GENERALIZED BLACK HOLE ENTROPY IS VON NEUMANN ENTROPY

New Perspectives in CFT and Gravity
DESY Theory Workshop
September 29, 2023

Sam Leutheusser
Princeton Gravity Initiative

Work with J. Kudler-Flam & G. Satishchandran: 2309.15897

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

- Originally derived by thermodynamic arguments

[Bekenstein, Hawking,…]

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

- Originally derived by thermodynamic arguments

[Bekenstein, Hawking,…]

- “Improved” to generalized entropy

$$S_{gen} = S_{BH} + S_{out}$$

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

- Originally derived by thermodynamic arguments
[Bekenstein, Hawking,…]
- “Improved” to generalized entropy
$$S_{gen} = S_{BH} + S_{out}$$
- Statistical Interpretation? $S_{BH} = \log \Omega_{BH}$?

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

- Originally derived by thermodynamic arguments [Bekenstein, Hawking,…]
- "Improved" to generalized entropy

$$S_{gen} = S_{BH} + S_{out}$$

- Statistical Interpretation? $S_{BH} = \log\Omega_{BH}$?
- Some approaches:
  - Microstate Counting [Strominger, Vafa,…]

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

- Originally derived by thermodynamic arguments [Bekenstein, Hawking,…]
- "Improved" to generalized entropy

$$S_{gen} = S_{BH} + S_{out}$$

- Statistical Interpretation? $S_{BH} = \log \Omega_{BH}$?
- Some approaches:
  - Microstate Counting [Strominger, Vafa,…] ← Symmetry!

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

- Originally derived by thermodynamic arguments [Bekenstein, Hawking,…]
- "Improved" to generalized entropy

$$S_{gen} = S_{BH} + S_{out}$$

- Statistical Interpretation? $S_{BH} = \log \Omega_{BH}$?
- Some approaches:
  - Microstate Counting [Strominger, Vafa,…] ← Symmetry!
  - Euclidean Path Integral [Gibbons, Hawking,…]

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

- Originally derived by thermodynamic arguments [Bekenstein, Hawking,…]
- “Improved” to generalized entropy

$$S_{gen} = S_{BH} + S_{out}$$

- Statistical Interpretation? $S_{BH} = \log \Omega_{BH}$?
- Some approaches:
  - Microstate Counting [Strominger, Vafa,…] ← Symmetry!
  - Euclidean Path Integral [Gibbons, Hawking,…] ← Static!

Black Hole Entropy

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

Widely Applicable!

- Originally derived by thermodynamic arguments [Bekenstein, Hawking,…]
- "Improved" to generalized entropy

$$S_{gen} = S_{BH} + S_{out}$$

- Statistical Interpretation? $S_{BH} = \log\Omega_{BH}$?
- Some approaches:
  - Microstate Counting [Strominger, Vafa,…] ← Symmetry!
  - Euclidean Path Integral [Gibbons, Hawking,…] ← Static!

A New Approach

- Explicit construction of an algebra of “dressed” observables outside a black hole horizon [Witten]

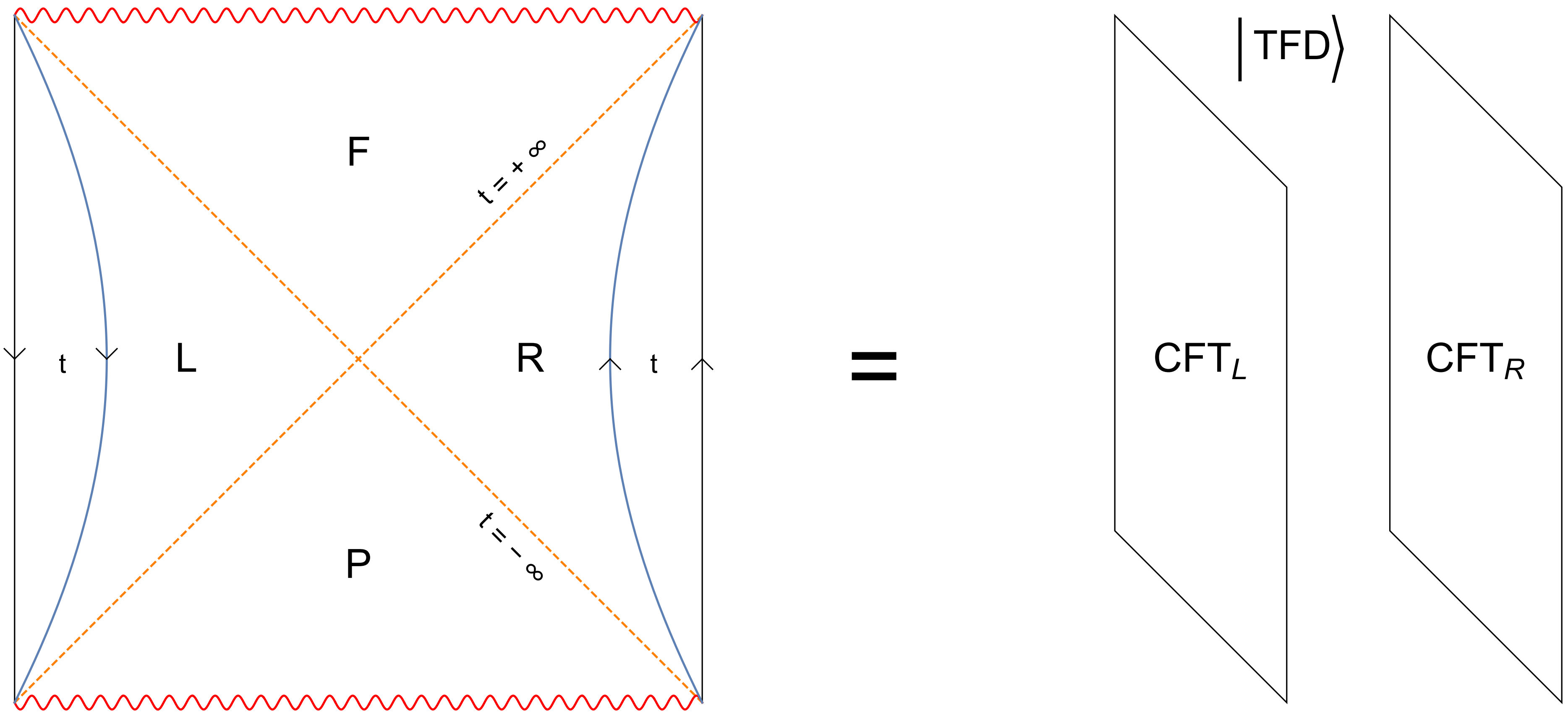
A New Approach

- Explicit construction of an algebra of “dressed” observables outside a black hole horizon [Witten]
- von Neumann entropy of dressed observables agrees with S_{gen} for AdS-Schwarzschild [Chandrasekaran, Penington, Witten (CPW)]

Entropy of AdS-Schwarzschild

- Eternal AdS black hole and thermofield double

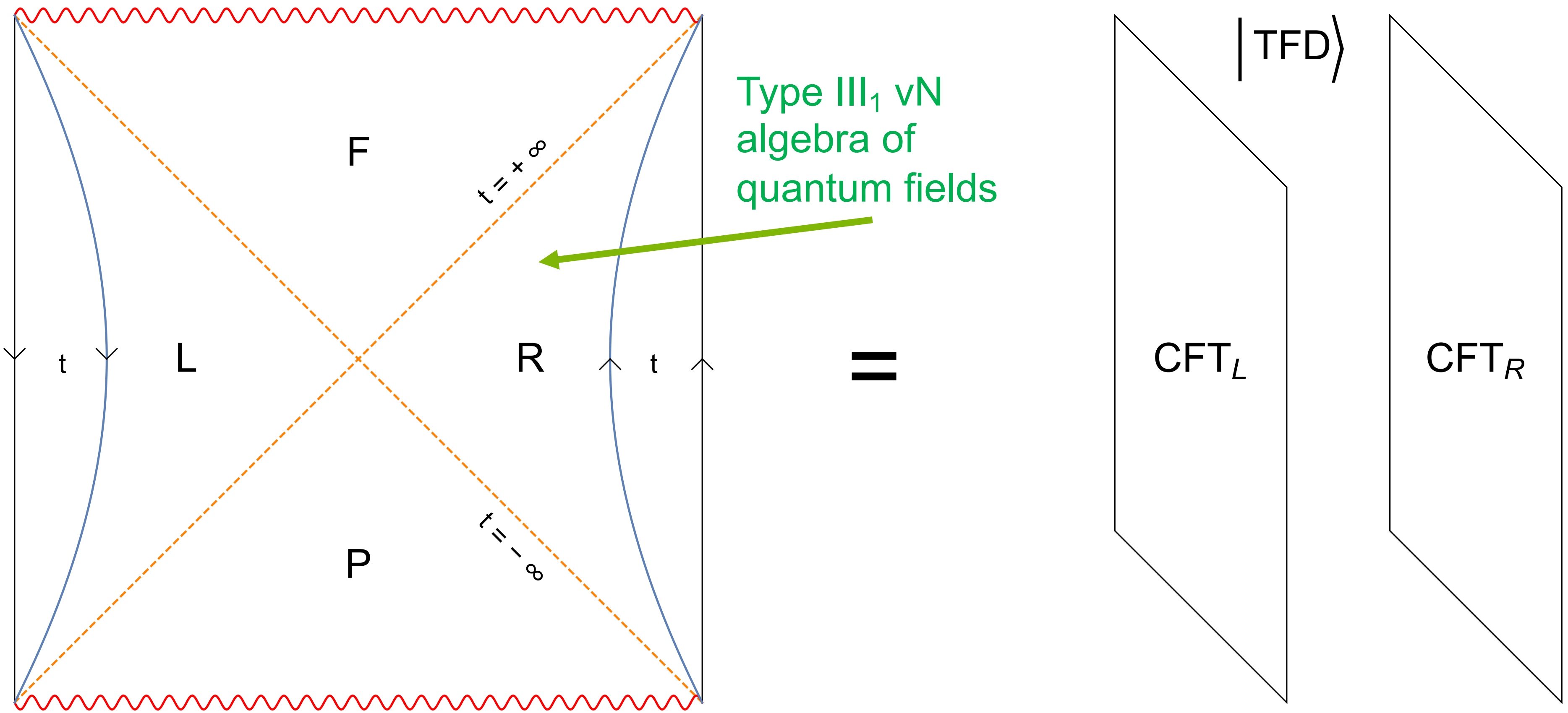
[Maldacena]



Entropy of AdS-Schwarzschild

- Eternal AdS black hole and thermofield double

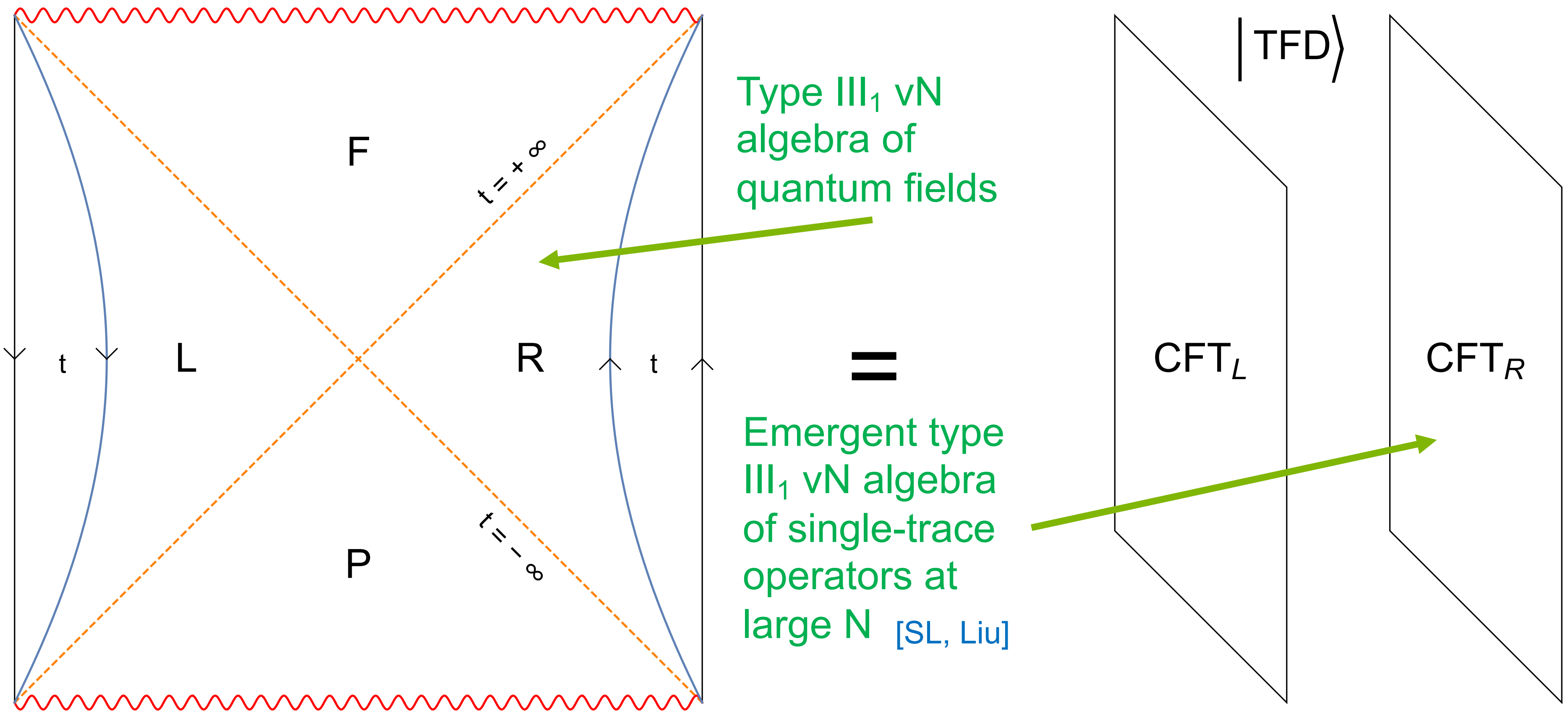
[Maldacena]



Entropy of AdS-Schwarzschild

- Eternal AdS black hole and thermofield double

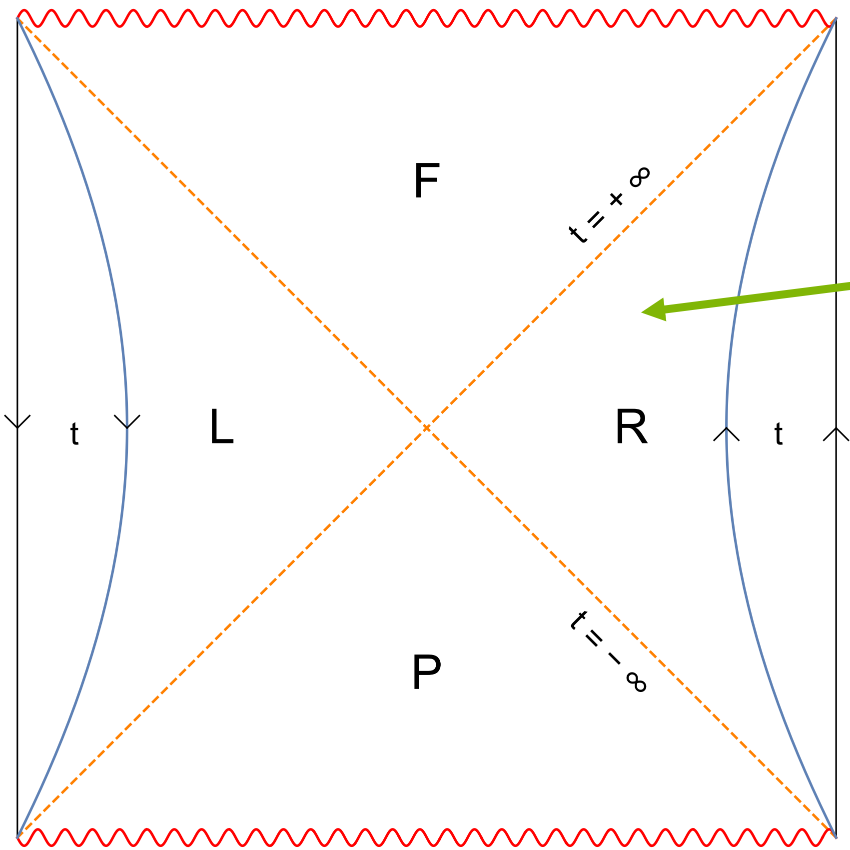
[Maldacena]



Entropy of AdS-Schwarzschild

- Eternal AdS black hole and thermofield double

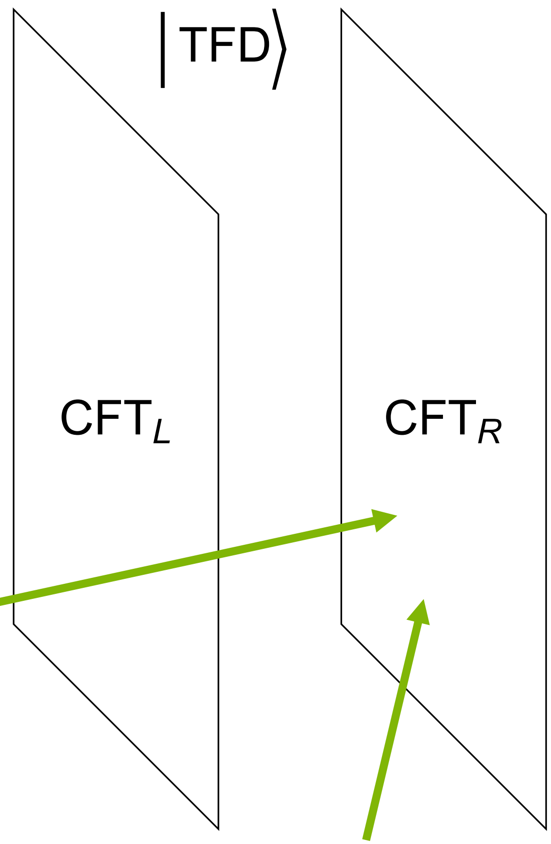
[Maldacena]



Type III_1 vN algebra of quantum fields

=

Emergent type III_1 vN algebra of single-trace operators at large N [SL, Liu]



Addition of CFT Hamiltonian deforms to type II_∞ ⇒ can compute entropy!

[Witten; Chandrasekaran, Penington, Witten (CPW)]

A New Approach

- Explicit construction of an algebra of “dressed” observables outside a black hole horizon [Witten]
- von Neumann entropy of dressed observables agrees with S_{gen} for AdS-Schwarzschild [Chandrasekaran, Penington, Witten (CPW)]

A New Approach

- Explicit construction of an algebra of “dressed” observables outside a black hole horizon [Witten]
- von Neumann entropy of dressed observables agrees with S_{gen} for AdS-Schwarzschild [Chandrasekaran, Penington, Witten (CPW)]
- With an observer one finds S_{gen} for de Sitter [Chandrasekaran, Longo, Penington, Witten (CLPW)]

A New Approach

- Explicit construction of an algebra of "dressed" observables outside a black hole horizon [Witten]
- von Neumann entropy of dressed observables agrees with S_{gen} for AdS-Schwarzschild [Chandrasekaran, Penington, Witten (CPW)]
- With an observer one finds S_{gen} for de Sitter [Chandrasekaran, Longo, Penington, Witten (CLPW)]

- Crucial ingredient:
  - Mathematical: "Crossed Product"

A New Approach

- Explicit construction of an algebra of "dressed" observables outside a black hole horizon [Witten]
- von Neumann entropy of dressed observables agrees with S_{gen} for AdS-Schwarzschild [Chandrasekaran, Penington, Witten (CPW)]
- With an observer one finds S_{gen} for de Sitter [Chandrasekaran, Longo, Penington, Witten (CLPW)]

- Crucial ingredient:
  - Mathematical: "Crossed Product"
  - Physical: Gravitational constraint equations

A New Approach

- Explicit construction of an algebra of “dressed” observables outside a black hole horizon [Witten]
- von Neumann entropy of dressed observables agrees with S_{gen} for AdS-Schwarzschild [Chandrasekaran, Penington, Witten (CPW)]
- With an observer one finds S_{gen} for de Sitter [Chandrasekaran, Longo, Penington, Witten (CLPW)]

- Crucial ingredient:
  - Mathematical: “Crossed Product”
  - Physical: Gravitational constraint equations
- How generally applicable is this construction?

Potential Obstructions

- Crossed product construction was applied to static subregions that admit equilibrium states

Potential Obstructions

- Crossed product construction was applied to static subregions that admit equilibrium states
- Many black hole spacetimes do not admit such states, e.g.

Potential Obstructions

- Crossed product construction was applied to static subregions that admit equilibrium states
- Many black hole spacetimes do not admit such states, e.g.
  - Asymptotically Flat Kerr

Potential Obstructions

- Crossed product construction was applied to static subregions that admit equilibrium states
- Many black hole spacetimes do not admit such states, e.g.
  - Asymptotically Flat Kerr
  - Schwarzschild de Sitter

Potential Obstructions

- Crossed product construction was applied to static subregions that admit equilibrium states
- Many black hole spacetimes do not admit such states, e.g.
  - Asymptotically Flat Kerr
  - Schwarzschild de Sitter
- Black holes formed from collapse will not be in equilibrium

Main Result

Perturbative Gravitational Constraints

⇓

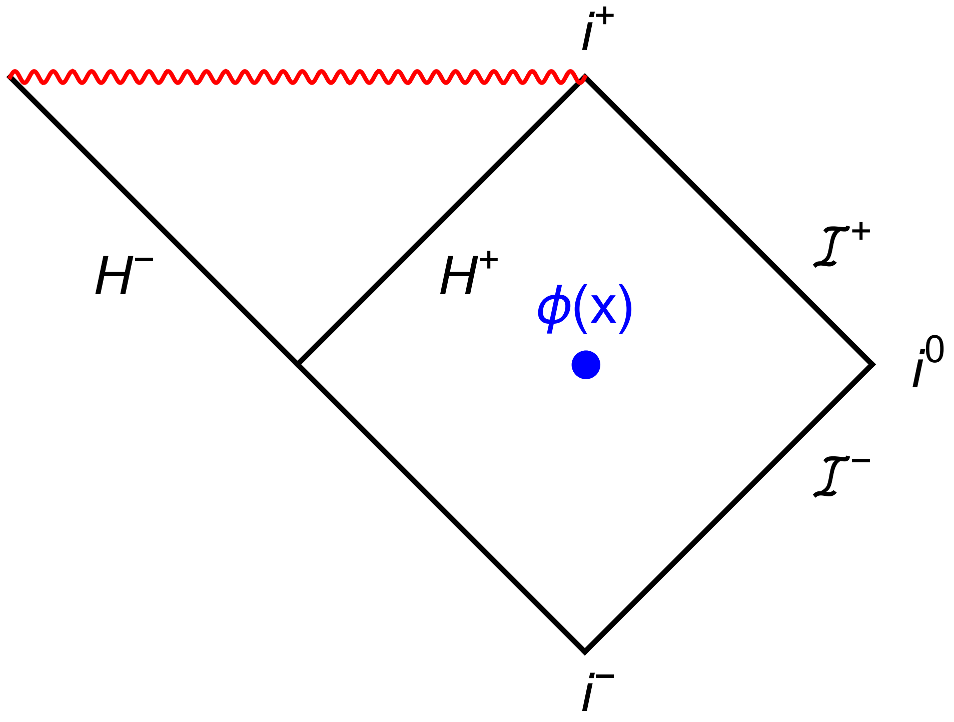
Crossed Product with *modular* group on spacetimes with Killing Horizons

⇓

Type II vN Algebra with $S_{vN} = S_{gen}$

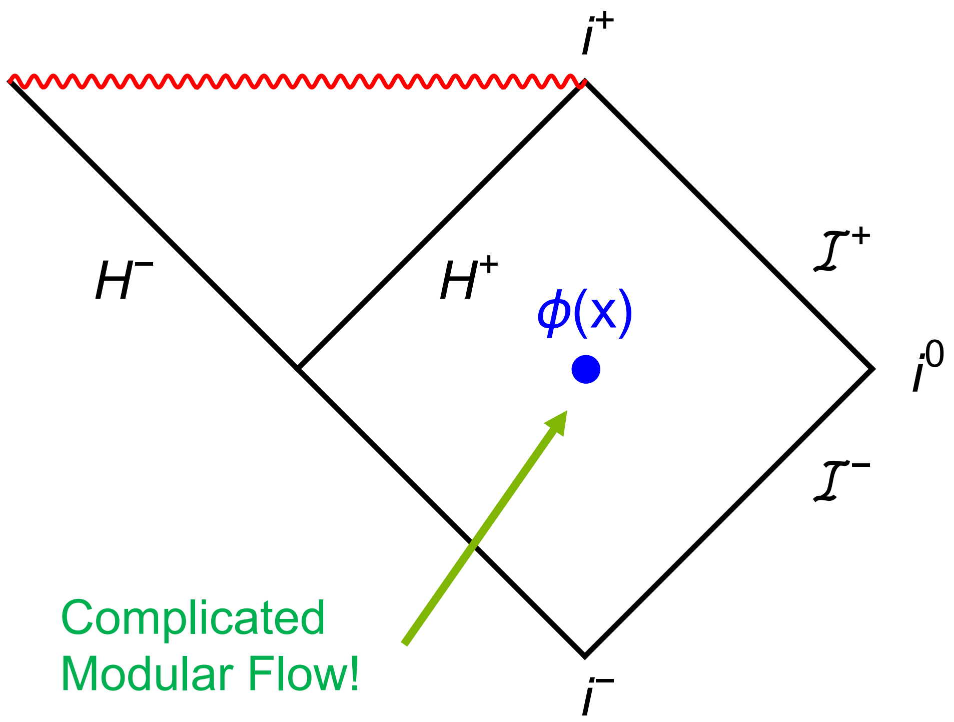
Key Technique

- E.g. Massless field on Schwarzschild



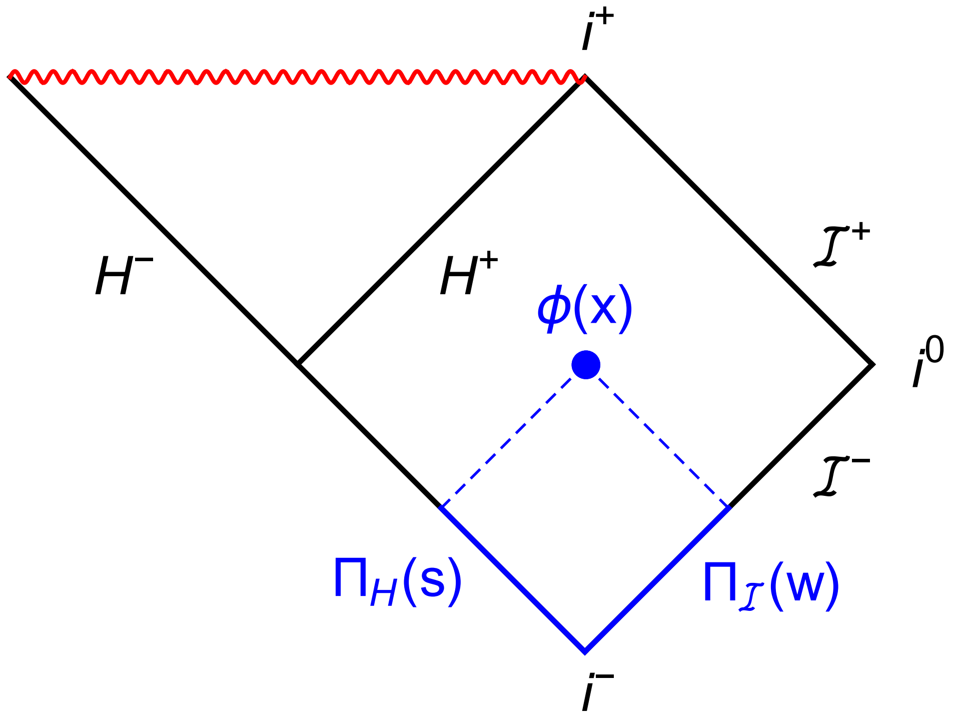
Key Technique

- E.g. Massless field on Schwarzschild



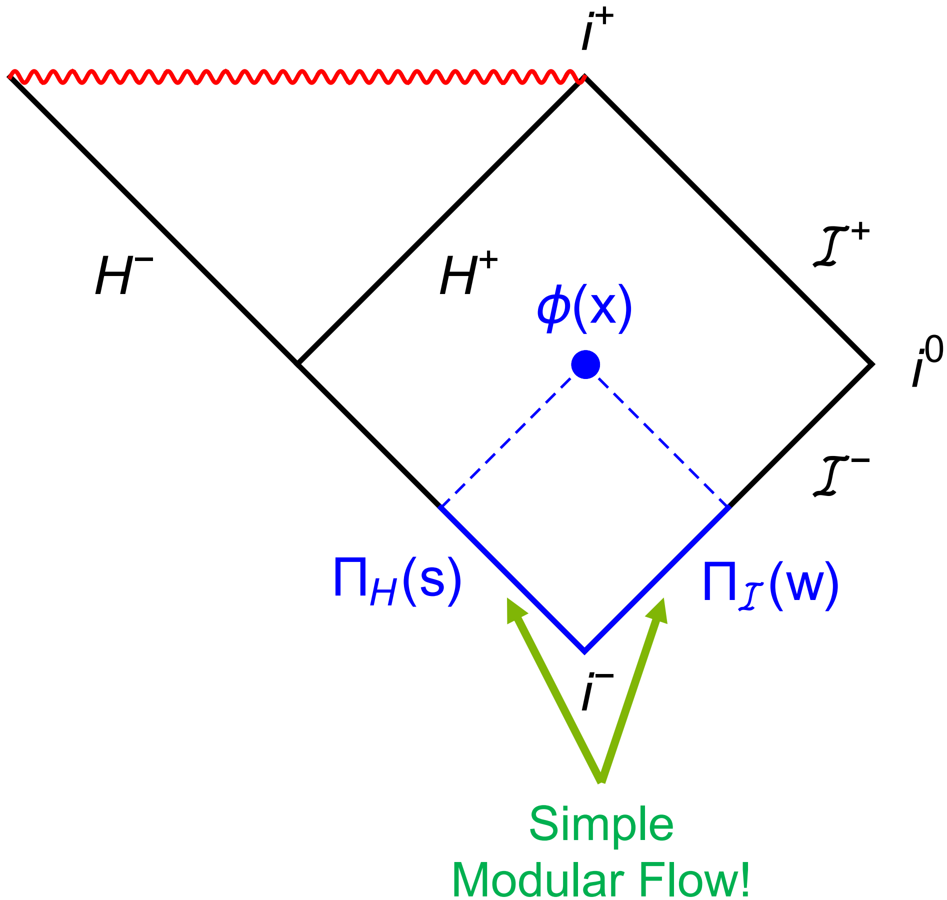
Key Technique

- E.g. Massless field on Schwarzschild



Key Technique

- E.g. Massless field on Schwarzschild



Talk Outline

1. vN Algebras and the Crossed Product
2. Quantization on Killing Horizons
3. Gravitational “Charges” and “Dressed” Operators
4. Asymptotically Flat Kerr Black Hole
5. Schwarzschild de Sitter Black Hole
6. Future directions

Local Algebras in QM

- Bipartite quantum system: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$

Local Algebras in QM

- Bipartite quantum system: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$
- Local operator algebra $\mathcal{B}(\mathcal{H}_R)$

Local Algebras in QM

- Bipartite quantum system: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$
- Local operator algebra $\mathcal{B}(\mathcal{H}_R)$
- Global state $|\Psi\rangle \in \mathcal{H} \rightarrow$ reduced density matrix $\rho_R \in \mathcal{B}(\mathcal{H}_R)$

Local Algebras in QM

- Bipartite quantum system: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$
- Local operator algebra $\mathcal{B}(\mathcal{H}_R)$
- Global state $|\Psi\rangle \in \mathcal{H} \rightarrow$ reduced density matrix $\rho_R \in \mathcal{B}(\mathcal{H}_R)$
- von Neumann entropy $S_R = -\mathrm{Tr}\rho_R \log \rho_R$

Local Algebras in QM

- Bipartite quantum system: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$
- Local operator algebra $\mathcal{B}(\mathcal{H}_R)$
- Global state $|\Psi\rangle \in \mathcal{H} \rightarrow$ reduced density matrix $\rho_R \in \mathcal{B}(\mathcal{H}_R)$
- von Neumann entropy $S_R = -\mathrm{Tr}\rho_R \log\rho_R$
- If $|\Psi\rangle$ is highly entangled, ρ_R and ρ_L will be full rank. In this highly entangled case can we define **modular flow**

$$\sigma_t(A) \equiv \Delta^{it} A \Delta^{-it}, \qquad A \in \mathcal{B}(\mathcal{H}), \qquad \Delta = \rho_L \otimes \rho_R^{-1}$$

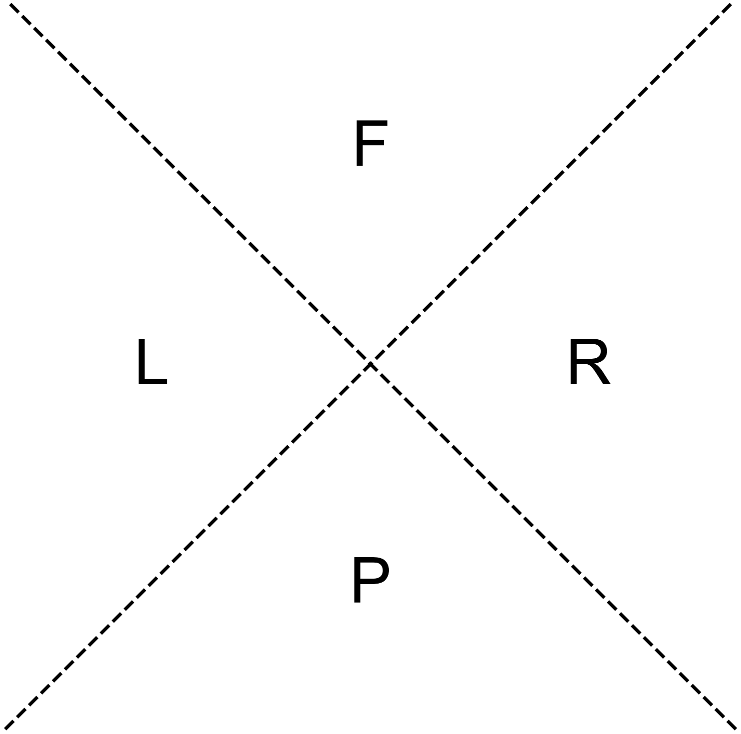
Local Algebras in QM

- Bipartite quantum system: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$
- Local operator algebra $\mathcal{B}(\mathcal{H}_R)$ ← Type I vN algebra
- Global state $|\Psi\rangle \in \mathcal{H} \rightarrow$ reduced density matrix $\rho_R \in \mathcal{B}(\mathcal{H}_R)$
- von Neumann entropy $S_R = -\mathrm{Tr}\rho_R \log\rho_R$
- If $|\Psi\rangle$ is highly entangled, ρ_R and ρ_L will be full rank. In this highly entangled case can we define **modular flow**

$$\sigma_t(A) \equiv \Delta^{it} A \Delta^{-it}, \qquad A \in \mathcal{B}(\mathcal{H}), \qquad \Delta = \rho_L \otimes \rho_R^{-1}$$

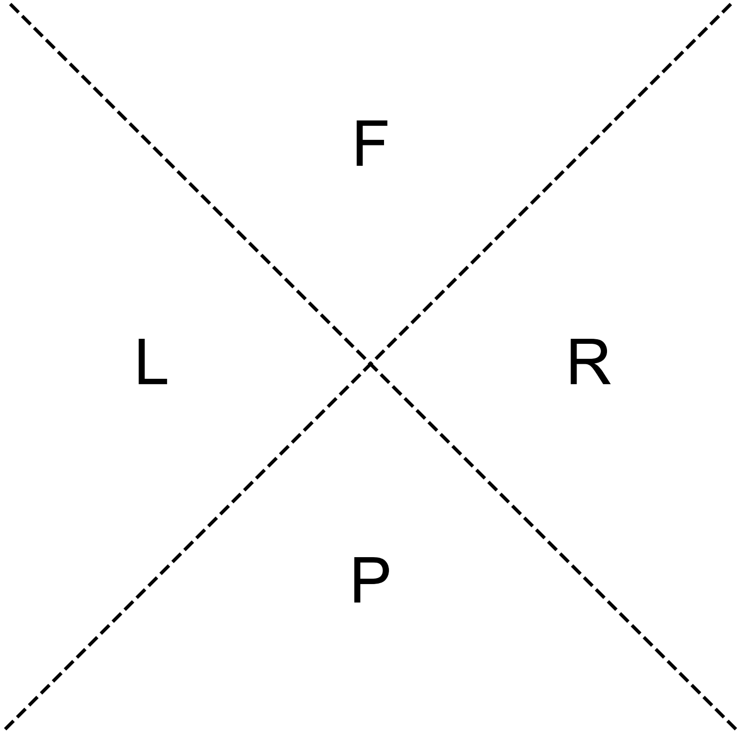
Local Algebras in Relativistic QFT

- E.g. Rindler decomposition of Minkowski space



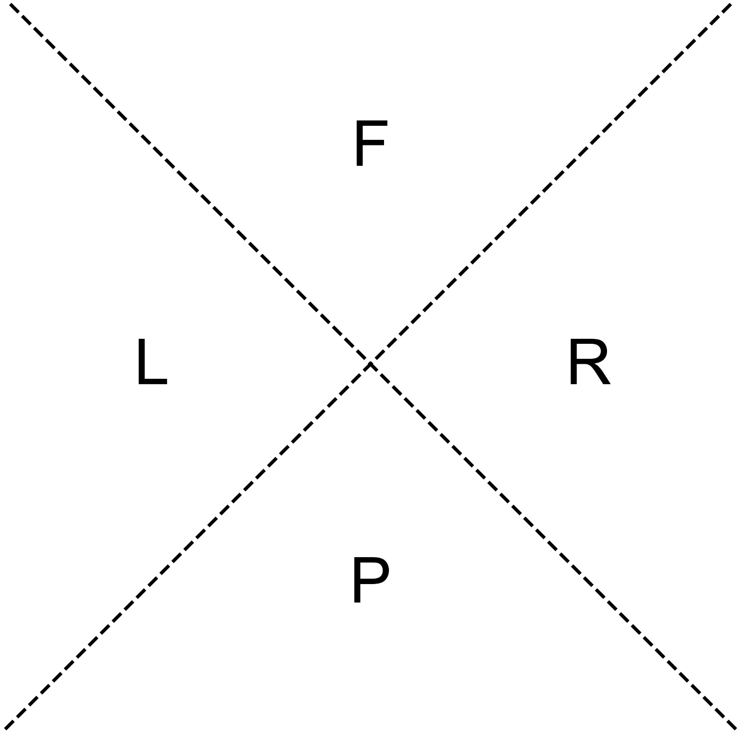
Local Algebras in Relativistic QFT

- E.g. Rindler decomposition of Minkowski space
- There do **not** exist $\mathcal{H}_L/\mathcal{H}_R$ on which fields localized in the L/R regions act



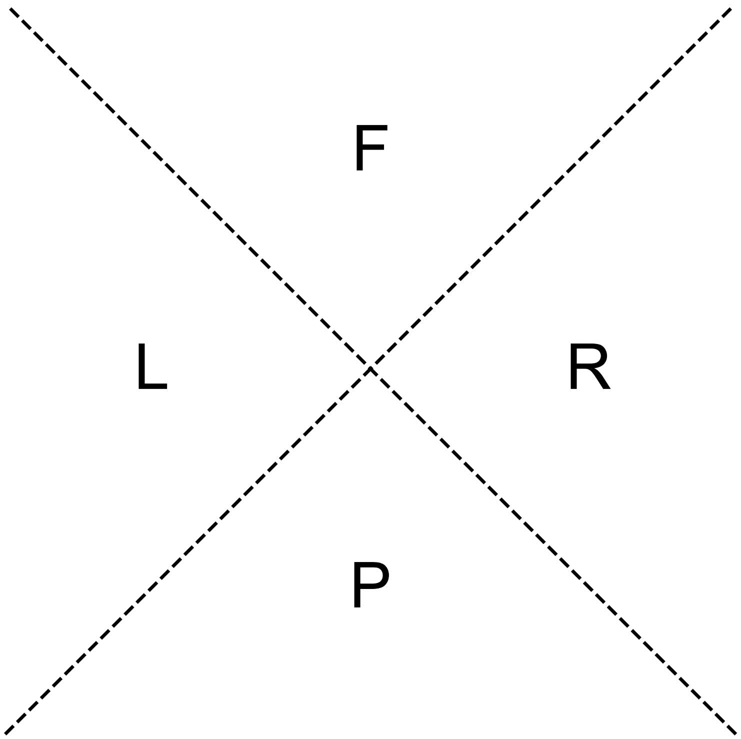
Local Algebras in Relativistic QFT

- E.g. Rindler decomposition of Minkowski space
- There do **not** exist $\mathcal{H}_L / \mathcal{H}_R$ on which fields localized in the L/R regions act
- There are **not** reduced density matrices for L/R



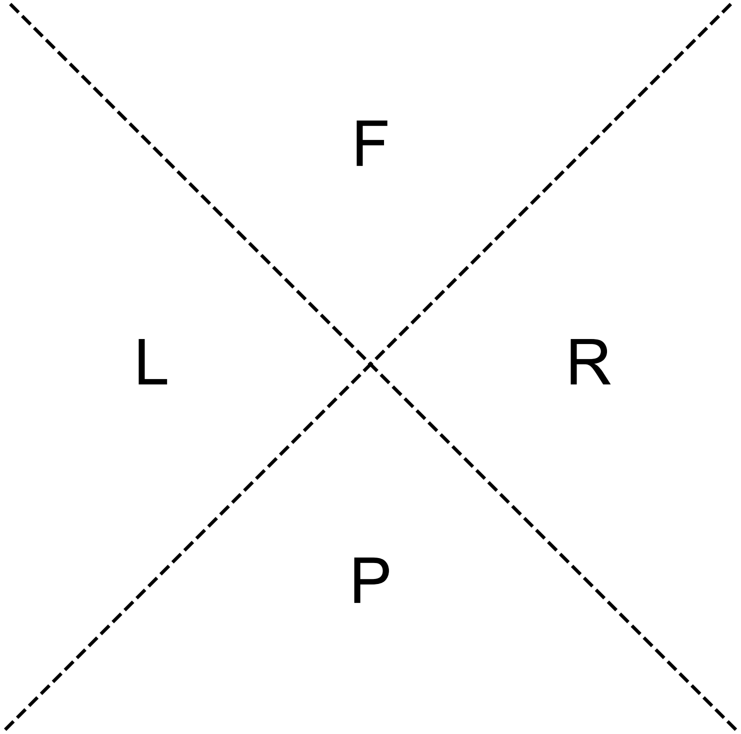
Local Algebras in Relativistic QFT

- E.g. Rindler decomposition of Minkowski space
- There do **not** exist $\mathcal{H}_L/\mathcal{H}_R$ on which fields localized in the L/R regions act
- There are **not** reduced density matrices for L/R
- There is **no** finite, well-defined entanglement entropy for L or R



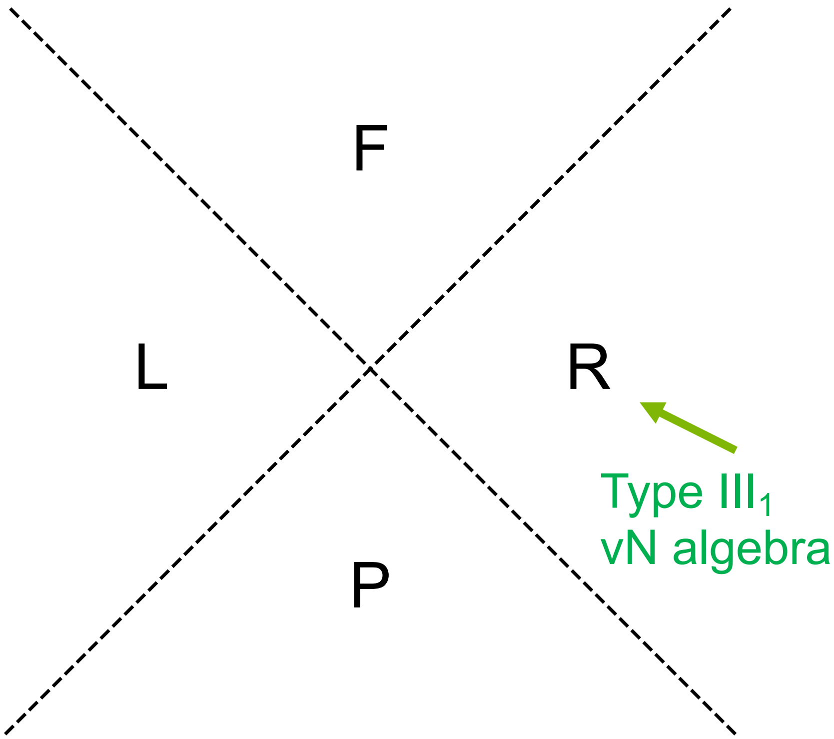
Local Algebras in Relativistic QFT

- E.g. Rindler decomposition of Minkowski space
- There do **not** exist $\mathcal{H}_L/\mathcal{H}_R$ on which fields localized in the L/R regions act
- There are **not** reduced density matrices for L/R
- There is **no** finite, well-defined entanglement entropy for L or R
- But, the modular operator, Δ, and modular flow, σ_t, still exist.



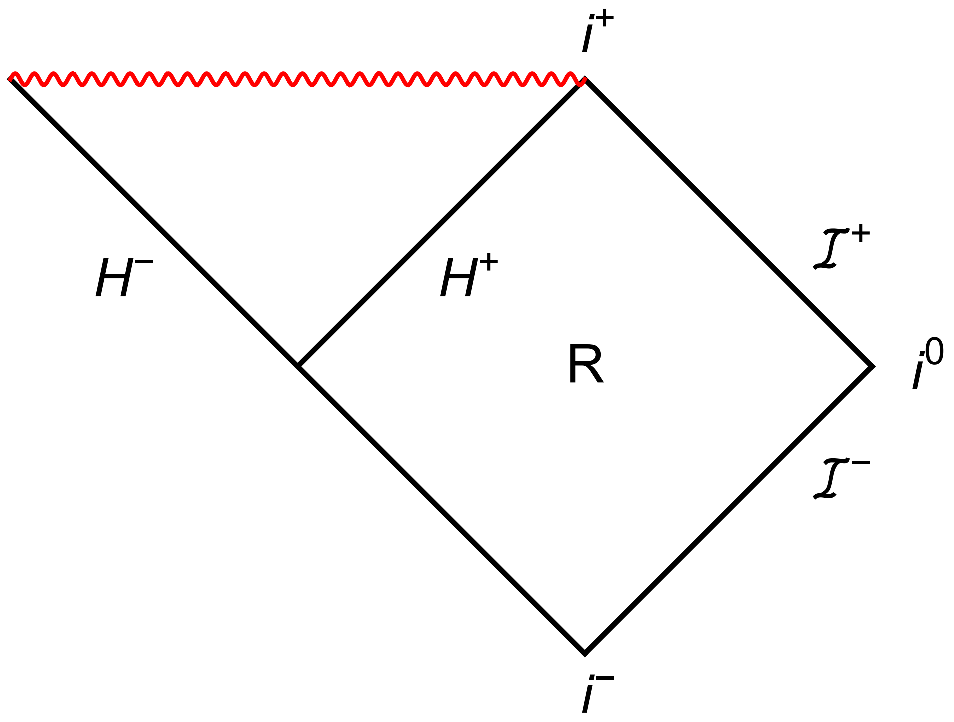
Local Algebras in Relativistic QFT

- E.g. Rindler decomposition of Minkowski space
- There do **not** exist $\mathcal{H}_L/\mathcal{H}_R$ on which fields localized in the L/R regions act
- There are **not** reduced density matrices for L/R
- There is **no** finite, well-defined entanglement entropy for L or R
- But, the modular operator, Δ, and modular flow, σ_t, still exist.



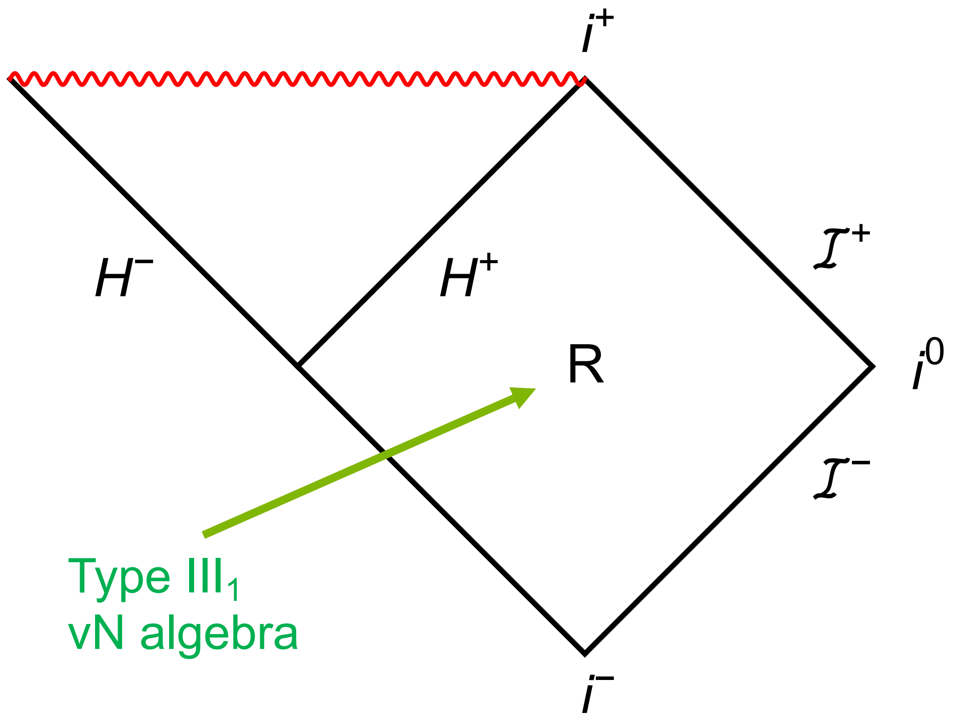
Type III Algebras and Black Holes

- E.g. QFT on Schwarzschild



Type III Algebras and Black Holes

- E.g. QFT on Schwarzschild



- A Type III algebra is associated to the exterior of any bifurcate Killing horizon!

Crossed Product

- vN Algebra $\mathfrak{U}$ + group G of automorphisms

Crossed Product

- vN Algebra $\mathfrak{U}$ + group G of automorphisms
- New algebra: $\mathfrak{U}_{ext} = \mathfrak{U} \rtimes G$ acting on $\mathcal{H} \otimes L^2(G)$

Crossed Product

- vN Algebra $\mathfrak{U}$ + group G of automorphisms
- New algebra: $\mathfrak{U}_{ext} = \mathfrak{U} \rtimes G$ acting on $\mathcal{H} \otimes L^2(G)$
- Key theorem [Takesaki]
  For $\mathfrak{U}$ of type III_1 and $G = \mathbb{R}$ the modular automorphism group, $\mathfrak{U}_{ext}$ is type II_∞

Crossed Product

- vN Algebra $\mathfrak{U}$ + group G of automorphisms
- New algebra: $\mathfrak{U}_{ext} = \mathfrak{U} \rtimes G$ acting on $\mathcal{H} \otimes L^2(G)$
- Key theorem [Takesaki]
  For $\mathfrak{U}$ of type III_1 and $G = \mathbb{R}$ the modular automorphism group, $\mathfrak{U}_{ext}$ is type II_∞
- Type II vN algebras admit density matrices and vN entropies (but not pure states)

Crossed Product

- vN Algebra $\mathfrak{U}$ + group G of automorphisms
- New algebra: $\mathfrak{U}_{ext} = \mathfrak{U} \rtimes G$ acting on $\mathcal{H} \otimes L^2(G)$
- Key theorem [Takesaki]
  For $\mathfrak{U}$ of type III_1 and $G = \mathbb{R}$ the modular automorphism group, $\mathfrak{U}_{ext}$ is type II_∞
- Type II vN algebras admit density matrices and vN entropies (but not pure states)
- Crossed product can take us from an algebra for which entropy cannot be defined to one that admits a definition of entropy!

Classical-Quantum States

- No entanglement between the QFT and extra degree of freedom

$$|\widehat{\omega}\rangle = |\omega\rangle \otimes |f\rangle \in \mathcal{H} \otimes L^2(\mathbb{R})$$

Classical-Quantum States

- No entanglement between the QFT and extra degree of freedom

$$|\hat{\omega}\rangle = |\omega\rangle \otimes |f\rangle \in \mathcal{H} \otimes L^2(\mathbb{R})$$

- Density matrix $\rho_{\hat{\omega}}$ can be explicitly computed

[CLPW, Jensen, Sorce, Speranza]

Classical-Quantum States

- No entanglement between the QFT and extra degree of freedom

$$|\hat{\omega}\rangle = |\omega\rangle \otimes |f\rangle \in \mathcal{H} \otimes L^2(\mathbb{R})$$

- Density matrix $\rho_{\hat{\omega}}$ can be explicitly computed
- For slowly-varying f

[CLPW, Jensen, Sorce, Speranza]

$$S_{vN}(\rho_{\hat{\omega}}) \approx \langle f|\boldsymbol{X}|f\rangle - S_{rel}(\omega|\omega_0) + S(\rho_f)$$

"Position" on $L^2(\mathbb{R})$

State used in crossed product algebra construction

"Classical" probability distribution $|f(X)|^2$

Classical-Quantum States

- No entanglement between the QFT and extra degree of freedom

$$|\hat{\omega}\rangle = |\omega\rangle \otimes |f\rangle \in \mathcal{H} \otimes L^2(\mathbb{R})$$

- Density matrix $\rho_{\hat{\omega}}$ can be explicitly computed
- For slowly-varying f

[CLPW, Jensen, Sorce, Speranza]

$$S_{vN}(\rho_{\hat{\omega}}) \approx \langle f|\boldsymbol{X}|f\rangle - S_{rel}(\omega|\omega_0) + S(\rho_f)$$

"Position" on $L^2(\mathbb{R})$

State used in crossed product algebra construction

"Classical" probability distribution $|f(X)|^2$

- For black holes, $\boldsymbol{X} = \delta^2 \boldsymbol{A}$ and one finds [Wall, CPW]

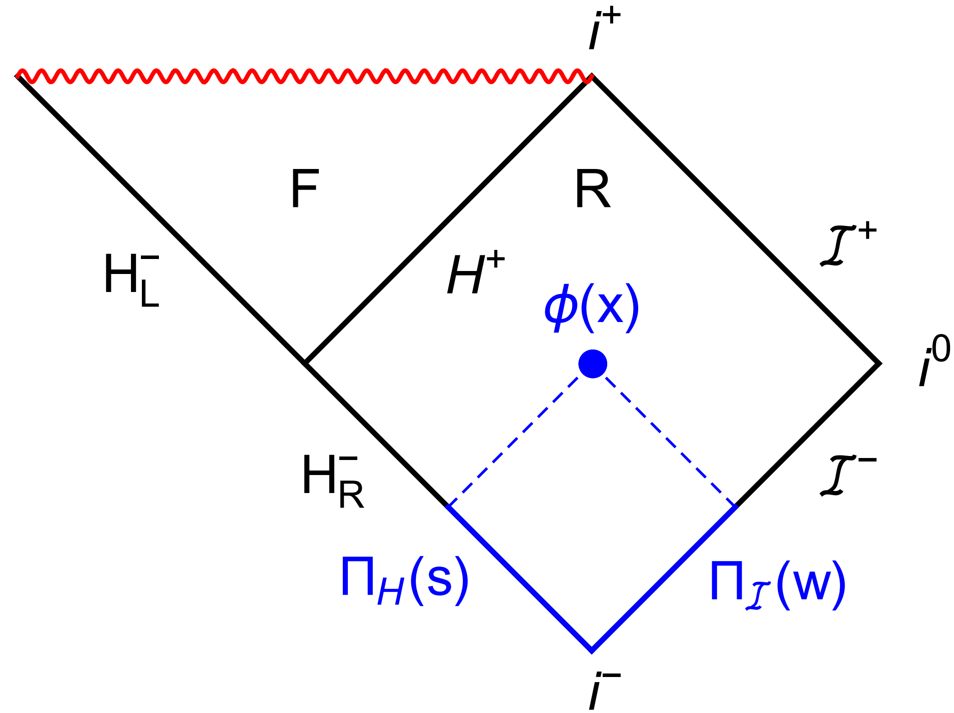
$$S_{vN}(\rho_{\hat{\omega}}) \approx S_{gen}(\mathcal{B}) + S(\rho_f) + C$$

Bifurcation surface

State-independent constant

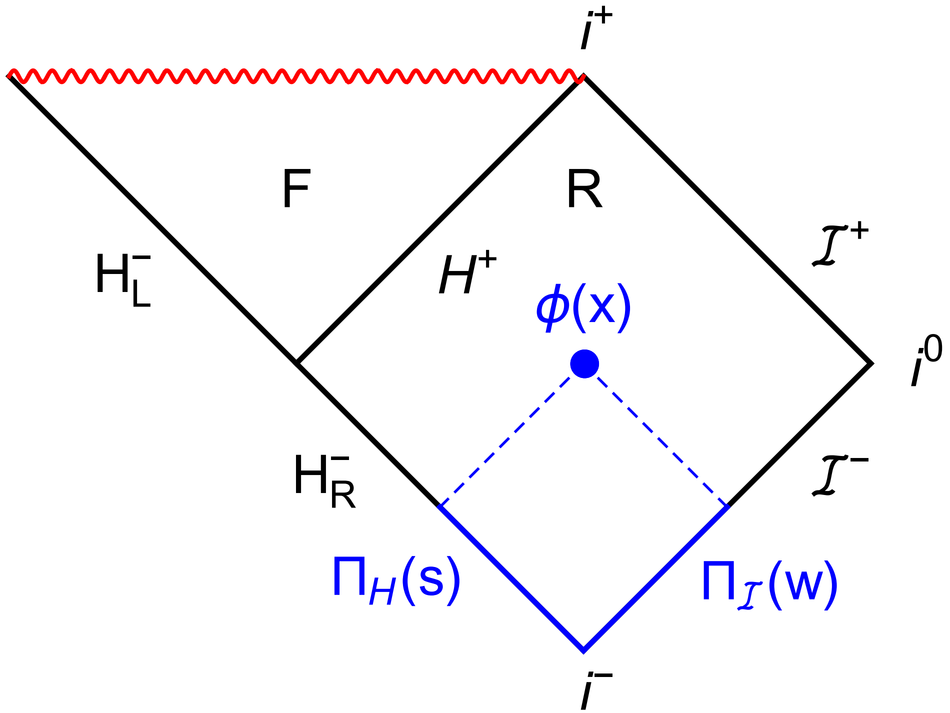
Quantization on Killing horizons

- For simplicity, treat a free massless scalar field



Quantization on Killing horizons

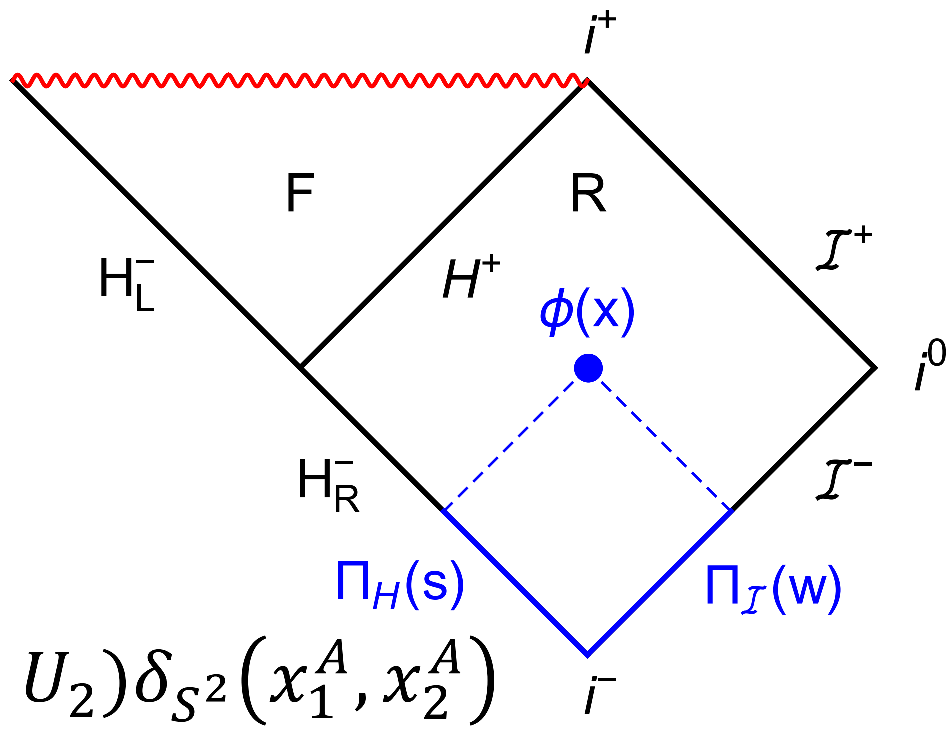
- For simplicity, treat a free massless scalar field
- Equation of motion gives
  $\boldsymbol{\phi}(f) = \boldsymbol{\Pi}_H(s) + \boldsymbol{\Pi}_{\mathcal{J}}(w)$
  $\Rightarrow \mathcal{A}_{R\cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{J}^-}$



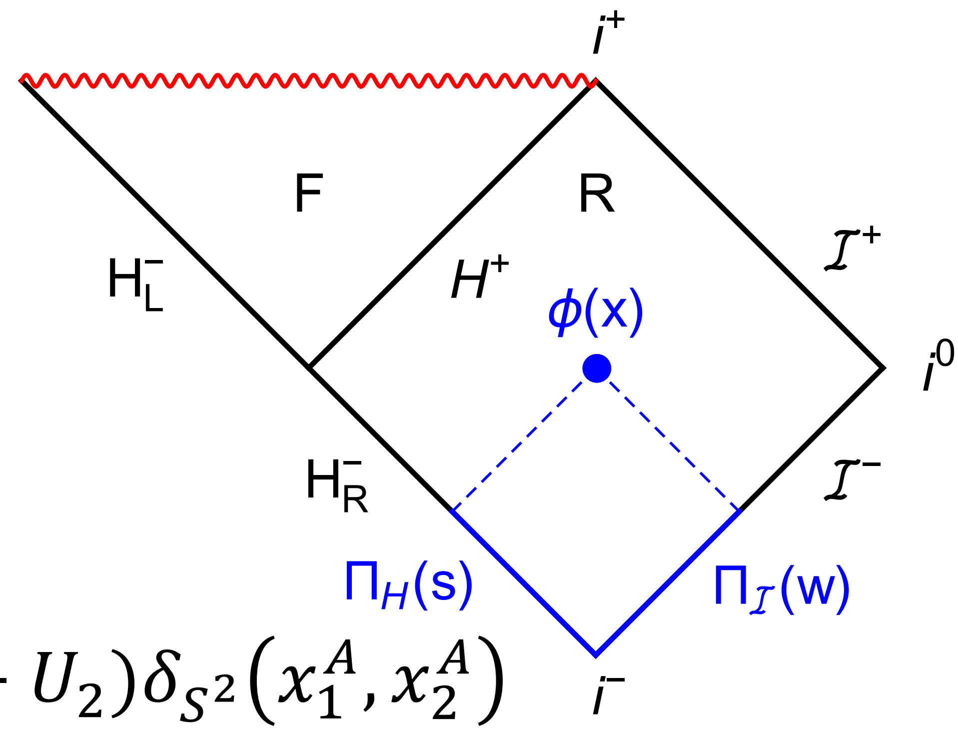
Quantization on Killing horizons

- For simplicity, treat a free massless scalar field
- Equation of motion gives $\boldsymbol{\phi}(f) = \boldsymbol{\Pi}_H(s) + \boldsymbol{\Pi}_{\mathcal{J}}(w)$
$\Rightarrow \mathcal{A}_{R\cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{J}^-}$
- One has algebra

$$[\boldsymbol{\Pi}_H(y_1), \boldsymbol{\Pi}_H(y_2)] = i\delta'(U_1 - U_2)\delta_{S^2}(x_1^A, x_2^A)$$



Quantization on Killing horizons

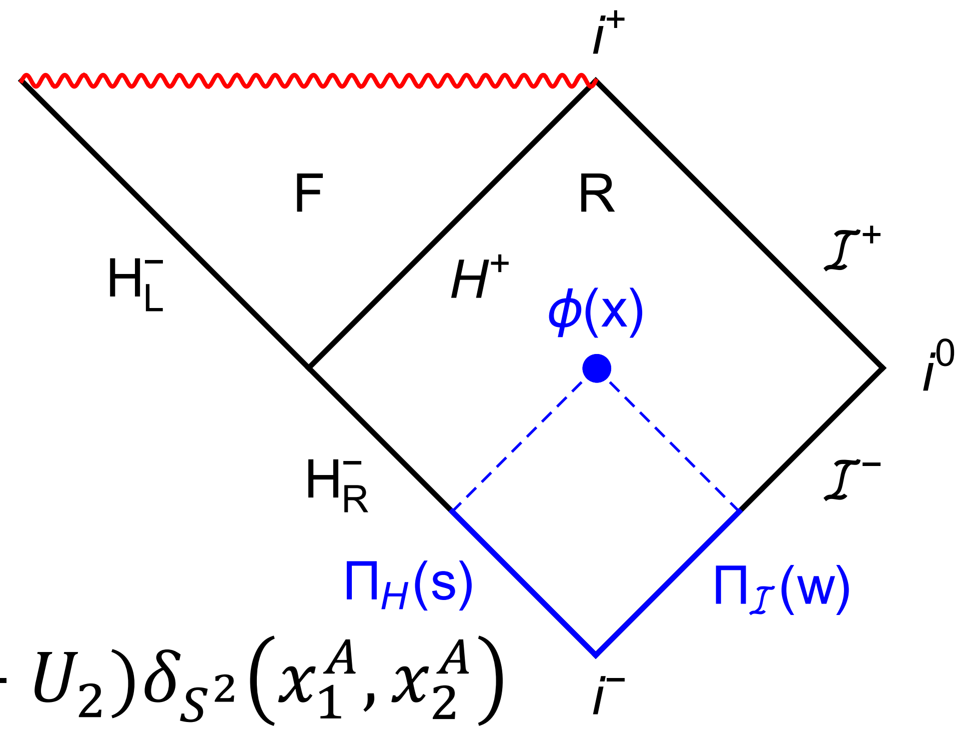


- For simplicity, treat a free massless scalar field
- Equation of motion gives $\boldsymbol{\phi}(f) = \mathbf{\Pi}_H(s) + \mathbf{\Pi}_{\mathcal{J}}(w)$ $\Rightarrow \mathcal{A}_{R\cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{J}^-}$
- One has algebra

$$[\mathbf{\Pi}_H(y_1), \mathbf{\Pi}_H(y_2)] = i\delta'(U_1 - U_2)\delta_{S^2}\left(x_1^A, x_2^A\right)$$

- The vacuum state $|\omega_0\rangle$ is thermal with $\beta = 2\pi/\kappa$ on the subalgebra $\mathfrak{U}(H_R^-)$

Quantization on Killing horizons



- For simplicity, treat a free massless scalar field
- Equation of motion gives
  $\boldsymbol{\phi}(f) = \boldsymbol{\Pi}_H(s) + \boldsymbol{\Pi}_{\mathcal{I}}(w)$
  $\Rightarrow \mathcal{A}_{R\cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{I}^-}$
- One has algebra

$$[\boldsymbol{\Pi}_H(y_1), \boldsymbol{\Pi}_H(y_2)] = i\delta'(U_1 - U_2)\delta_{S^2}\left(x_1^A, x_2^A\right)$$

- The vacuum state $|\omega_0\rangle$ is thermal with $\beta = 2\pi/\kappa$ on the subalgebra $\mathfrak{U}(H_R^-)$
- On H^- modular flow is Killing time translation:
  $-\log\Delta_{\omega_0} \equiv \boldsymbol{H}_{\omega_0} = \beta \boldsymbol{F}_\xi^H, \quad \left[\boldsymbol{F}_\xi^H, \boldsymbol{\Pi}_H(f)\right] = i\boldsymbol{\Pi}_H\left(\mathcal{L}_\xi f\right)$

Perturbation Theory

- Consider $g_{ab}(\lambda)$, $\Phi(\lambda)$ solving

$$G_{ab}(\lambda) = 8\pi G_N T_{ab}(\lambda), \qquad \Box_{g(\lambda)}\Phi(\lambda) = 0$$

Perturbation Theory

- Consider $g_{ab}(\lambda)$, $\Phi(\lambda)$ solving
$$G_{ab}(\lambda) = 8\pi G_N T_{ab}(\lambda), \qquad \Box_{g(\lambda)}\Phi(\lambda) = 0$$
- Expand around a known vacuum solution of Einstein's equation, i.e. $g_{ab}(0) = g^0_{ab}, \Phi(0) = 0$

Perturbation Theory

- Consider $g_{ab}(\lambda)$, $\Phi(\lambda)$ solving
$$G_{ab}(\lambda) = 8\pi G_N T_{ab}(\lambda), \qquad \Box_{g(\lambda)}\Phi(\lambda) = 0$$
- Expand around a known vacuum solution of Einstein's equation, i.e. $g_{ab}(0) = g^0_{ab}, \Phi(0) = 0$
- Write $\delta^n g_{ab} \equiv \left.\frac{d^n g_{ab}}{d\lambda^n}\right|_{\lambda=0}$ and $\delta^n \Phi \equiv \left.\frac{d^n \Phi}{d\lambda^n}\right|_{\lambda=0}$. One obtains free scalar and gravitons on g^0_{ab} at $O(\lambda)$

Perturbation Theory

- Consider $g_{ab}(\lambda)$, $\Phi(\lambda)$ solving
$$G_{ab}(\lambda) = 8\pi G_N T_{ab}(\lambda), \qquad \Box_{g(\lambda)}\Phi(\lambda) = 0$$
- Expand around a known vacuum solution of Einstein's equation, i.e. $g_{ab}(0) = g^0_{ab}, \Phi(0) = 0$
- Write $\delta^n g_{ab} \equiv \left.\frac{d^n g_{ab}}{d\lambda^n}\right|_{\lambda=0}$ and $\delta^n \Phi \equiv \left.\frac{d^n \Phi}{d\lambda^n}\right|_{\lambda=0}$. One obtains free scalar and gravitons on g^0_{ab} at $O(\lambda)$
- With $\phi \equiv \delta\Phi$, for any Killing vector X of g^0_{ab} we get a "flux-charge relation" $F_X = \int_\Sigma \, dQ_X(g^0, \phi, \delta^2 g, \delta^2\Phi)$, with $[\boldsymbol{F}_X, \boldsymbol{\phi}(f)] = i\boldsymbol{\phi}(\mathcal{L}_X f)$ [Hollands, Wald]

Perturbation Theory

- Consider $g_{ab}(\lambda)$, $\Phi(\lambda)$ solving

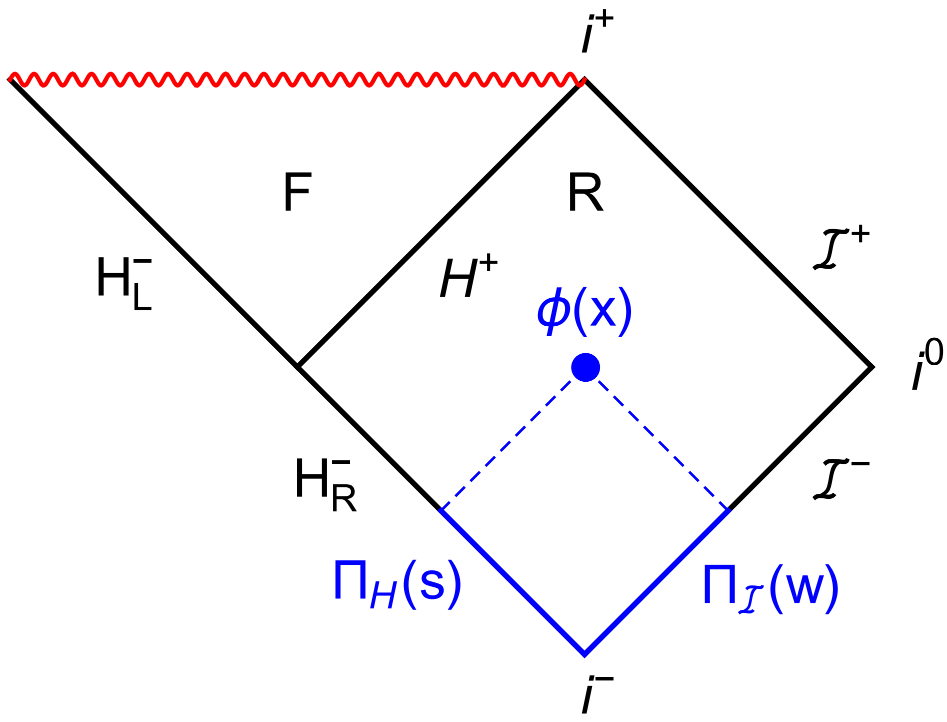
$$G_{ab}(\lambda) = 8\pi G_N T_{ab}(\lambda), \qquad \Box_{g(\lambda)}\Phi(\lambda) = 0$$

- Expand around a known vacuum solution of Einstein's equation, i.e. $g_{ab}(0) = g^0_{ab}, \Phi(0) = 0$
- Write $\delta^n g_{ab} \equiv \left.\frac{d^n g_{ab}}{d\lambda^n}\right|_{\lambda=0}$ and $\delta^n \Phi \equiv \left.\frac{d^n \Phi}{d\lambda^n}\right|_{\lambda=0}$. One obtains free scalar and gravitons on g^0_{ab} at $O(\lambda)$
- With $\phi \equiv \delta\Phi$, for any Killing vector X of g^0_{ab} we get a "flux-charge relation" $F_X = \int_\Sigma \; dQ_X(g^0, \phi, \delta^2 g, \delta^2\Phi)$, with $[\boldsymbol{F}_X, \boldsymbol{\phi}(f)] = i\boldsymbol{\phi}(\mathcal{L}_X f)$ [Hollands, Wald]
- Background isometries related to gravity charges!

Gravitational Charges

- On a Killing horizon for the horizon Killing vector ξ

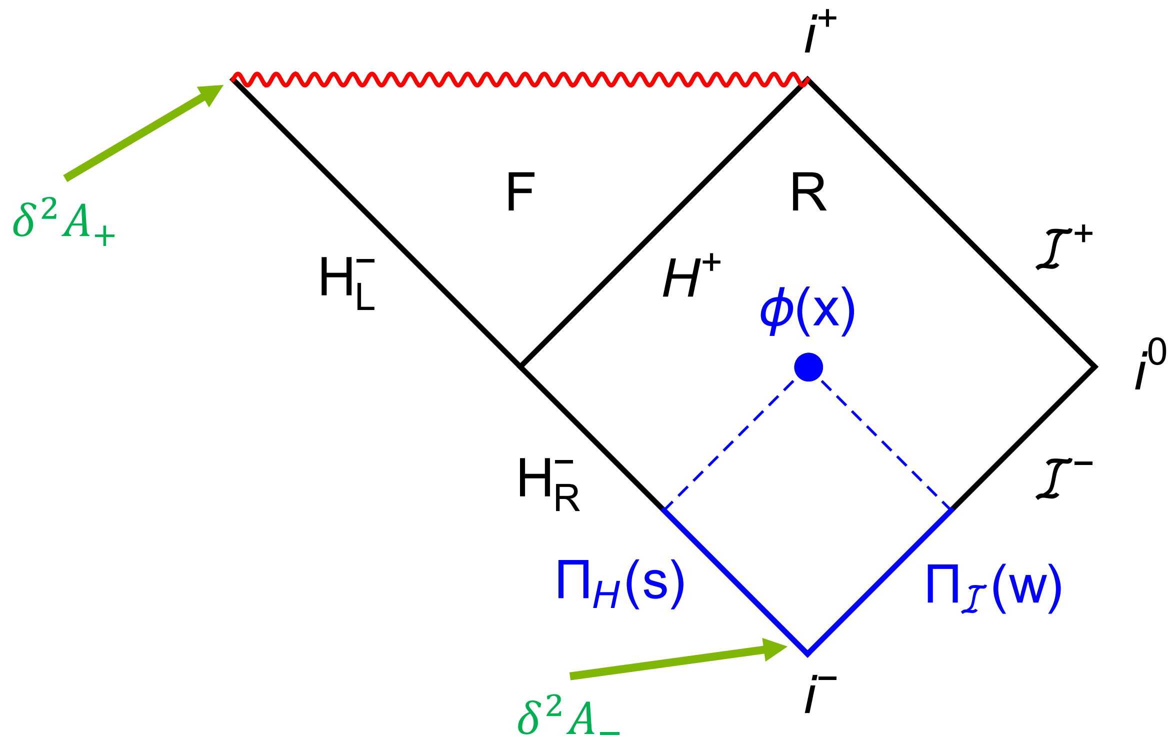
$$-4G_N \beta F_\xi^H = \delta^2 A_+ - \delta^2 A_-$$



Gravitational Charges

- On a Killing horizon for the horizon Killing vector ξ

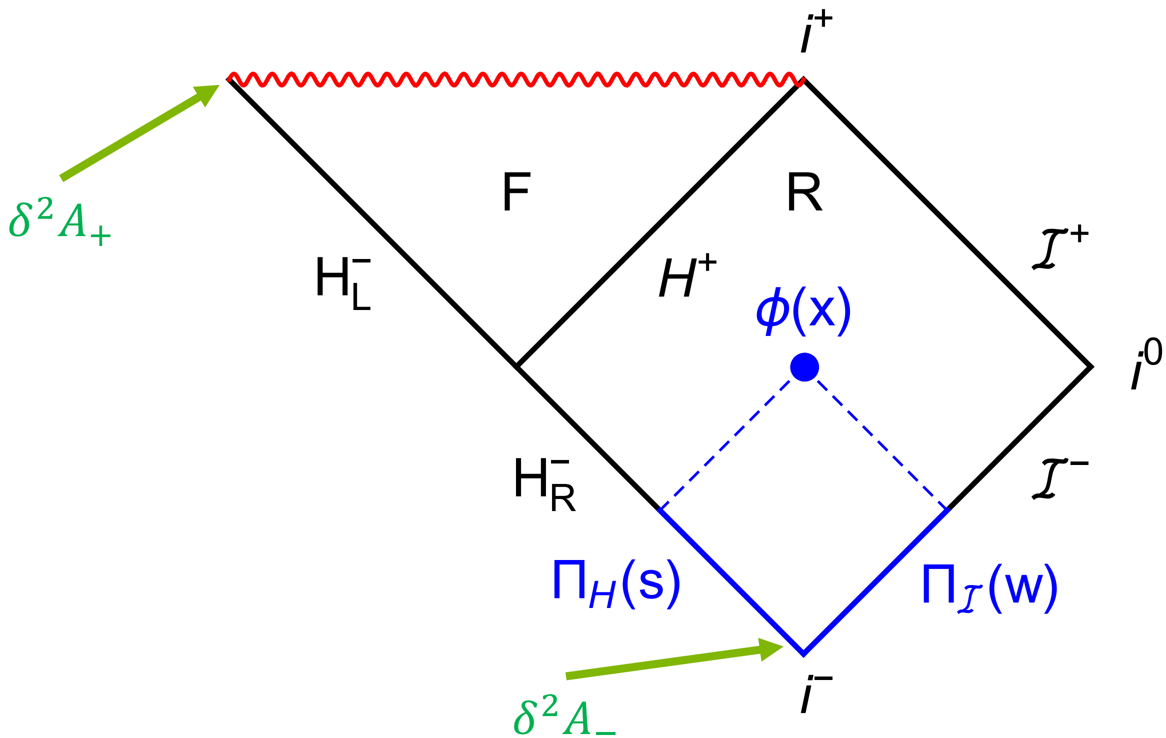
$$-4G_N \beta F_\xi^H = \delta^2 A_+ - \delta^2 A_-$$



Gravitational Charges

- On a Killing horizon for the horizon Killing vector ξ

$$-4G_N\beta F_\xi^H = \delta^2 A_+ - \delta^2 A_-$$



- Operators in R should commute with $\delta^2 A_+$

Dressed Observables

- Quantize the second-order gravitational charges as independent degrees of freedom on $L^2(\mathbb{R})$

Dressed Observables

- Quantize the second-order gravitational charges as independent degrees of freedom on $L^2(\mathbb{R})$
- Flux-charge relation $\Rightarrow$ only $\delta^2 \boldsymbol{A}_-$ is an independent degree of freedom, $\delta^2 \boldsymbol{A}_+ \equiv \delta^2 \boldsymbol{A}_- - 4G_N \beta \boldsymbol{F}_\xi^H$

Dressed Observables

- Quantize the second-order gravitational charges as independent degrees of freedom on $L^2(\mathbb{R})$
- Flux-charge relation $\Rightarrow$ only $\delta^2 \boldsymbol{A}_-$ is an independent degree of freedom, $\delta^2 \boldsymbol{A}_+ \equiv \delta^2 \boldsymbol{A}_- - 4G_N \beta \boldsymbol{F}_\xi^H$
- Extended Hilbert space is $\mathcal{H} \otimes L^2\left(\mathbb{R}_{\delta^2 \boldsymbol{A}_-}\right)$

Dressed Observables

- Quantize the second-order gravitational charges as independent degrees of freedom on $L^2(\mathbb{R})$
- Flux-charge relation ⇒ only $\delta^2 \boldsymbol{A}_-$ is an independent degree of freedom, $\delta^2 \boldsymbol{A}_+ \equiv \delta^2 \boldsymbol{A}_- - 4G_N \beta \boldsymbol{F}_\xi^H$
- Extended Hilbert space is $\mathcal{H} \otimes L^2\left(\mathbb{R}_{\delta^2 \boldsymbol{A}_-}\right)$
- Conjugate operator $\boldsymbol{t}_-$ is such that $\left[\frac{\delta^2 \boldsymbol{A}_-}{4G_N \beta}, \boldsymbol{t}_-\right] = i$

Dressed Observables

- Quantize the second-order gravitational charges as independent degrees of freedom on $L^2(\mathbb{R})$
- Flux-charge relation ⇒ only $\delta^2\boldsymbol{A}_-$ is an independent degree of freedom, $\delta^2\boldsymbol{A}_+ \equiv \delta^2\boldsymbol{A}_- - 4G_N\beta\boldsymbol{F}_\xi^H$
- Extended Hilbert space is $\mathcal{H} \otimes L^2\left(\mathbb{R}_{\delta^2\boldsymbol{A}_-}\right)$
- Conjugate operator $\boldsymbol{t}_-$ is such that $\left[\frac{\delta^2\boldsymbol{A}_-}{4G_N\beta}, \boldsymbol{t}_-\right] = i$
- "Dressed" operators that commute with $\delta^2\boldsymbol{A}_+$

$$\boldsymbol{\Pi}(s;\boldsymbol{t}_-) \equiv e^{i\boldsymbol{F}_\xi^H\boldsymbol{t}_-}\boldsymbol{\Pi}(s)e^{-i\boldsymbol{F}_\xi^H\boldsymbol{t}_-}$$

Dressed Observables

- Algebra of "dressed" operators on H_R^-

$$\mathfrak{A}_{ext}(H_R^-) = \{\boldsymbol{\Pi}(s; \boldsymbol{t}_-), \delta^2 \boldsymbol{A}_-\}'', \qquad \mathrm{supp}(s) \subset H_R^-$$

Dressed Observables

- Algebra of "dressed" operators on H_R^-

  $$\mathfrak{U}_{ext}(H_R^-) = \{\boldsymbol{\Pi}(s; \boldsymbol{t}_-), \delta^2 \boldsymbol{A}_-\}'', \qquad \mathrm{supp}(s) \subset H_R^-$$

- Unitarily equivalent to

$$\widehat{\mathfrak{U}}_{ext}(H_R^-) = \left\{\boldsymbol{\Pi}(s), \frac{\delta^2 \boldsymbol{A}_-}{4G_N} + \beta \boldsymbol{F}_\xi^H\right\}''$$

Dressed Observables

- Algebra of "dressed" operators on H_R^-

$$\mathfrak{A}_{ext}(H_R^-) = \{\boldsymbol{\Pi}(s;\boldsymbol{t}_-), \delta^2 \boldsymbol{A}_-\}'', \qquad \mathrm{supp}(s) \subset H_R^-$$

- Unitarily equivalent to

$$\hat{\mathfrak{A}}_{ext}(H_R^-) = \left\{\boldsymbol{\Pi}(s), \frac{\delta^2 \boldsymbol{A}_-}{4G_N} + \beta \boldsymbol{F}_\xi^H\right\}''$$

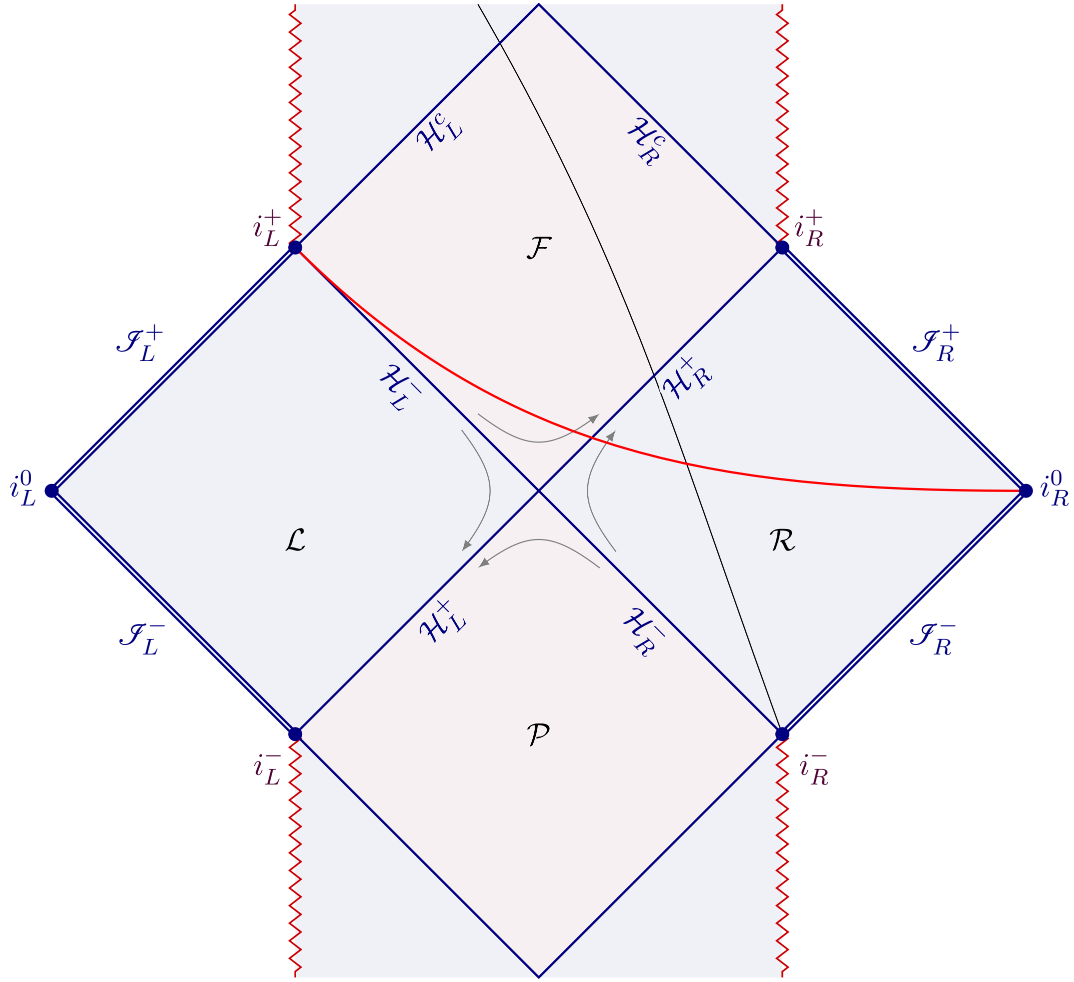
- Since $\beta \boldsymbol{F}_\xi^H = \boldsymbol{H}_{\omega_0}$ on H^- and $\mathfrak{A}(H_R^-)$ is type III_1, Takesaki's theorem implies $\mathfrak{A}_{ext}(H_R^-)$ is type II_∞

Dressed Observables

- Algebra of "dressed" operators on H_R^-
  $$\mathfrak{A}_{ext}(H_R^-) = \{\boldsymbol{\Pi}(s;\boldsymbol{t}_-), \delta^2 \boldsymbol{A}_-\}'', \qquad \mathrm{supp}(s) \subset H_R^-$$
- Unitarily equivalent to
  $$\widehat{\mathfrak{A}}_{ext}(H_R^-) = \left\{\boldsymbol{\Pi}(s), \frac{\delta^2 \boldsymbol{A}_-}{4G_N} + \beta \boldsymbol{F}_\xi^H\right\}''$$
- Since $\beta \boldsymbol{F}_\xi^H = \boldsymbol{H}_{\omega_0}$ on H^- and $\mathfrak{A}(H_R^-)$ is type III_1, Takesaki's theorem implies $\mathfrak{A}_{ext}(H_R^-)$ is type II_∞
- Thus, perturbative gravitational constraints deform the algebra on the "right" half of Killing horizon from type III_1 to type II_∞ allowing vN entropies to be defined

Asymptotically Flat Kerr Black Hole

- Consider the region $R \cup F$



Asymptotically Flat Kerr Black Hole

- Field algebra decomposition $\mathcal{A}_{R \cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{J}^-}$

Asymptotically Flat Kerr Black Hole

- Field algebra decomposition $\mathcal{A}_{R \cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{J}^-}$
- No equilibrium state exists on R!

Asymptotically Flat Kerr Black Hole

- Field algebra decomposition $\mathcal{A}_{R\cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{J}^-}$
- No equilibrium state exists on R!
- Instead, use the affine time vacuum on H^- and advanced time vacuum on $\mathcal{J}^-$ to obtain a GNS Hilbert space with von Neumann algebra
$$\mathfrak{U}(R \cup F) \simeq \mathfrak{U}(H^-) \otimes \mathfrak{U}(\mathcal{J}^-)$$

Asymptotically Flat Kerr Black Hole

- Field algebra decomposition $\mathcal{A}_{R\cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{J}^-}$
- No equilibrium state exists on R!
- Instead, use the affine time vacuum on H^- and advanced time vacuum on $\mathcal{J}^-$ to obtain a GNS Hilbert space with von Neumann algebra
$$\mathfrak{A}(R \cup F) \simeq \mathfrak{A}(H^-) \otimes \mathfrak{A}(\mathcal{J}^-)$$
- Algebra in R, $\mathfrak{A}(R) \simeq \mathfrak{A}(H_R^-) \otimes \mathfrak{A}(\mathcal{J}^-)$ is type III_1

Asymptotically Flat Kerr Black Hole

- Field algebra decomposition $\mathcal{A}_{R\cup F} \simeq \mathcal{A}_{H^-} \otimes \mathcal{A}_{\mathcal{J}^-}$
- No equilibrium state exists on R!
- Instead, use the affine time vacuum on H^- and advanced time vacuum on $\mathcal{J}^-$ to obtain a GNS Hilbert space with von Neumann algebra
$$\mathfrak{U}(R \cup F) \simeq \mathfrak{U}(H^-) \otimes \mathfrak{U}(\mathcal{J}^-)$$
- Algebra in R, $\mathfrak{U}(R) \simeq \mathfrak{U}(H_R^-) \otimes \mathfrak{U}(\mathcal{J}^-)$ is type III_1
- Flux-charge relations

$$\delta^2 \boldsymbol{A}_+ - \delta^2 \boldsymbol{A}_- = -4 G_N \beta \boldsymbol{F}_\xi^H, \qquad \delta^2 \boldsymbol{J}_+ - \delta^2 \boldsymbol{J}_- = \boldsymbol{F}_\psi^H$$
$$\xi = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \psi}$$

Asymptotically Flat Kerr Black Hole

- "Dressed" observables on H_R^-

$$\boldsymbol{\Pi}_H(s;\boldsymbol{t}_-,\boldsymbol{\psi}) \equiv e^{i\boldsymbol{F}_\psi^H\boldsymbol{\psi}} e^{i\boldsymbol{F}_\xi^H\boldsymbol{t}_-} \boldsymbol{\Pi}(s) e^{-i\boldsymbol{F}_\xi^H\boldsymbol{t}_-} e^{-i\boldsymbol{F}_\psi^H\boldsymbol{\psi}}$$

Asymptotically Flat Kerr Black Hole

- "Dressed" observables on H_R^-

$$\boldsymbol{\Pi}_H(s;\boldsymbol{t}_-,\boldsymbol{\psi}) \equiv e^{i\boldsymbol{F}_\psi^H\boldsymbol{\psi}}e^{i\boldsymbol{F}_\xi^H\boldsymbol{t}_-}\boldsymbol{\Pi}(s)e^{-i\boldsymbol{F}_\xi^H\boldsymbol{t}_-}e^{-i\boldsymbol{F}_\psi^H\boldsymbol{\psi}}$$

- Algebra of "dressed" operators in R

$$\mathfrak{U}_{ext}(R) = \{\boldsymbol{\Pi}_H(s;\boldsymbol{t}_-,\boldsymbol{\psi}),\boldsymbol{\Pi}_{\mathcal{J}}(w),\delta^2\boldsymbol{A}_-\}''$$

Asymptotically Flat Kerr Black Hole

- "Dressed" observables on H_R^-

$$\boldsymbol{\Pi}_H(s;\boldsymbol{t}_-,\boldsymbol{\psi}) \equiv e^{i\boldsymbol{F}_\psi^H \boldsymbol{\psi}} e^{i\boldsymbol{F}_\xi^H \boldsymbol{t}_-} \boldsymbol{\Pi}(s) e^{-i\boldsymbol{F}_\xi^H \boldsymbol{t}_-} e^{-i\boldsymbol{F}_\psi^H \boldsymbol{\psi}}$$

- Algebra of "dressed" operators in R

$$\mathfrak{U}_{ext}(R) = \{\boldsymbol{\Pi}_H(s;\boldsymbol{t}_-,\boldsymbol{\psi}), \boldsymbol{\Pi}_{\mathcal{J}}(w), \delta^2 \boldsymbol{A}_-\}''$$

- Tensor product of II_∞ on H_R^- and I_∞ on $\mathcal{J}^-$

Asymptotically Flat Kerr Black Hole

- "Dressed" observables on H_R^-

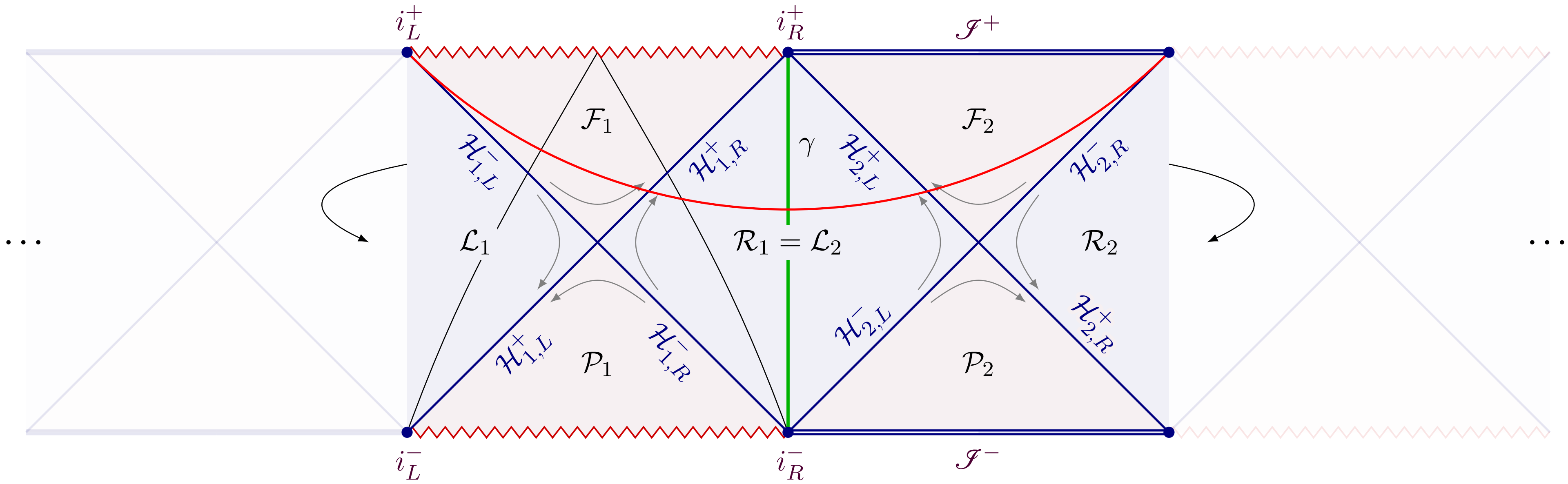
$$\boldsymbol{\Pi}_H(s;\boldsymbol{t}_-,\boldsymbol{\psi}) \equiv e^{i\boldsymbol{F}_\psi^H \boldsymbol{\psi}} e^{i\boldsymbol{F}_\xi^H \boldsymbol{t}_-} \boldsymbol{\Pi}(s) e^{-i\boldsymbol{F}_\xi^H \boldsymbol{t}_-} e^{-i\boldsymbol{F}_\psi^H \boldsymbol{\psi}}$$

- Algebra of "dressed" operators in R

$$\mathfrak{U}_{ext}(R) = \{\boldsymbol{\Pi}_H(s;\boldsymbol{t}_-,\boldsymbol{\psi}), \boldsymbol{\Pi}_{\mathcal{J}}(w), \delta^2 \boldsymbol{A}_-\}''$$

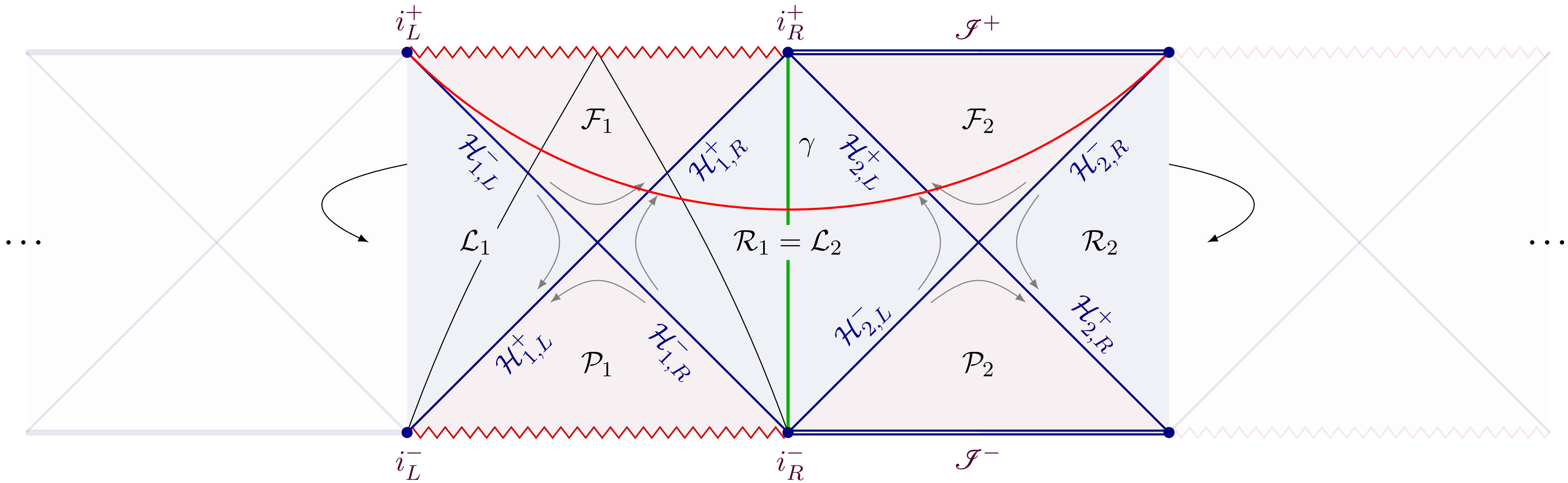
- Tensor product of II_∞ on H_R^- and I_∞ on $\mathcal{J}^-$
- For classical-quantum states with slowly-varying wavefunctions for the charges, the von Neumann entropy gives the generalized black hole entropy!

Schwarzschild de Sitter



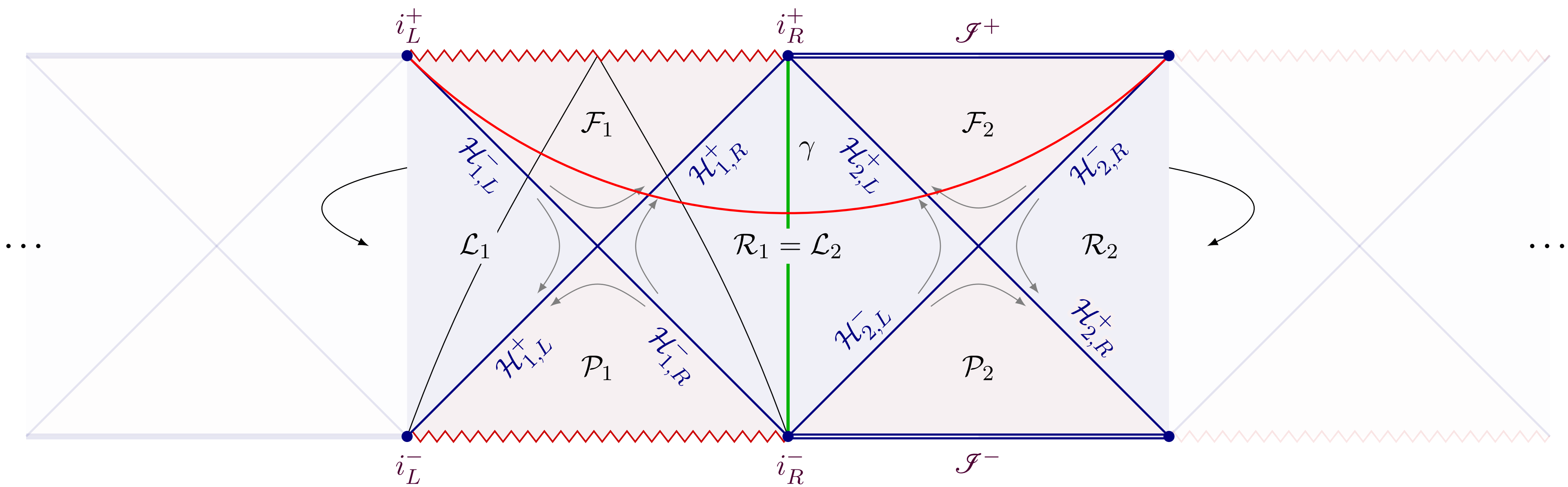
- Decomposition $\mathcal{A}_{R_1 \cup F_1 \cup F_2} \simeq \mathcal{A}_{H_1^-} \otimes \mathcal{A}_{H_2^-}$

Schwarzschild de Sitter



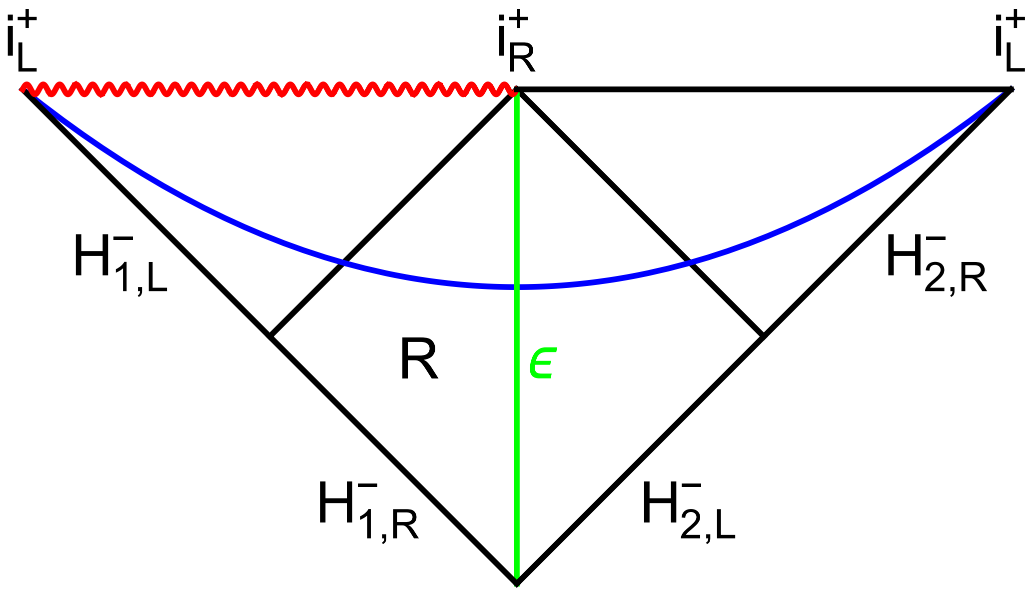
- Decomposition $\mathcal{A}_{R_1 \cup F_1 \cup F_2} \simeq \mathcal{A}_{H_1^-} \otimes \mathcal{A}_{H_2^-}$
- No equilibrium state in R_1 due to horizons of different temperatures!

Schwarzschild de Sitter



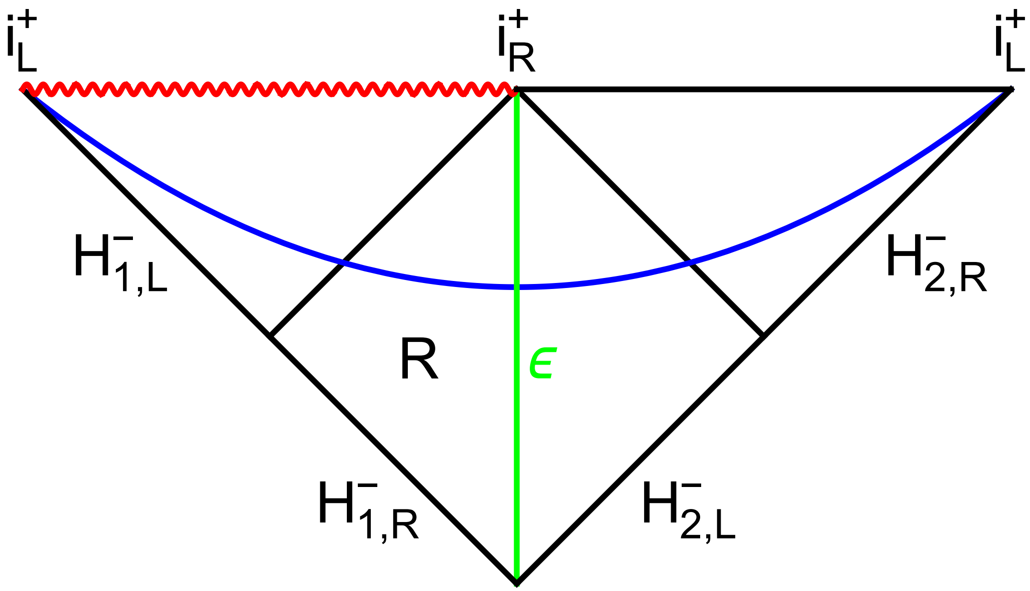
- Decomposition $\mathcal{A}_{R_1 \cup F_1 \cup F_2} \simeq \mathcal{A}_{H_1^-} \otimes \mathcal{A}_{H_2^-}$
- No equilibrium state in R_1 due to horizons of different temperatures!
- Instead, use vacuum state for affine time translations along each horizon *separately*

Schwarzschild de Sitter



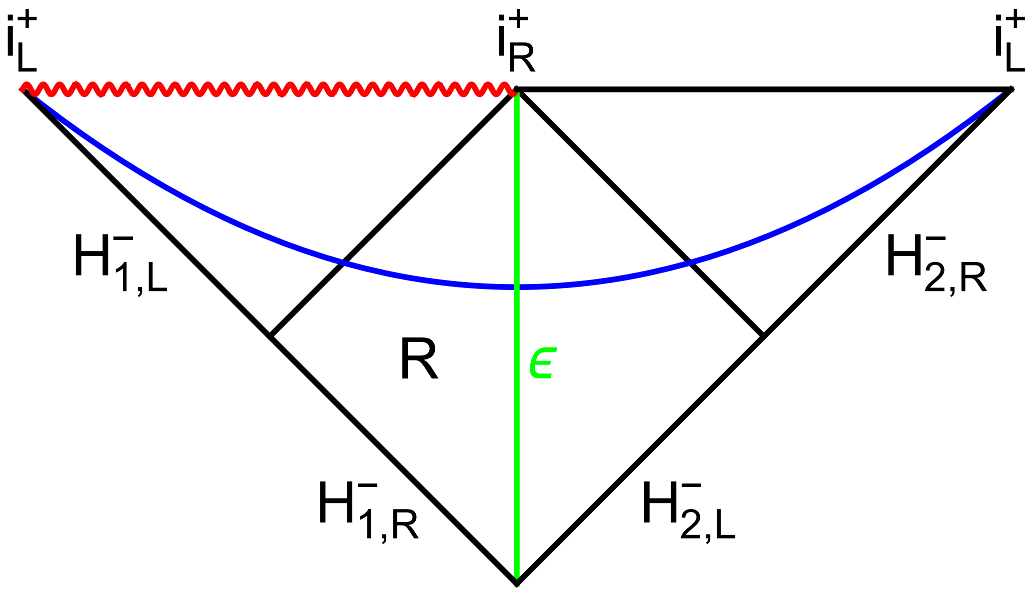
- Global constraint ⇒ $\left[\boldsymbol{F}_\xi, \boldsymbol{\phi}(f)\right] = 0$

Schwarzschild de Sitter



- Global constraint $\Rightarrow \left[\boldsymbol{F}_\xi, \boldsymbol{\phi}(f)\right] = 0$
- Must add an observer to obtain nontrivial algebra

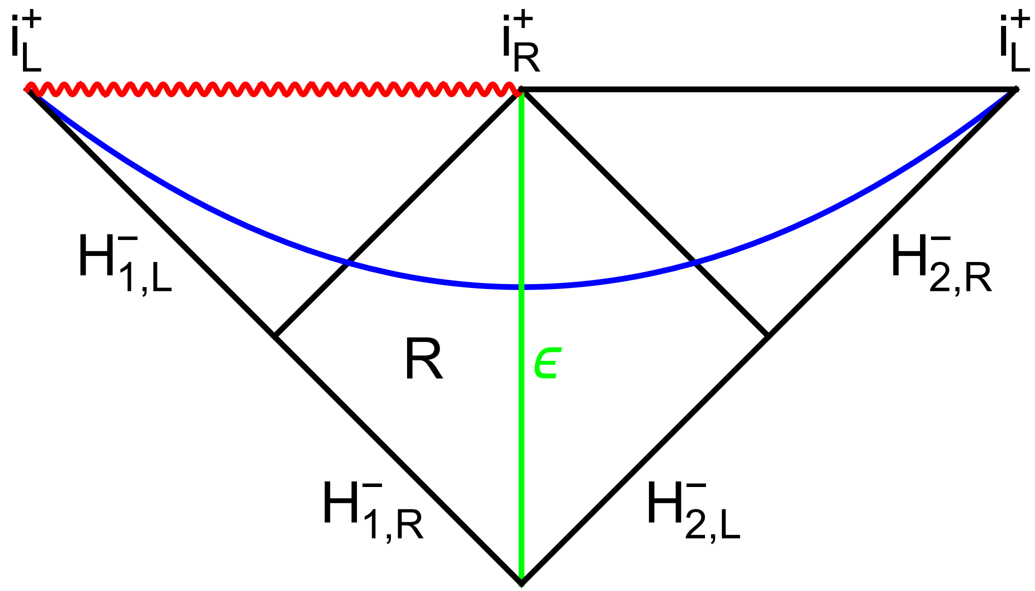
Schwarzschild de Sitter



- Global constraint ⇒ $\left[\boldsymbol{F}_\xi, \boldsymbol{\phi}(f)\right] = 0$
- Must add an observer to obtain nontrivial algebra
- Flux-charge relations on each horizon ⇒ algebra

$$\mathfrak{U}_{ext}(R) = \mathfrak{U}_{ext}\left(H_{1,R}^-\right) \otimes \mathfrak{U}_{ext}\left(H_{2,L}^-\right)$$

Schwarzschild de Sitter

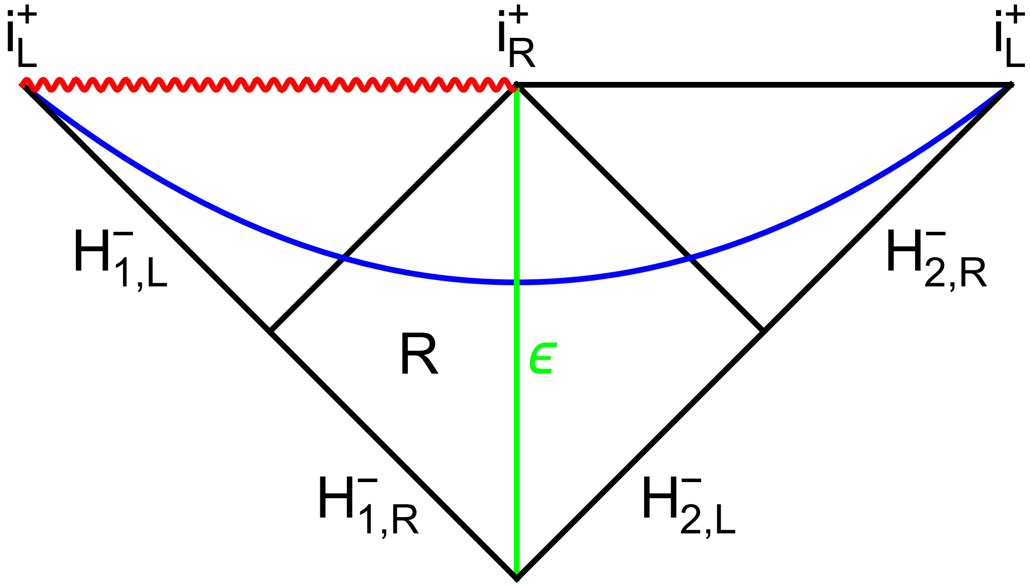


- Global constraint ⇒ $\left[\boldsymbol{F}_\xi, \boldsymbol{\phi}(f)\right] = 0$
- Must add an observer to obtain nontrivial algebra
- Flux-charge relations on each horizon ⇒ algebra

$$\mathfrak{U}_{ext}(R) = \mathfrak{U}_{ext}\left(H^-_{1,R}\right) \otimes \mathfrak{U}_{ext}\left(H^-_{2,L}\right)$$

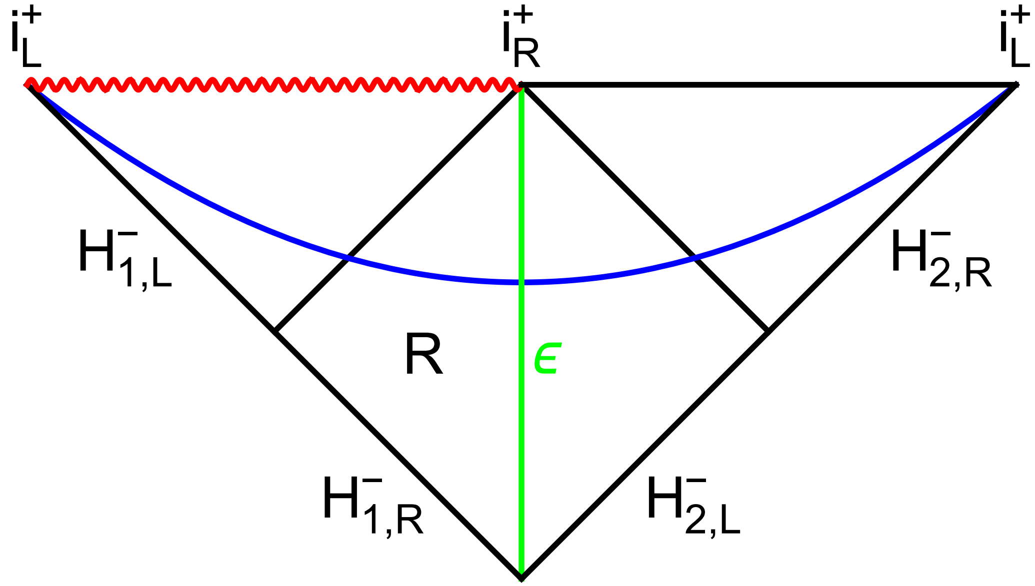
- "Matching condition" $\boldsymbol{\epsilon} = -\frac{\delta^2 \boldsymbol{A}^1_-}{4G_N\beta_1} - \frac{\delta^2 \boldsymbol{A}^2_-}{4G_N\beta_2}$

Schwarzschild de Sitter



- Resulting algebra is type II_∞

Schwarzschild de Sitter



- Resulting algebra is type II_∞
- For classical-quantum states slowly varying in both perturbed areas, the von Neumann entropy is the generalized entropy

$$S_{gen}(\hat{\omega}) = \hat{\omega}\left(\frac{\boldsymbol{A}_{B_1}}{4G_N}\right) + \hat{\omega}\left(\frac{\boldsymbol{A}_{B_2}}{4G_N}\right) + S_{vN}\left(\omega\Big|_R\right)$$

Future directions

- Construction for general horizon cuts and applications to the generalized second law
- Understand notions of entropy associated to general bulk surfaces [Jensen, Sorce, Speranza `23]
- Subtleties for near-extremal black holes?
- Recovering a type I algebra from the bulk perspective?

Thank you!