresponding topological sectors $Z(\mathcal{H}_3)$ according to Table I. For the non-trivial twists, the conformal spins are only integers up to a topological spin correction (Table II). Upper right: the identity sector on $L = 18$ sites. The first excited states of the vacuum ($\Delta = 2, s = -2, 2$) are circled in black.

Vanhove et al., PRL '22

Part II: Symmetries & Dualities

- Tensor network is nothing but a discrete path integral
- Crucial aspect are the symmetries (local / nonlocal / higher form) and associated anomalies, …
- Recent years: a systematic study of all different representations of a given tensor network / partition function : bimodule categories
  - Unified treatment of TFTs and CFTs



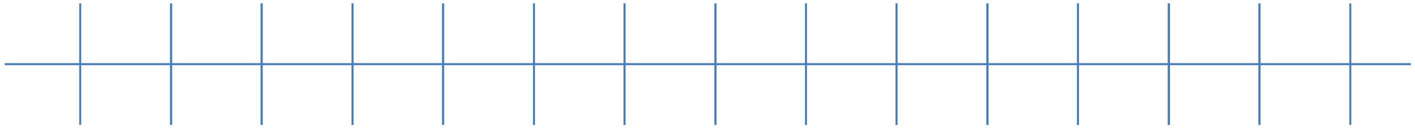
Dualities & (Categorical) symmetries

- Critical tensor networks exhibit new symmetries:
  - Kramers-Wannier duality becomes a symmetry

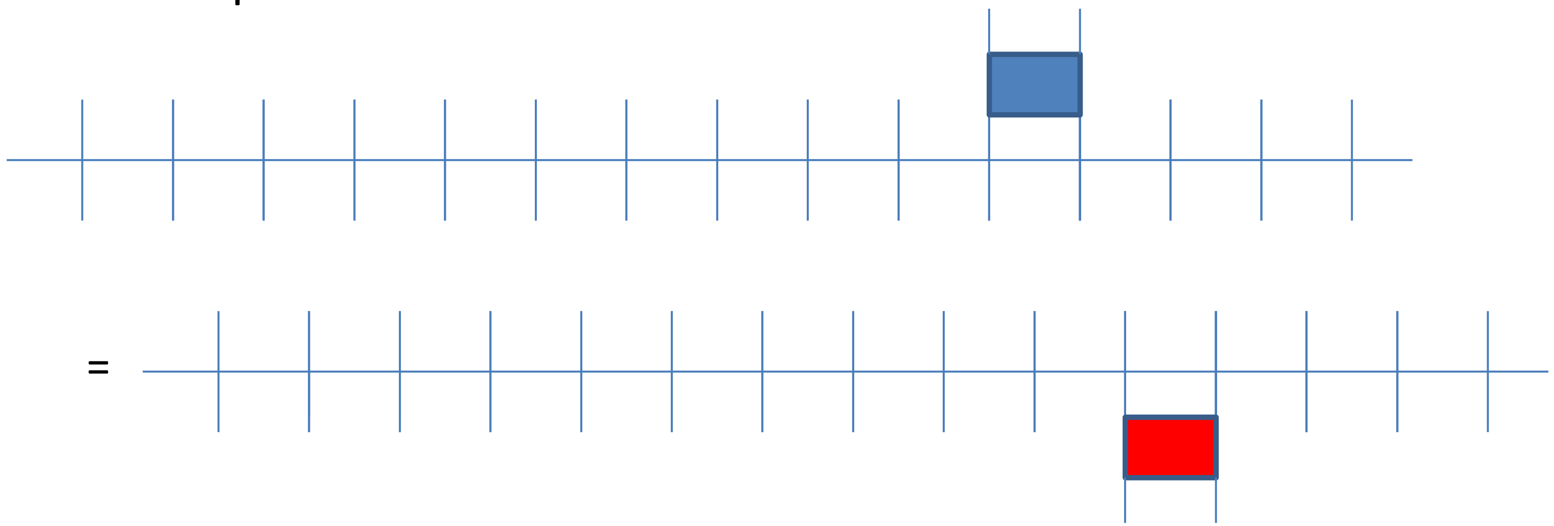
$$\mathcal{H}_1 = J\sum_i X_i X_{i+1} + K\sum_i Z_i$$

$$\mathcal{H}_2 = J\sum_i X_i + K\sum_i Z_i Z_{i+1}$$

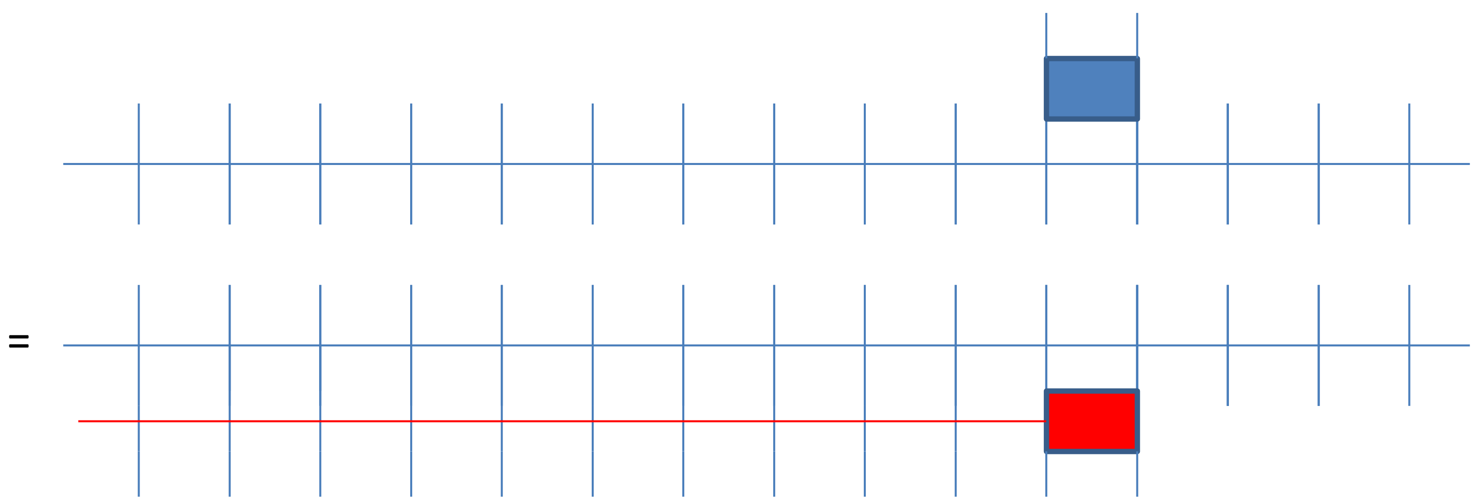
- Intertwiner as Matrix product operator:



- Symmetric operators remain local

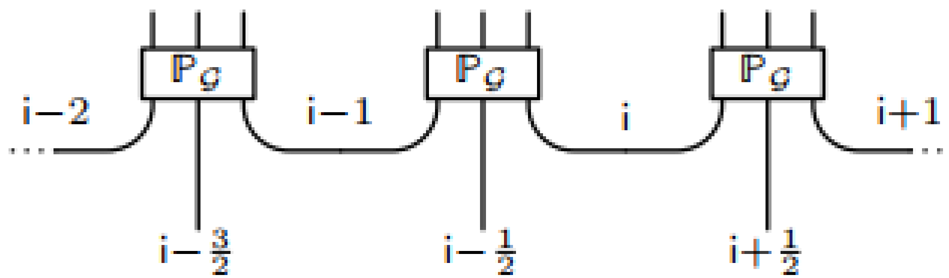


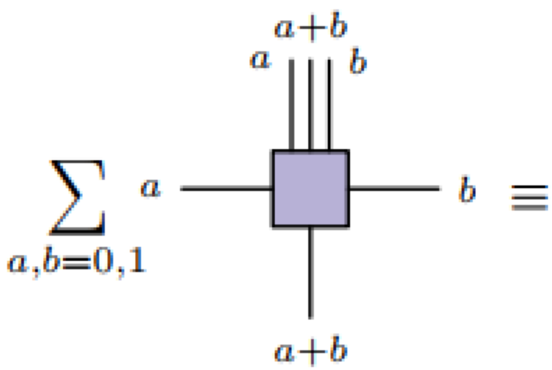
- Non-symmetric operators become nonlocal (string)



Kramers-Wannier duality of Ising model

- KW is obtained by “gauging” the Z_2 symmetry and then disentangling matter fields:

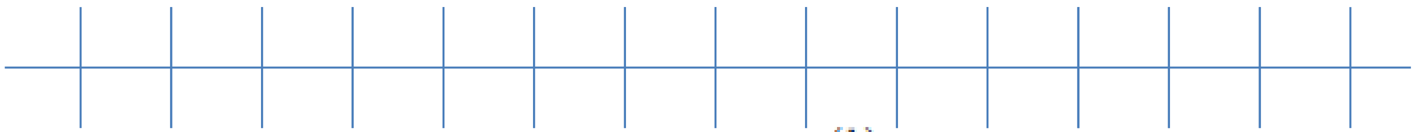




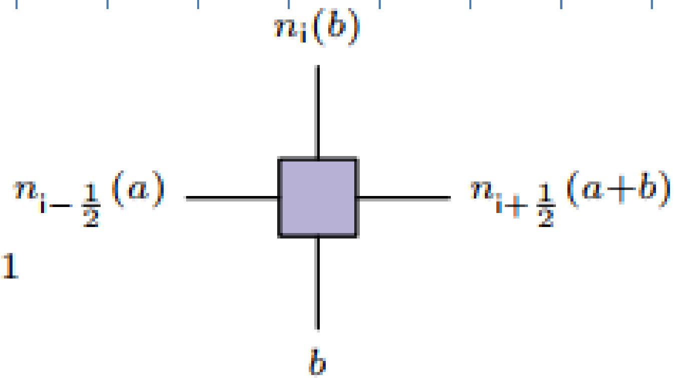
$$\equiv \sum_{a,b=0,1} |a\rangle |a, a+b, b\rangle \langle b| \langle a+b|$$

Lootens et al., PRXQ ‘22

Jordan-Wigner transform as an MPO intertwiner



- Local tensor: $\sum_{a,b=0,1}$



$$\equiv \sum_{a,b=0,1} |n_{\mathrm{i}-\frac{1}{2}}(a)\rangle |n_{\mathrm{i}}(b)\rangle \langle n_{\mathrm{i}+\frac{1}{2}}(a+b)| \langle b|$$

- Transformed Hamiltonian:

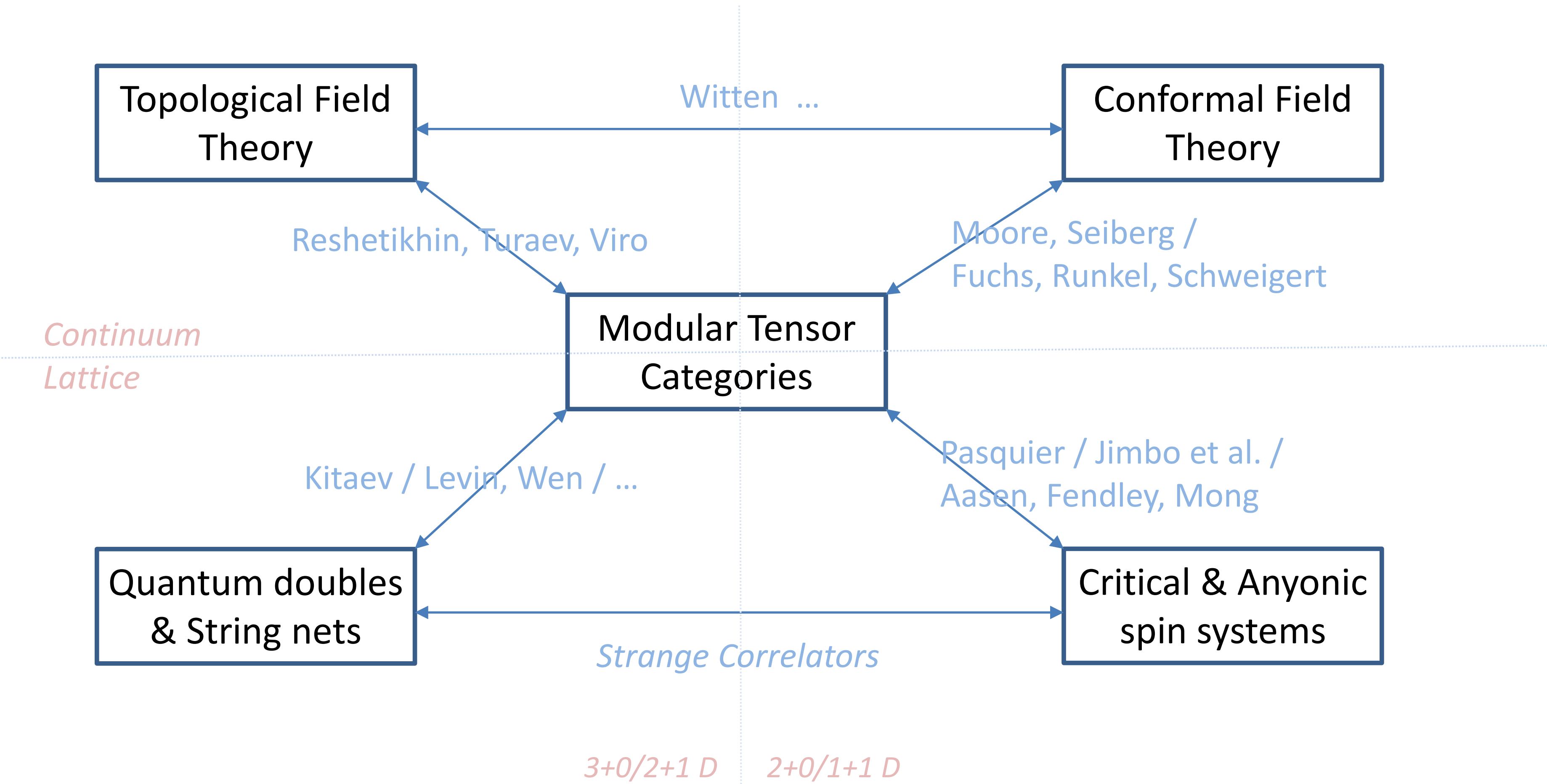
$$\mathbb{H}_C = -J \sum_{\mathrm{i}} \Big(c^\dagger_{\mathrm{i}-\frac{1}{2}} c_{\mathrm{i}+\frac{1}{2}} + c^\dagger_{\mathrm{i}-\frac{1}{2}} c^\dagger_{\mathrm{i}+\frac{1}{2}} + \mathrm{h.c.} - g\big(2c^\dagger_{\mathrm{i}+\frac{1}{2}} c_{\mathrm{i}+\frac{1}{2}} - 1\big)\Big),$$

- JW is a genuine duality!

Lootens et al., PRXQ ‘22

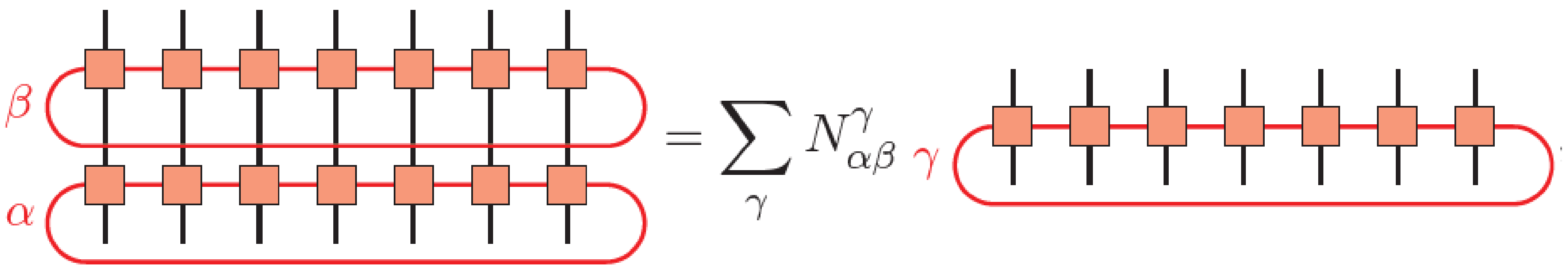
Critical systems and categorical symmetries

- Can we construct tensor networks which automatically exhibit categorical symmetries?



MPO symmetries

- Central ideas:
  - Symmetries are represented by scale-invariant (possibly correlated) operators O_α (e.g. $\hat{O}_g = U_g^{\otimes N}$)
  - Those operators form a closed algebra (as representation of that symmetry), independent of the system size
  - For 1D systems: such symmetries are guaranteed to be Matrix Product Operator (MPO) symmetries, but do not necessarily form a group (could be a fusion algebra); in 2D: PEPO algebras



  - Such MPOs generalize the representation theory of groups
    - Development of character theory, irreps, ... [Bridgeman, Lootens, FV '22]

Fuchs-Runkel-Schweigert machinery for bimodule categories

- Input: Moore-Seiberg data concerning the representations of a chiral algebra in the form of a modular tensor category D (simple objects and fusion rules):

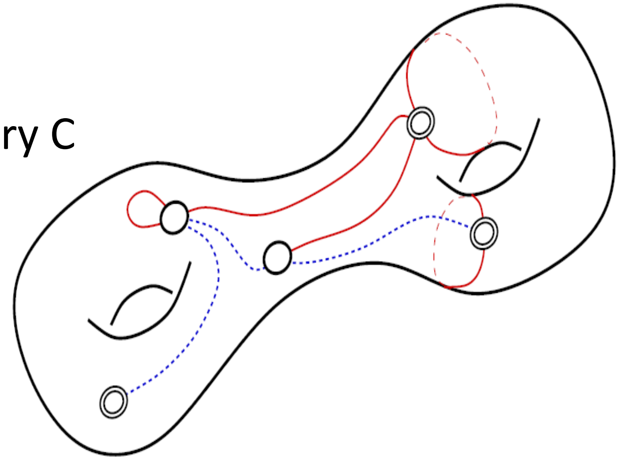
$$\alpha \otimes \beta \cong \bigoplus_{\gamma \in \mathcal{D}} N_{\alpha\beta}^{\gamma}\, \gamma$$

  – Many different CFTs correspond to the same D; we need an additional piece of data in the form of a right module category M to specify CFT:

$$A \triangleleft \alpha \cong \bigoplus_{B \in \mathcal{M}} N_{A\alpha}^{B}\, B;$$

    - This M allows to define a CFT on any closed surface such that it remains invariant under mapping class group

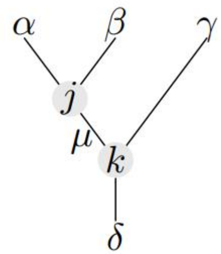
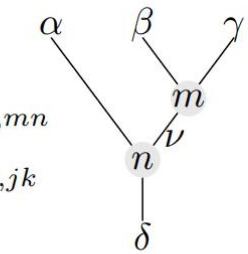
  – M turns out to be a left module of a different category C which is *"equivalent"* to D, and C defines the topological defect lines in the CFT

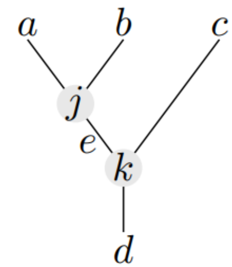
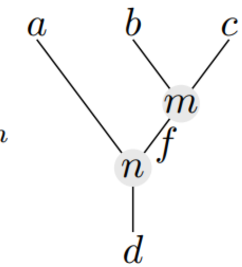


- Summary: CFT determined by a D,M,C such that M is a (C,D) is an invertible bimodule category

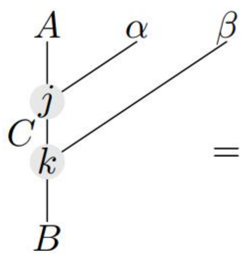
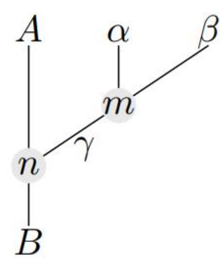
F-symbols

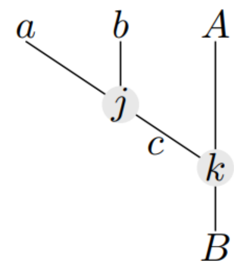
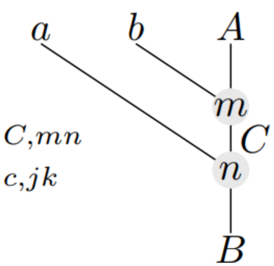
- Fusion categories C and D are endowed with functors: $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$
  - Associativity of fusion rules implies the existence of F-symbols:


$= \sum_{\mu,mn} \left({}^4F_\delta^{\alpha\beta\gamma}\right)_{\mu,jk}^{\nu,mn}$


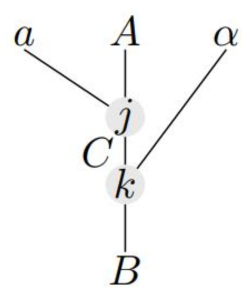
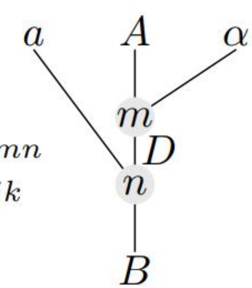

$= \sum_{f,mn} \left({}^0F_d^{abc}\right)_{e,jk}^{f,mn}$


- Similar: functors $\triangleleft : \mathcal{M} \times \mathcal{D} \to \mathcal{M}$ and $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$


$= \sum_{\gamma,mn} \left({}^3F_B^{A\alpha\beta}\right)_{C,jk}^{\gamma,mn}$



$= \sum_{C,mn} \left({}^1F_B^{abA}\right)_{c,jk}^{C,mn}$


- Compatibility of C and D:


$= \sum_{D,mn} \left({}^2F_B^{aA\alpha}\right)_{C,jk}^{D,mn}$


Big Pentagon equation

- Compatibility of those F-symbols leads to a set of 6 coupled equations:

$$\sum_o \left({}^0F^{fcd}_e\right)^{h,no}_{g,lm} \left({}^0F^{abh}_e\right)^{i,pq}_{f,ko} = \sum_{j,rst} \left({}^0F^{abc}_g\right)^{j,rs}_{f,kl} \left({}^0F^{ajd}_e\right)^{i,tq}_{g,sm} \left({}^0F^{bcd}_i\right)^{h,np}_{j,rt}$$

$((a \otimes b) \otimes c) \triangleright A \xrightarrow{\cong} a \triangleright (b \triangleright (c \triangleright A))$

$$\sum_o \left({}^1F^{fcA}_B\right)^{C,no}_{g,lm} \left({}^1F^{abC}_B\right)^{D,pq}_{f,ko} = \sum_{j,rst} \left({}^0F^{abc}_g\right)^{j,rs}_{f,kl} \left({}^1F^{ajA}_B\right)^{D,tq}_{g,sm} \left({}^1F^{bcA}_D\right)^{C,np}_{j,rt}$$

$((a \otimes b) \triangleright A) \triangleleft \alpha \xrightarrow{\cong} a \triangleright (b \triangleright (A \triangleleft \alpha))$

$$\sum_o \left({}^2F^{fA\alpha}_B\right)^{D,no}_{C,lm} \left({}^1F^{abD}_B\right)^{E,pq}_{f,ko} = \sum_{F,rst} \left({}^1F^{abA}_C\right)^{F,rs}_{f,kl} \left({}^2F^{aF\alpha}_B\right)^{E,tq}_{C,sm} \left({}^2F^{bA\alpha}_E\right)^{D,np}_{F,rt}$$

$((a \triangleright A) \triangleleft \alpha) \triangleleft \beta \xrightarrow{\cong} a \triangleright (A \triangleleft (\alpha \otimes \beta))$

$$\sum_o \left({}^3F^{C\alpha\beta}_B\right)^{\gamma,no}_{D,lm} \left({}^2F^{aA\gamma}_B\right)^{E,pq}_{C,ko} = \sum_{F,rst} \left({}^2F^{aA\alpha}_D\right)^{F,rs}_{C,kl} \left({}^2F^{aF\beta}_B\right)^{E,tq}_{D,sm} \left({}^3F^{A\alpha\beta}_E\right)^{\gamma,np}_{F,rt}$$

$((A \triangleleft \alpha) \triangleleft \beta) \triangleleft \gamma \xrightarrow{\cong} A \triangleleft (\alpha \otimes (\beta \otimes \gamma))$

$$\sum_o \left({}^3F^{C\beta\gamma}_B\right)^{\mu,no}_{D,lm} \left({}^3F^{A\alpha\mu}_B\right)^{\nu,pq}_{C,ko} = \sum_{\delta,rst} \left({}^3F^{A\alpha\beta}_D\right)^{\delta,rs}_{C,kl} \left({}^3F^{A\delta\gamma}_B\right)^{\nu,tq}_{D,sm} \left({}^4F^{\alpha\beta\gamma}_\nu\right)^{\mu,np}_{\delta,rt}$$

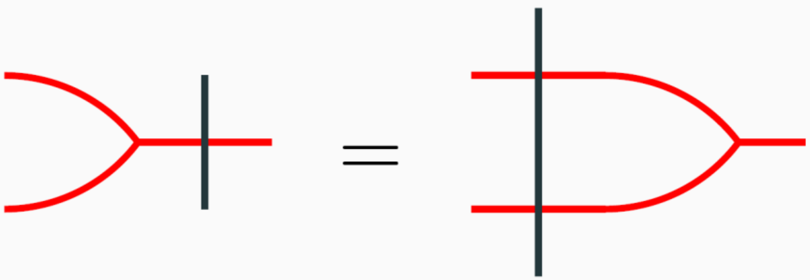
$$\sum_o \left({}^4F^{\eta\gamma\delta}_\rho\right)^{\mu,no}_{\lambda,lm} \left({}^4F^{\alpha\beta\mu}_\rho\right)^{\nu,pq}_{\eta,ko} = \sum_{\kappa,rst} \left({}^4F^{\alpha\beta\gamma}_\lambda\right)^{\kappa,rs}_{\eta,kl} \left({}^4F^{\alpha\kappa\delta}_\rho\right)^{\nu,tq}_{\lambda,sm} \left({}^4F^{\beta\gamma\delta}_\nu\right)^{\mu,np}_{\kappa,rt}$$

- Given fusion rules, (nontrivial) algorithms can be constructed to solve those equations; the fusion rules are consistent iff there is at least 1 nontrivial solution

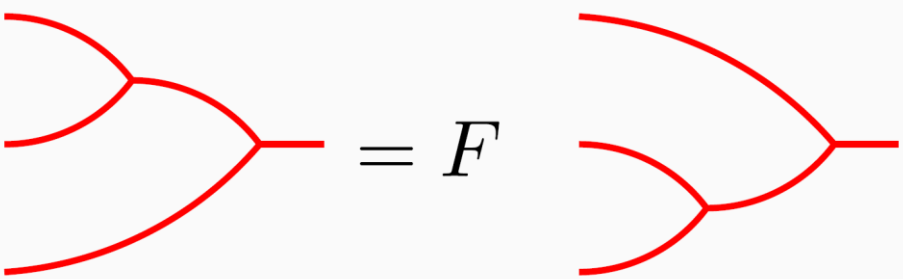
MPO symmetries : tensor formalism

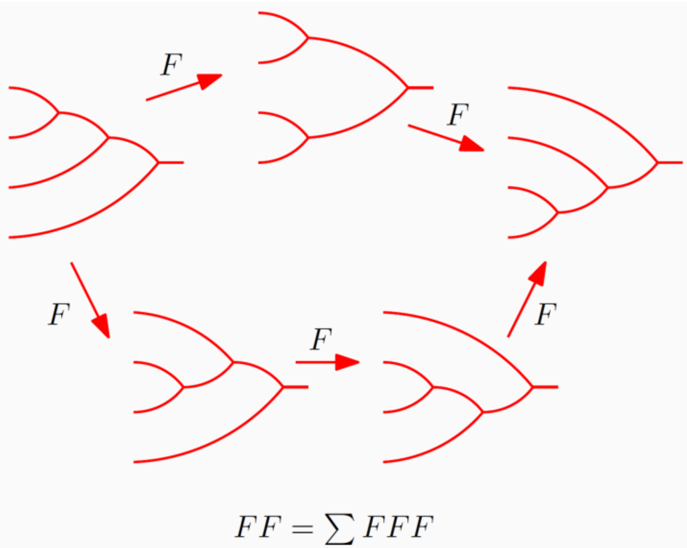
Pulling through equation

Zipper equation



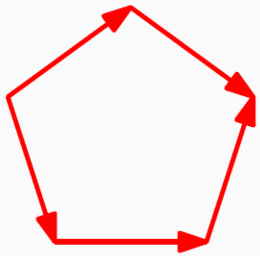
Associativity

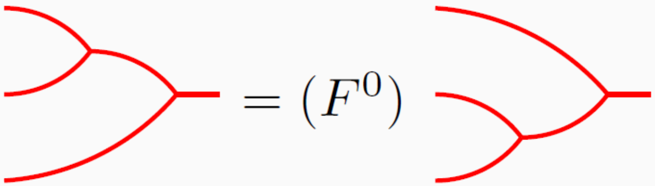




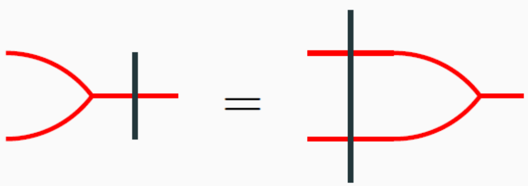
Lootens, Fuchs, Haegeman, Schweigert, FV '20

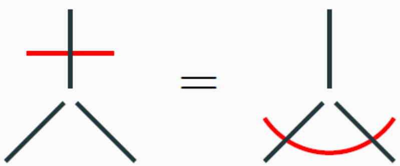
Six coupled pentagon equations: M(C,D) bimodules



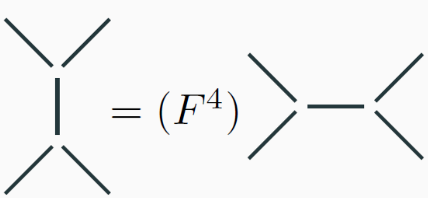


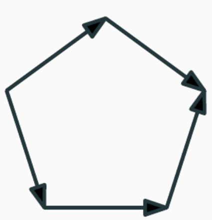
$$(F^0)(F^0) = \sum (F^0)(F^0)(F^0) \qquad (F^1)(F^1) = \sum (F^0)(F^1)(F^1)$$





$$(F^1)(F^2) = \sum (F^2)(F^2)(F^1) \qquad (F^2)(F^3) = \sum (F^3)(F^2)(F^2)$$

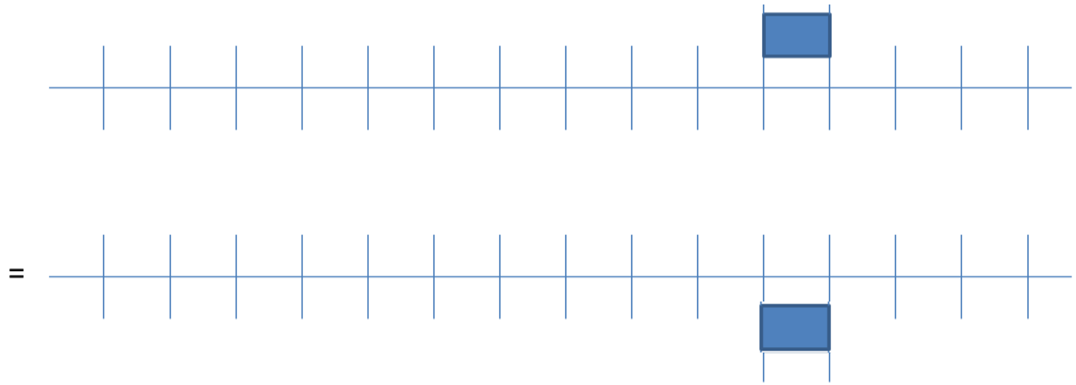




$$(F^3)(F^3) = \sum (F^4)(F^3)(F^3) \qquad (F^4)(F^4) = \sum (F^4)(F^4)(F^4)$$

Lootens, Fuchs, Haegeman, Schweigert, FV ‘20

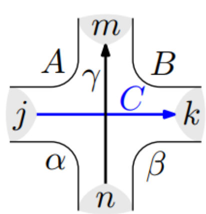
- Central premise: given a MPO symmetry, all local symmetric operators can be constructed in terms of the 3F symbols (generalized Clebsh-Gordan coefficients)



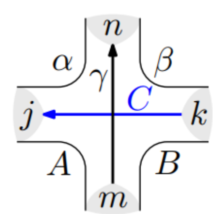
  - ***KEY INSIGHT: The structure factors of those symmetric observables are completely determined by the 4F symbols and hence the fusion category D***

- Different MPO algebras can have the same 4F symbols: those are obtained by choosing different modules M / Morita-equivalent categories C

- Bonus: we can construct explicit MPO intertwiners between such theories



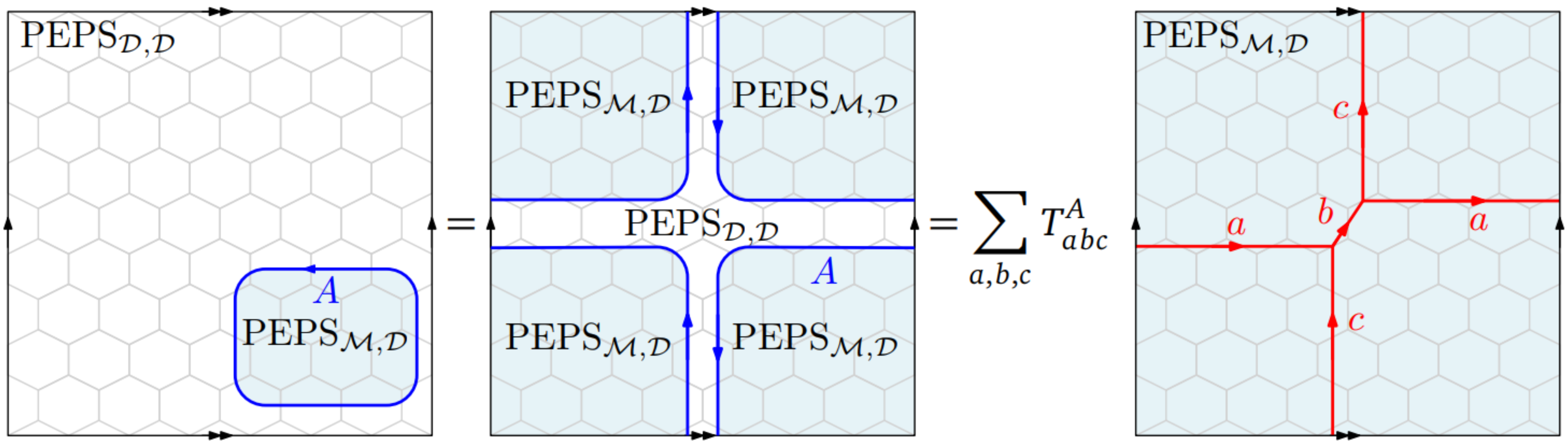
$$:= \frac{\left({}^{3}F^{C\alpha\gamma}_{B}\right)^{\beta,nk}_{A,jm}}{\sqrt{d_A d_\beta}},$$



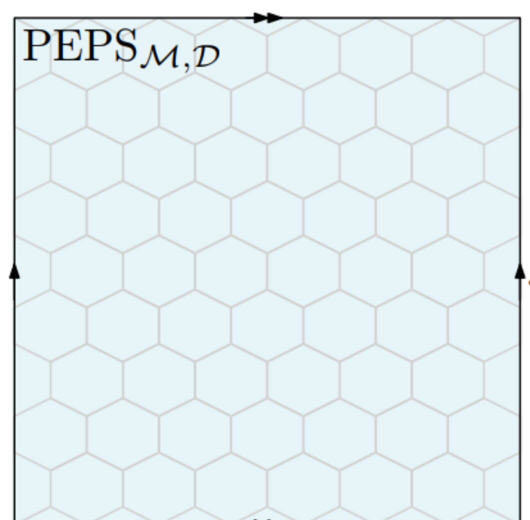
$$:= \frac{\left({}_{3}F^{C\alpha\gamma}_{B}\right)^{\beta,nk}_{A,jm}}{\sqrt{d_A d_\beta}},$$

Intertwiners and non-diagonal partition functions

- Such intertwiners can be used to construct all modular partition function of stat. mech models (e.g. vertex vs RSOS)



- Different modular invariant :



Dualities in quantum spin chains

- General scheme:
  - Given a system with a MPO symmetry C and related bimodule M(C,D) plus 4F symbol (only depending on D), all dual theories are specified by the possible invertible bimodules M'(C',D) having the same 4F
  - The Hamiltonians of the dual theories are specified in terms of the respective 3F Clebsch-Gordan coefficients, as are the intertwiners
    - Note that the symmetries of the dual theories are specified by C and C' respectively, so can be different!
    - Note also that a special case is obtained by MPOs which are just tensor products of local symmetries

- PS: the same machinery can be used to construct quantum circuits between Morita-equivalent string nets (2+1D TFT), to find all inequivalent PEPS representations of string nets, to construct modular invariant partition functions of stat mech models, …

Examples

1. $\mathcal{D} = \mathsf{Vec}_{\mathbb{Z}_2}$: $\mathbb{Z}_2$ symmetry
   - $\mathcal{M} = \mathsf{Vec}$: transverse field Ising model
   - $\mathcal{M} = \mathsf{Vec}_{\mathbb{Z}_2}$: Kramers-Wannier dual
   - $\mathcal{M} = \mathsf{sVec}/\langle\psi \simeq \mathbb{1}\rangle$: free fermion
2. $\mathcal{D} = \mathsf{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}$: $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry
   - $\mathcal{M} = \mathsf{Vec}$: spin 1 Heisenberg model, non-trivial SPT (Haldane phase)
   - $\mathcal{M} = \mathsf{Vec}^{\psi}$: Kennedy-Tasaki dual (trivial SPT), related to SPT entangler
3. $\mathcal{D} = \mathsf{Ising}$: $\mathbb{Z}_2$ symmetry $+$ Kramers-Wannier self-duality
   - $\mathcal{M} = \mathsf{Ising}$: critical transverse field Ising model
   - $\mathcal{M} = \mathsf{Ising}/\langle\psi \simeq \mathbb{1}\rangle$: massless free fermion
4. $\mathcal{D} = \mathsf{Ising}^{\boxtimes 2}$: $(\mathbb{Z}_2 +$ Kramers-Wannier self-duality$)^{\otimes 2}$
   - $\mathcal{M} = \mathsf{Ising}^2$: two decoupled critical transverse field Ising models
   - $\mathcal{M} = \mathsf{Ising}$: critical XY model
   - $\mathcal{M} = \mathsf{Ising}/\langle\psi \simeq \mathbb{1}\rangle$: massless Dirac fermion

5. $\mathcal{D} = \mathsf{Vec}_{\mathbb{Z}_2}$: $\mathbb{Z}_2$ symmetry
    - $\mathcal{M} = \mathsf{Vec}$: XXZ model
    - $\mathcal{M} = \mathsf{sVec}$: t-J_z model
6. $\mathcal{D} = \mathsf{Rep}(\mathrm{U}_q(\mathfrak{sl}_2))$: quantum deformed $SU(2)$ symmetry
    - $\mathcal{M} = \mathsf{Rep}(\mathrm{U}_q(\mathfrak{sl}_2))$: solid-on-solid (SOS) models
    - $\mathcal{M} = \mathsf{Vec}$: 6-vertex model (XXZ)
7. $\mathcal{D} = \mathcal{H}_3$: exotic fusion category, "Haagerup subfactor"
    - $\mathcal{M} = \mathcal{H}_3$: ?[1][2]
    - $\mathcal{M} = \mathcal{M}_{3,2}$: ?
    - $\mathcal{M} = \mathcal{M}_{3,1}$: ?

8. $\mathcal{D} = \mathsf{Rep}(S_3)$
    - $\mathcal{M} = \mathsf{Vec}$: XXZ model
    - $\mathcal{M} = \mathsf{Rep}(\mathbb{Z}_2)$:

$$\mathbb{H} = \sum_{\mathsf{i}} Z_{\mathsf{i}-1} Z_{\mathsf{i}+1} + Z_{\mathsf{i}-1} X_{\mathsf{i}} Z_{\mathsf{i}+1} + \Delta X_{\mathsf{i}}$$

    Interestingly, this model has a non-invertible $\mathsf{Rep}(S_3)$ symmetry!
    - $\mathcal{M} = \mathsf{Rep}(\mathbb{Z}_3)$: modified 3-state Potts model
    - $\mathcal{M} = \mathsf{Rep}(S_3)$: $\mathsf{Rep}(S_3)$ anyonic spin chain

Duality is isometry on superselection sectors

Hilbert space and Hamiltonian split into superselection sectors, which have to match between models:

$$\mathcal{H}_A = \bigoplus_i^n \mathcal{H}_{A,i} \quad \text{and} \quad \mathcal{H}_B = \bigoplus_i^n \mathcal{H}_{B,i}\,,$$
$$\mathbb{H}_A = \bigoplus_i^n \mathbb{H}_{A,i} \quad \text{and} \quad \mathbb{H}_B = \bigoplus_i^n \mathbb{H}_{B,i}\,.$$

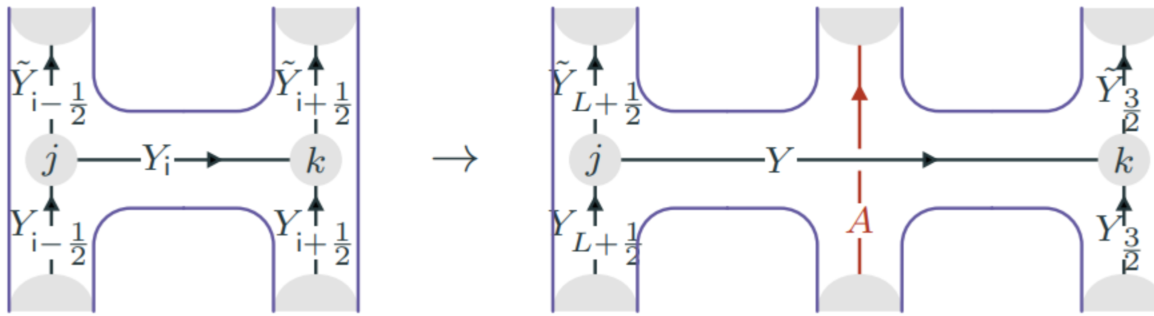
although they need not be the same size (different degeneracies). Dualities are isometries that interchange these sectors:

$$\mathbb{U}_i : \mathcal{H}_{A,i} \to \mathcal{H}_{B,i}, \quad \text{s.t.} \quad \mathbb{U}_i(\mathbb{H}_{A,i})\mathbb{U}_i^\dagger = \mathbb{H}_{B,i}$$

Here, superselection sectors refer to symmetry charges *and* boundary conditions

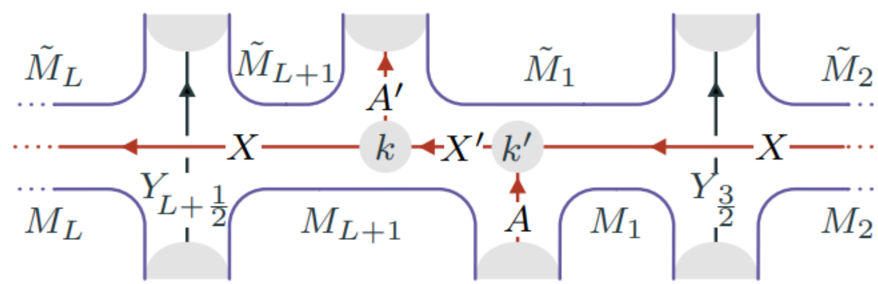
Lootens, Delcamp et al., ‘22

Symmetry twisted boundary conditions locally change the bonds in the Hamiltonian



in such a way that translation invariance is preserved up to local unitaries. The symmetry operators now act as symmetry "tubes":

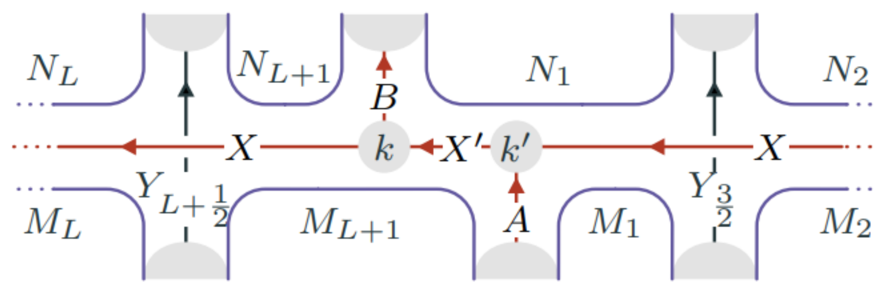
$$\mathfrak{T}^{A,A',X,X',k,k'}_{\mathcal{M}|\mathcal{M}} =$$



These symmetry tubes span an algebra, known as the tube algebra. Superselection sectors are irreducible representations of this algebra, which are described by the Drinfel'd center $\mathcal{Z}(\mathcal{C})$ of the fusion category $\mathcal{C}$ that describes the symmetry operators.

We can similarly define intertwiner tubes, that will implement the duality operator in the presence of a symmetry twist:

$$\mathfrak{T}^{A,B,X,X',k,k'}_{\mathcal{M}|\mathcal{N}} =$$



These intertwiner tubes allow us to relate dual topological sectors labeled by $\mathcal{Z}(\mathcal{C}_i) \simeq \mathcal{Z}(\mathcal{C}_j)$, which in general involves a permutation of the topological sectors.

The simplest example is the interchange of charges and fluxes for the $\mathbb{Z}_2$ Kramers-Wannier duality:

$$
\begin{aligned}
(\text{periodic}, \mathbb{Z}_2 - \text{even}) &\rightarrow (\text{periodic}, \mathbb{Z}_2 - \text{even}) \\
(\text{periodic}, \mathbb{Z}_2 - \text{odd}) &\rightarrow (\text{anti-periodic}, \mathbb{Z}_2 - \text{even}) \\
(\text{anti-periodic}, \mathbb{Z}_2 - \text{even}) &\rightarrow (\text{periodic}, \mathbb{Z}_2 - \text{odd}) \\
(\text{anti-periodic}, \mathbb{Z}_2 - \text{odd}) &\rightarrow (\text{anti-periodic}, \mathbb{Z}_2 - \text{odd})
\end{aligned}
$$

These maps can be computed explicitly for any duality[4]

[4] LL, Delcamp, Verstraete, *Dualities in one-dimensional quantum lattice models: topological sectors* 2211.03777

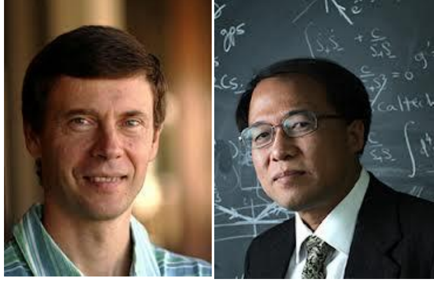
- Given Hamiltonian $\mathbb{H}$, its duals can be systematically constructed via identification of underlying categorical structures describing its symmetries:

$$\begin{tikzcd} \mathbb{H}(\mathcal{D},\mathcal{M}) \arrow[loop above, "\mathsf{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{M})"] \arrow[r, "\mathsf{Fun}_{\mathcal{D}}(\mathcal{M},\mathcal{M}')"] & \mathbb{H}(\mathcal{D},\mathcal{M}') \arrow[loop above, "\mathsf{Fun}_{\mathcal{D}}(\mathcal{M}',\mathcal{M}')"] \end{tikzcd}$$

- Dualities and symmetries are realized as MPOs
- Generalization to **higher dimensions** is **systematic**[5][6]

[5]Delcamp, *Tensor network approach to electromagnetic duality in (3+1)d topological gauge models*, JHEP 149 (2022)
[6]Delcamp, Tiwari, *Higher categorical symmetries and gauging in two-dimensional spin systems*, 2301.01259

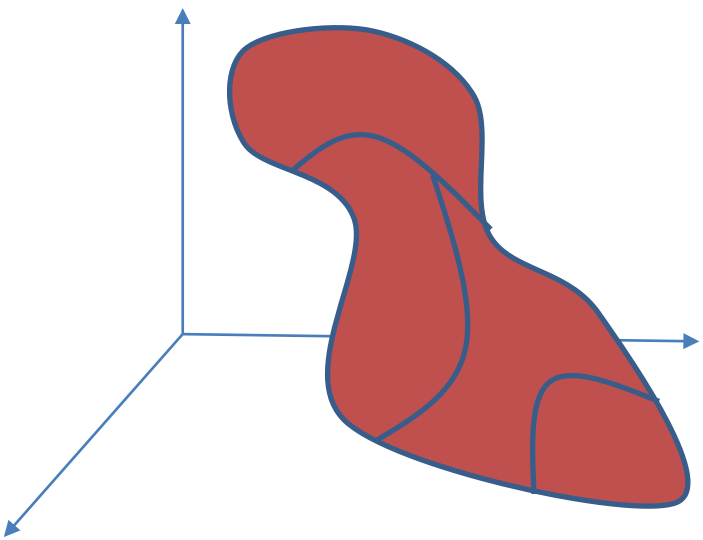
Topological phases of Matter

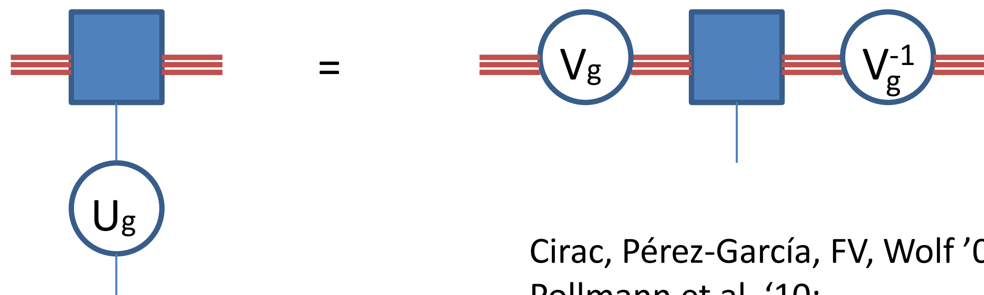


- Last decades has seen a revolution understanding topological phases of matter
  - Realization in Quantum Hall systems, observation of Majorana fermions, …
  - Topological phases of matter: there is no LOCAL order parameter distinguishing topological phases from trivial ones
    - Phase is characterized by *long range entanglement*
    - This entanglement can be used to built a fault-tolerant quantum computer
- Tensor network approach: topological order is all about symmetries of the entanglement degrees of freedom
  - Landau paradigm of order parameters is recovered in symmetries of LOCAL tensors

1D interacting SPT phases of matter: MPS

- Classification of phases of matter of 1-D spin chains under adiabatic paths preserving a symmetry: manifold breaks into pieces (symmetry protected topological order)



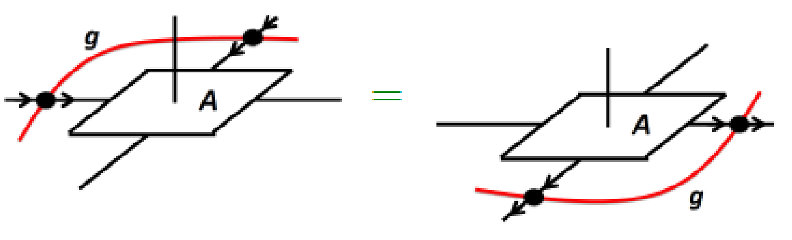


Cirac, Pérez-García, FV, Wolf '08
Pollmann et al. '10;
Chen, Wen '11
Cirac, Pérez-García, Schuch '11

  - Different phases are characterized by projective representations of physical symmetry group ($H^2(G,U(1))$)
  - In case of fermions: graded tensor algebras, and already topological phases without imposing symmetries (Majorana / Kitaev spin chain)

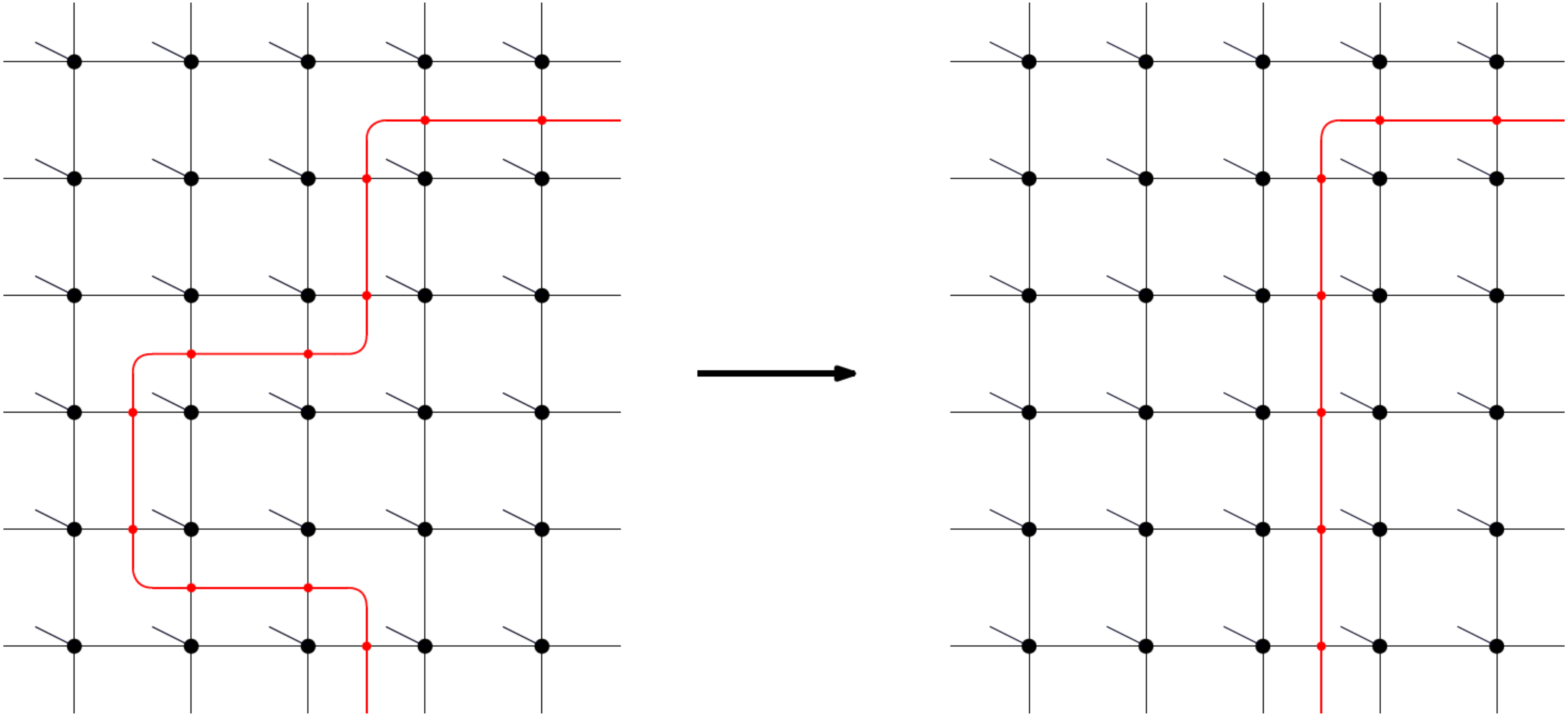
Symmetries in PEPS

- Symmetries and topological order is much richer in 2 dimensions: existence of anyons, Wilson loop operators, …
  - 2 dimensions is where the most surprising things can happen: 2 is low enough to have a lot of entanglement (3 dimensions is already much closer to mean field theory), but 2 is large enough to have nontrivial statistics (e.g. fractional quantum Hall effect)
  - All those exotic materials exhibit a special entanglement structure which is locally reflected in symmetries of the microscopic tensors



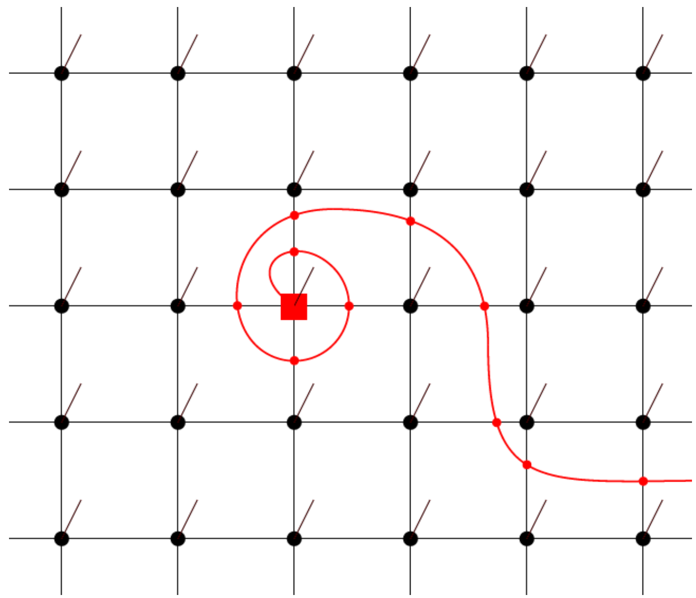
  - Probing entanglement reveals nonlocal order parameters: Landau symmetry breaking, but now on the entanglement degrees of freedom

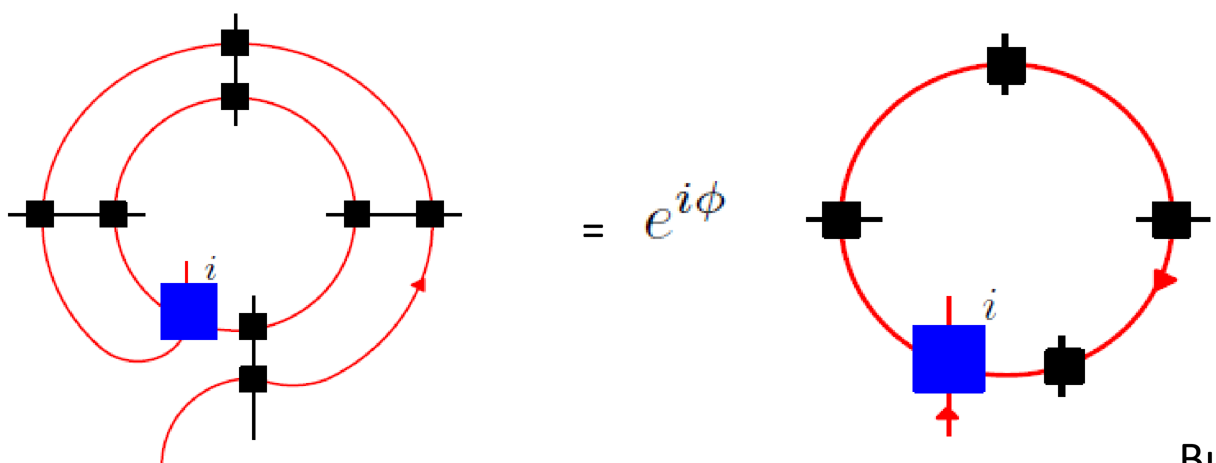
- Those symmetries give rise to Wilson loops that can be pulled through the tensor network: tensor network representations of Levin-Wen models (lattice versions of topological phases of matter with gappable boundaries)



- Critical spin systems are obtained as “strange correlators” of those (conjecture of P. Fendley: correlators have to satisfy discrete holomorphicity condition)

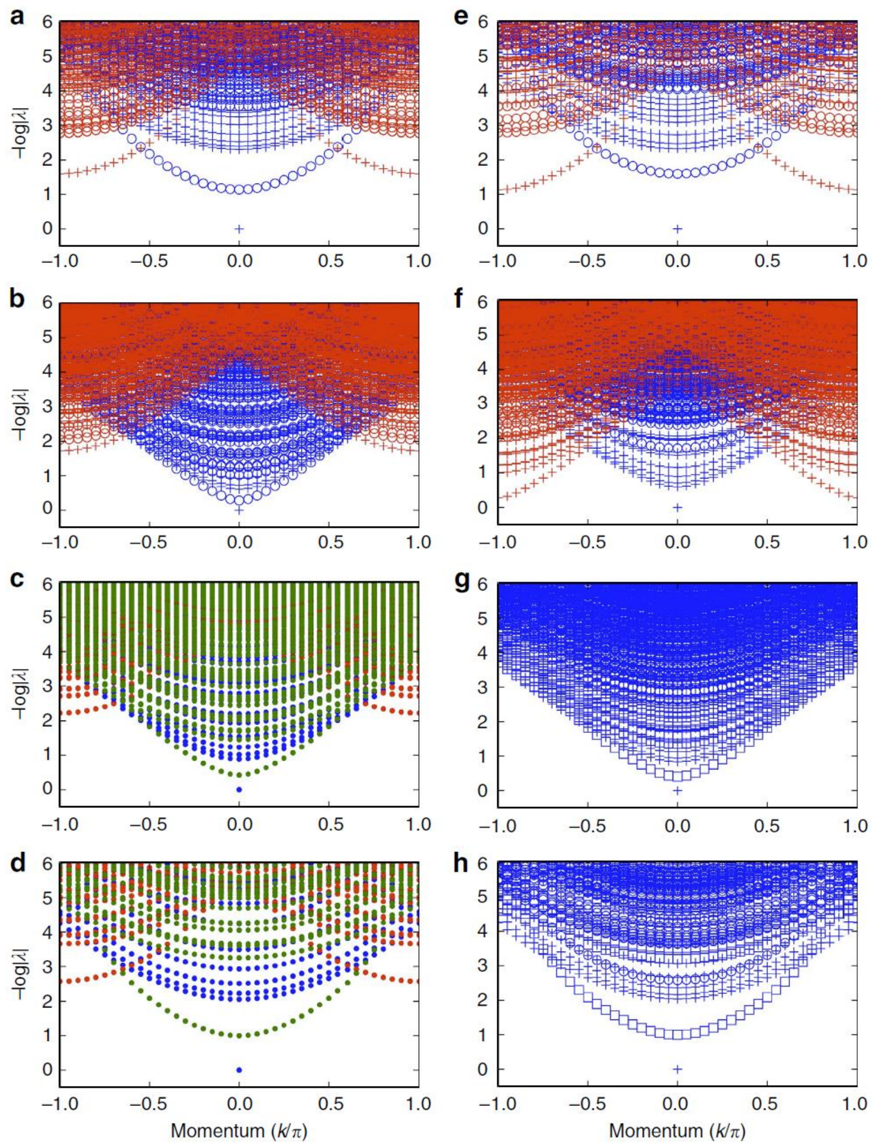
- Elementary excitations (anyons) in the system consist of end points of those strings: those necessarily come in pairs (cfr. Fermions: simplest type of anyon)





Bultinck et al., ‘17

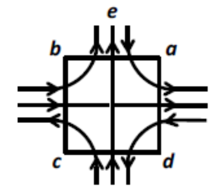
- Entanglement spectrum and confinement/deconfinement phase transition by anyon condensation in a Z2 gauge theory (Shenker/Fradkin == toric code with string tension)



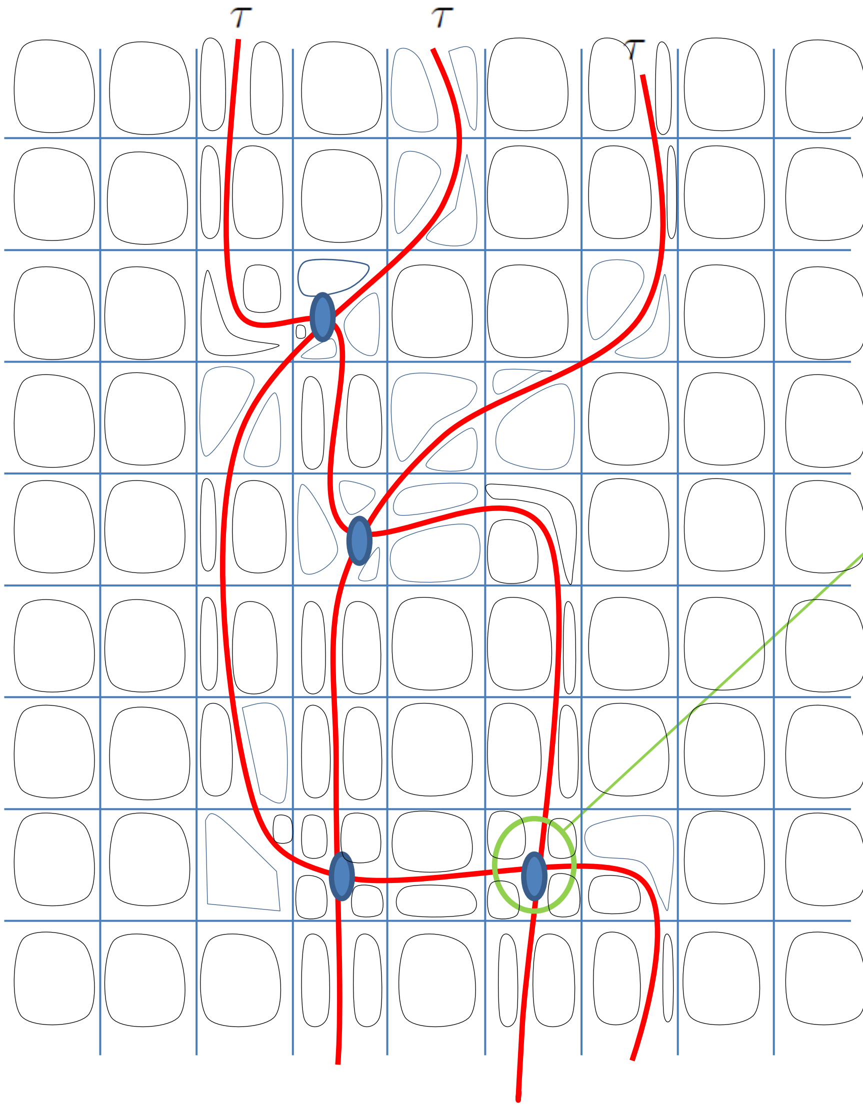
Haegeman, Zauner, Schuch, FV ‘15

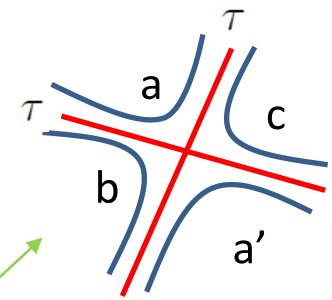
Topological Quantum Computation in the shadow world

We can identify a tensor product structure of logical qubits with the entanglement (virtual) degrees of freedom; e.g. Fibonacci string net

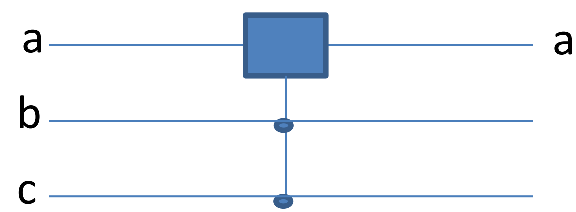


$:= G^{ab^*e}_{cdf}$





Braiding tensor is F-symbol

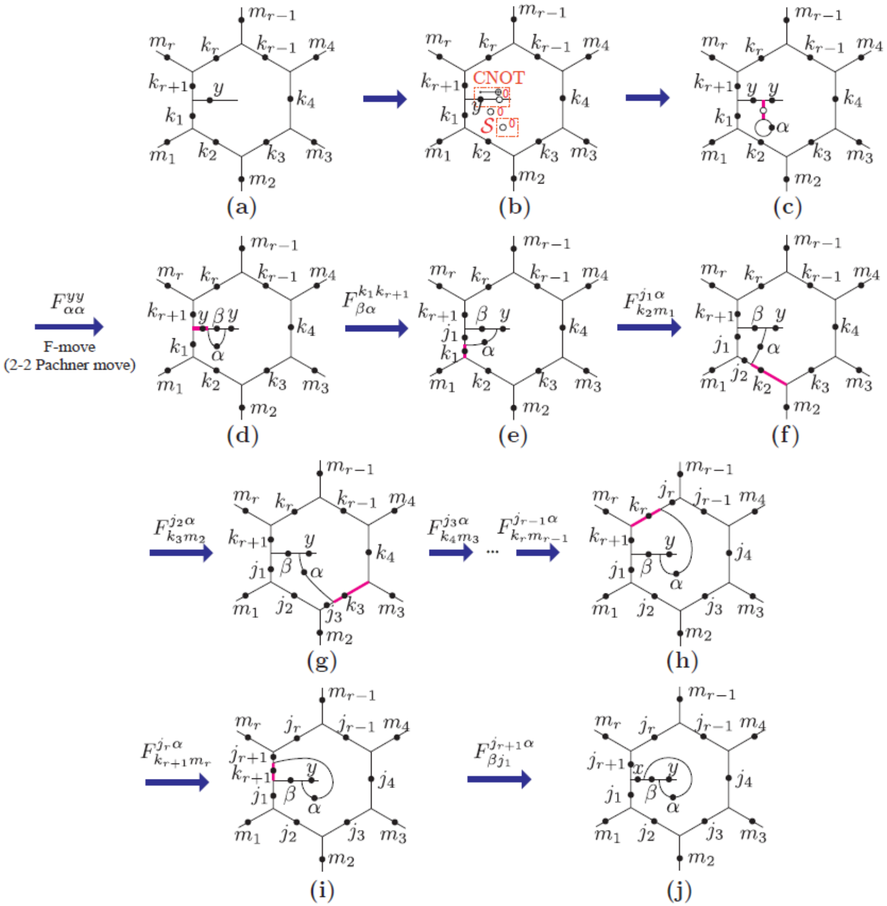


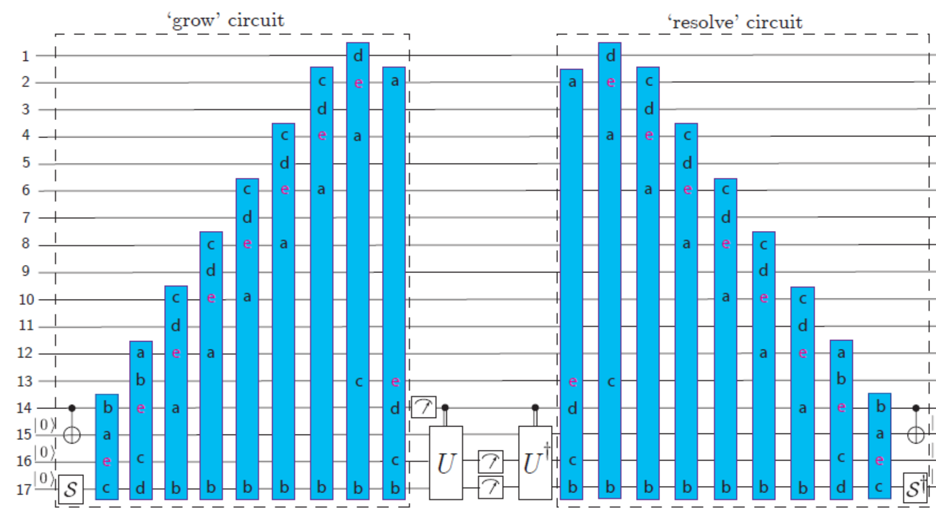
Controlled-Controlled-U gate

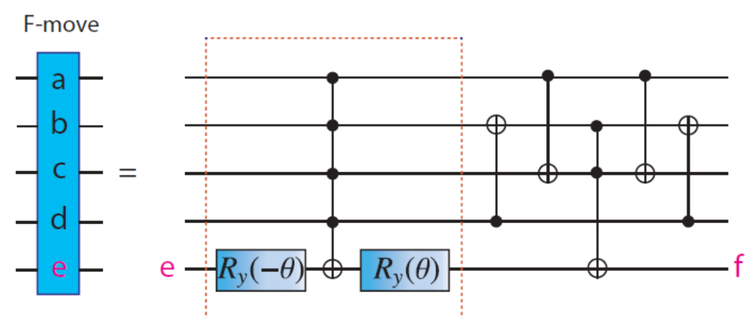
$$\begin{pmatrix} \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} \\ \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} \end{pmatrix}$$

Freedman, Kitaev, Wang, …

Application: Fibonacci Turaev-Viro error correcting code







Measuring anyon charges

Schotte, Zhu, Burgelman, FV PRX '22

Conclusion

- Tensor networks yield natural language for representing topological and critical systems

MPO symmetries and Levin-Wen models

- Tensors with MPO symmetries can be used to find tensor network representations of Levin-Wen models (lattice versions of topological phases of matter with gappable boundaries)