Complexity Equals (Almost) Anything

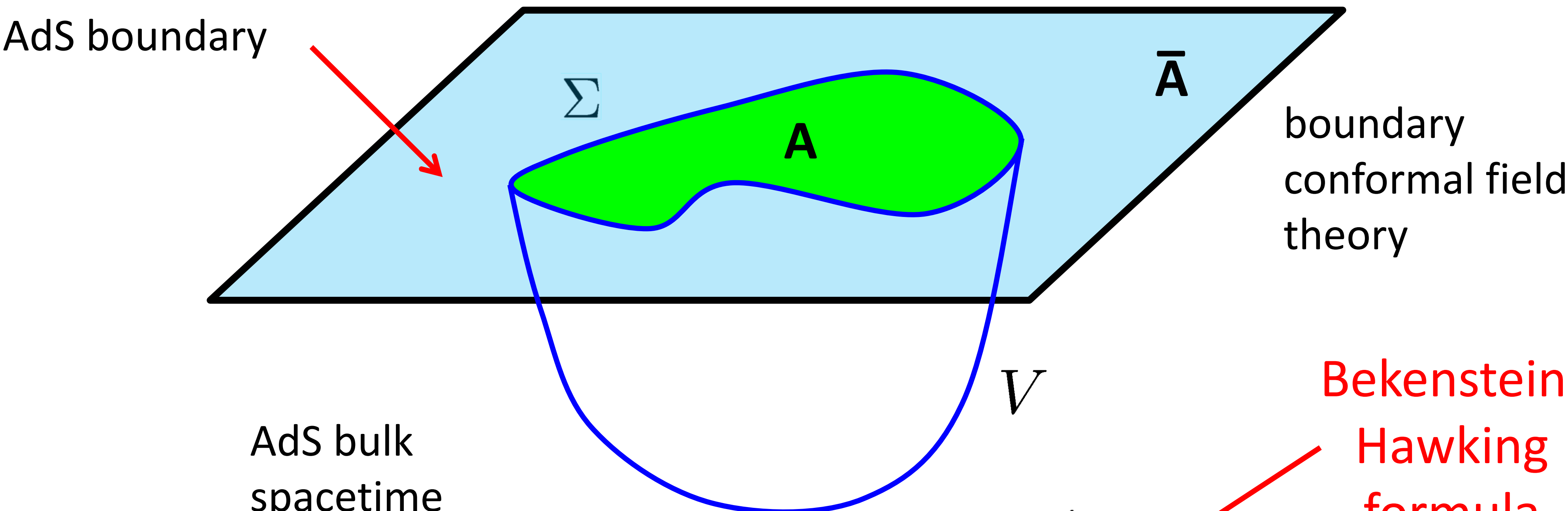
Alexandre Belin, RCM, Shan-Ming Ruan, Gabor Sarosi & Antony Speranza
[arXiv:2111.02429; arXiv:2210.09647]
Eivind Jorstad, RCM & Shan-Ming Ruan [arXiv:2304.05453]

Holographic Entanglement Entropy:

(Ryu & Takayanagi `06)

- CFT dof within **A** described by density matrix $\rho_A = Tr_{\bar{A}}(|\psi\rangle\langle\psi|)$

→ calculate von Neumann entropy: $S(A) = -Tr\left[\rho_A \, \log \rho_A\right]$



$$S(A) \;=\; \min_{\partial V \sim \Sigma} \; \frac{A_V}{4G_N}$$

Bekenstein-Hawking formula

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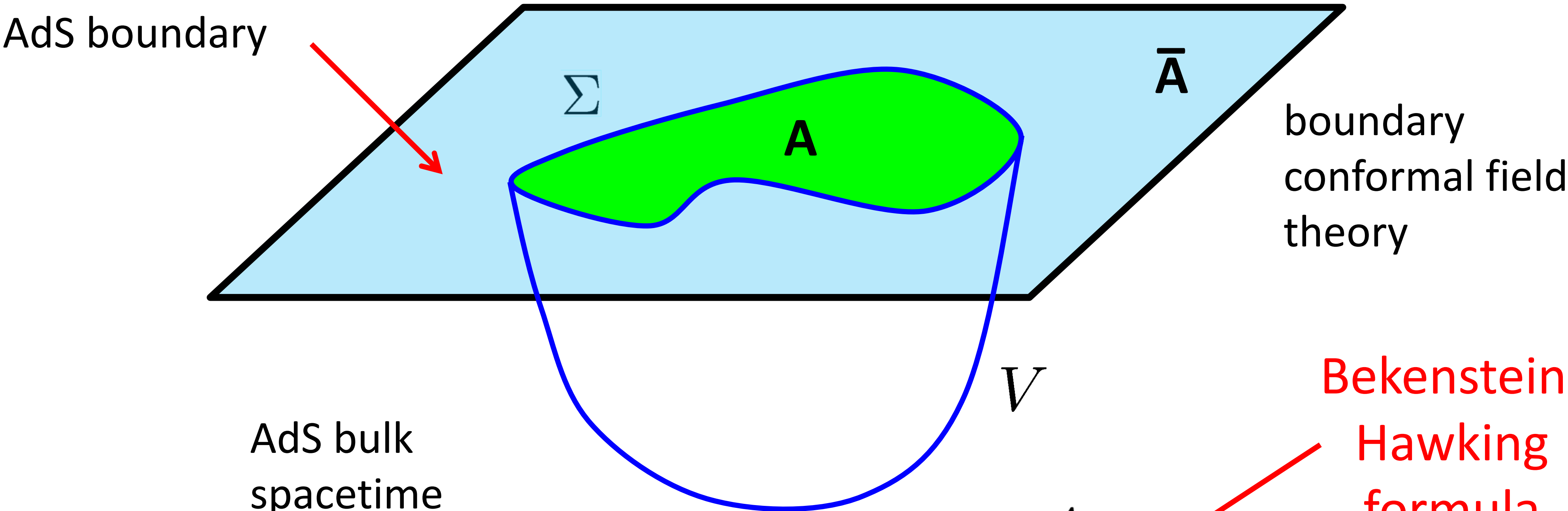

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- holographic EE is a fruitful forum for bulk-boundary dialogue:

⟶ new lessons about quantum field theories

⟶ new lessons about quantum gravity

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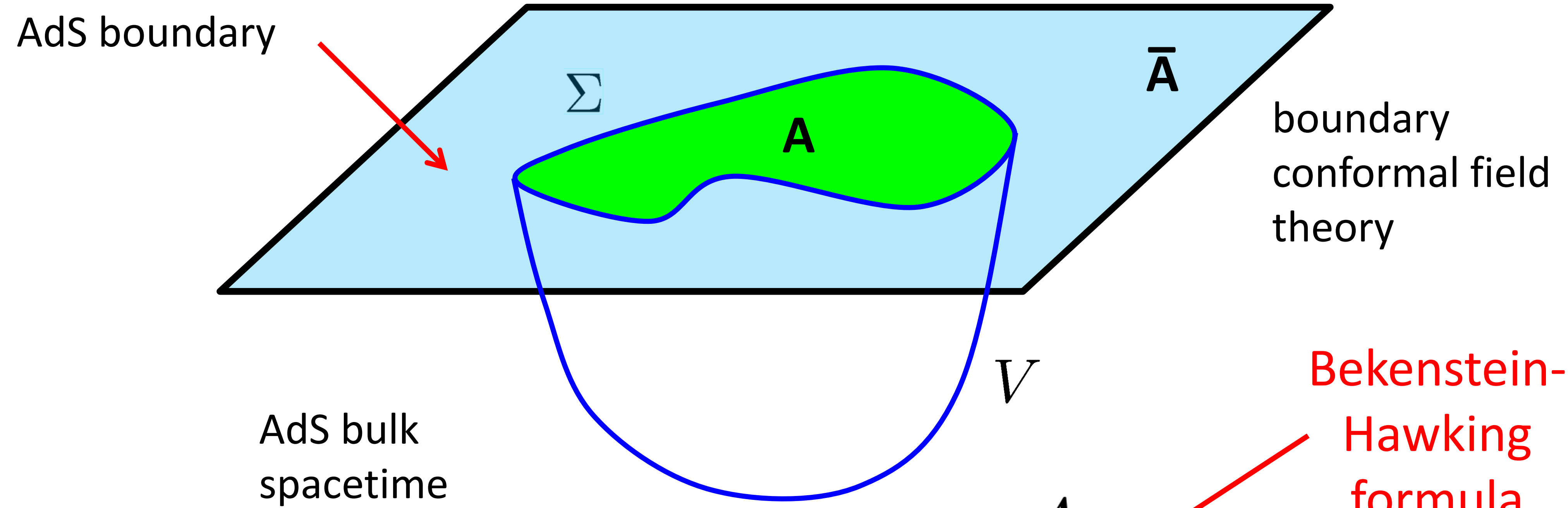

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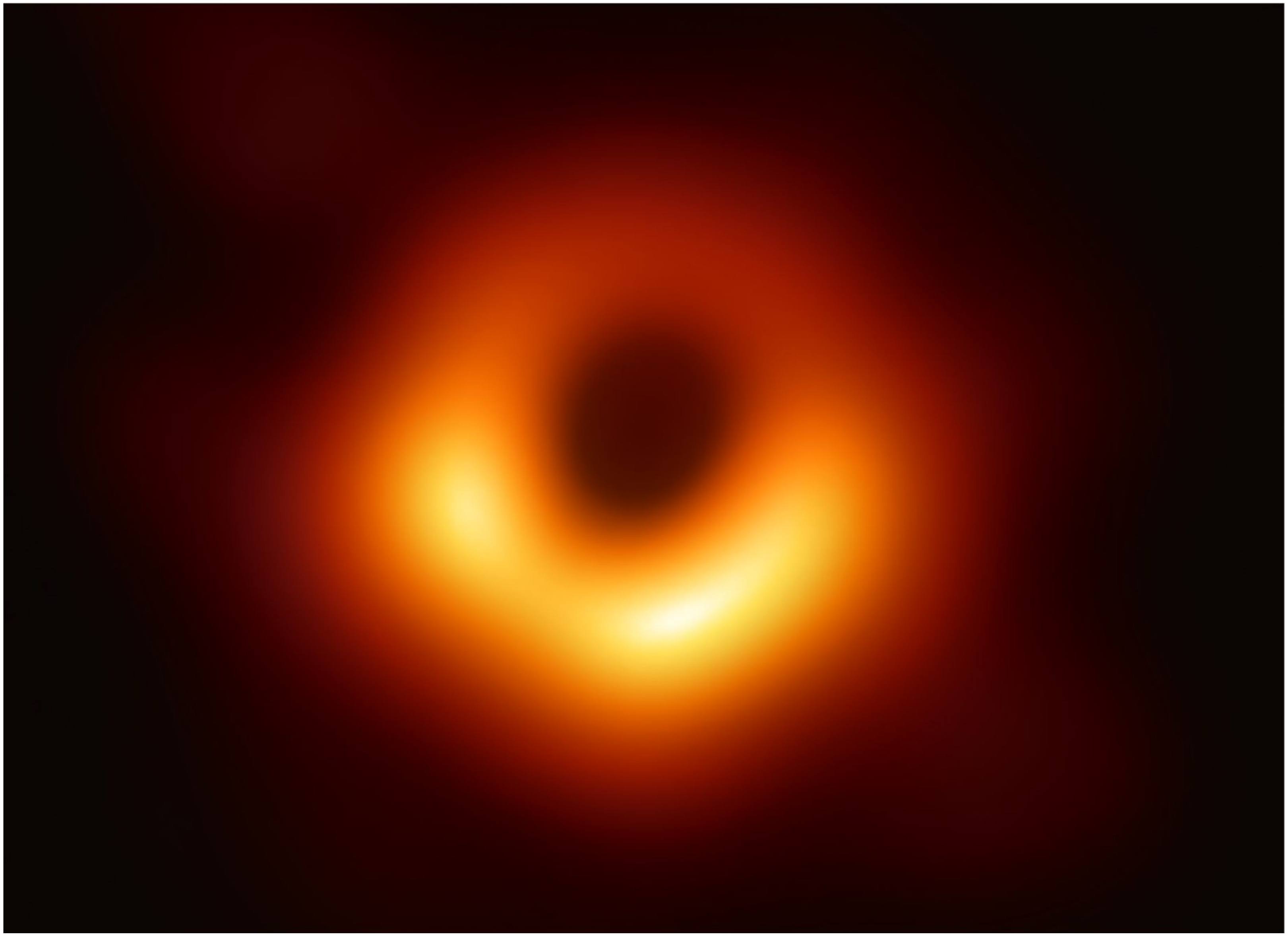
Spacetime Geometry = Entanglement

(van Raamsdonk `10)

Susskind: Entanglement Entropy is not enough!

- “to understand the rich geometric structures that exist behind black hole horizons and which are predicted by general relativity.”



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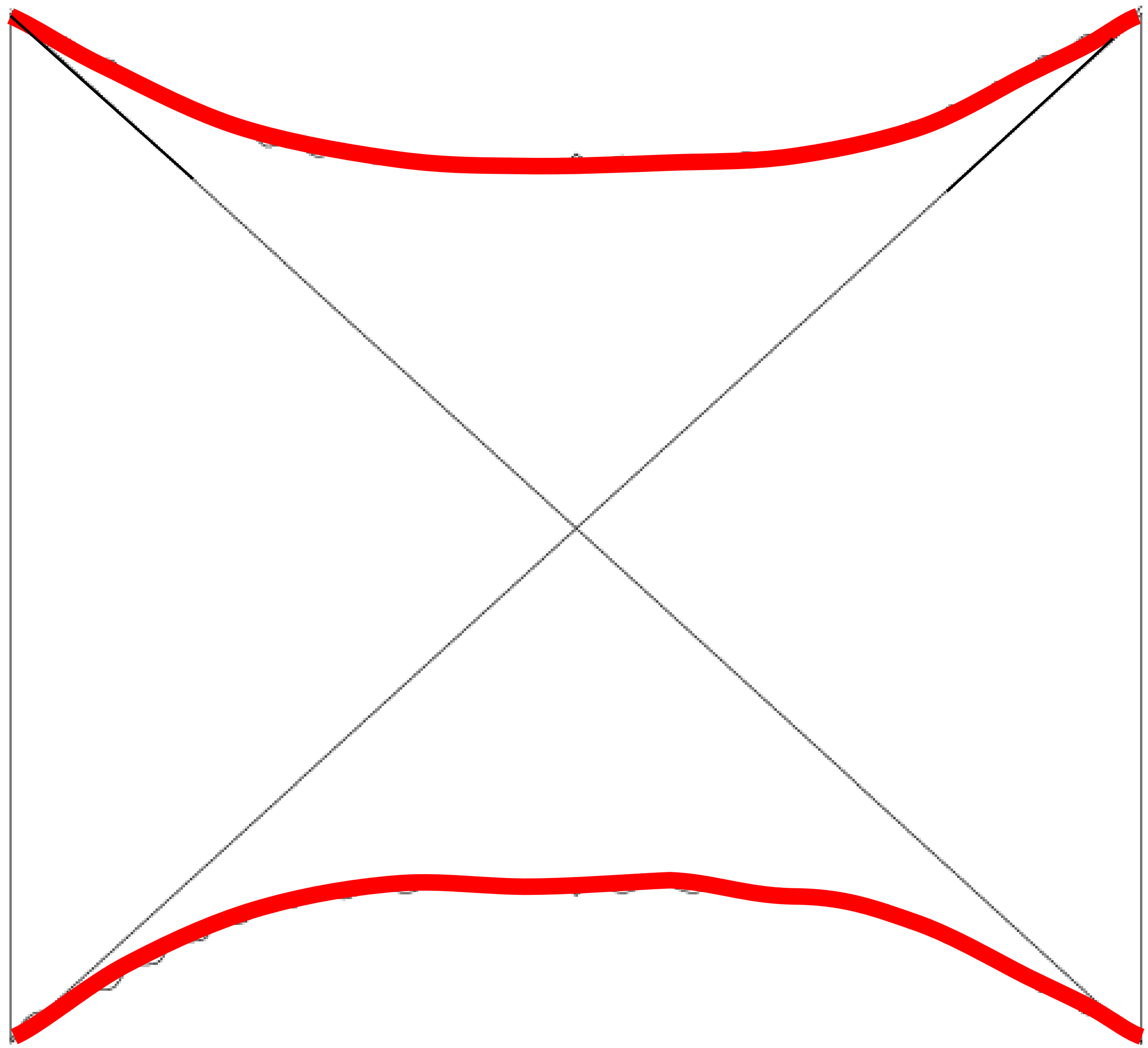

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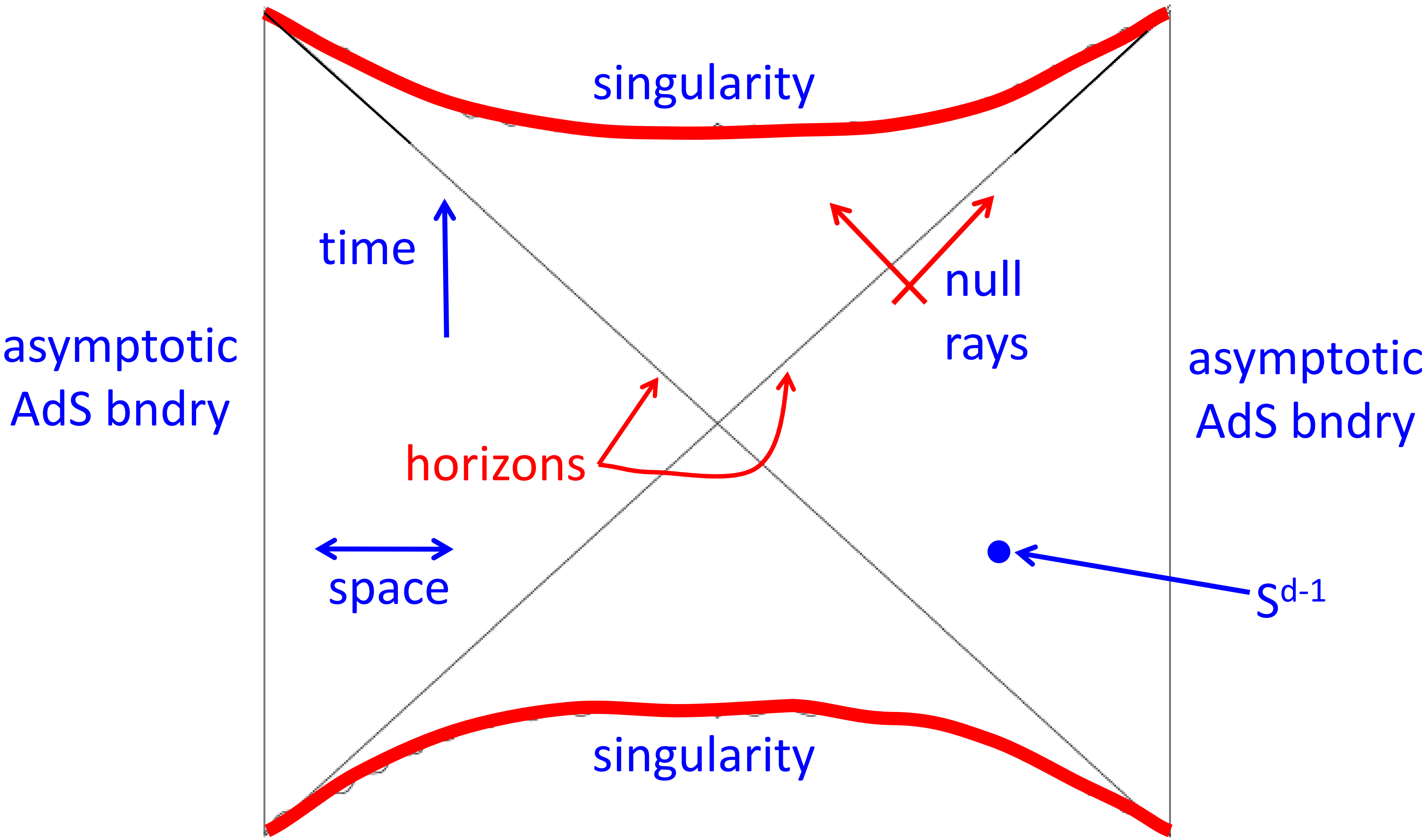

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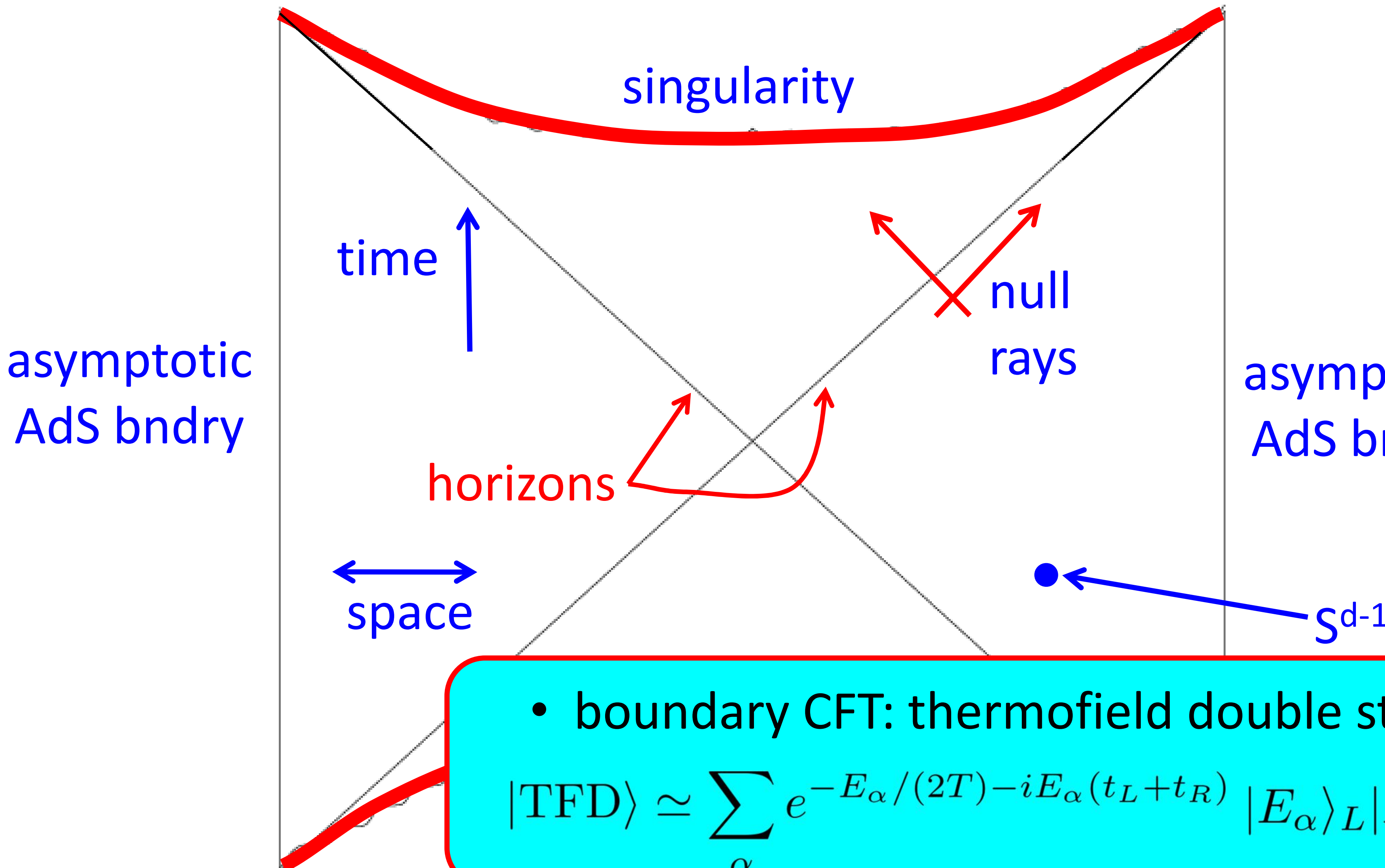

- boundary CFT: thermofield double state

$$|\mathrm{TFD}\rangle \simeq \sum_{\alpha} e^{-E_\alpha/(2T)-iE_\alpha(t_L+t_R)}\,|E_\alpha\rangle_L |E_\alpha\rangle_R$$

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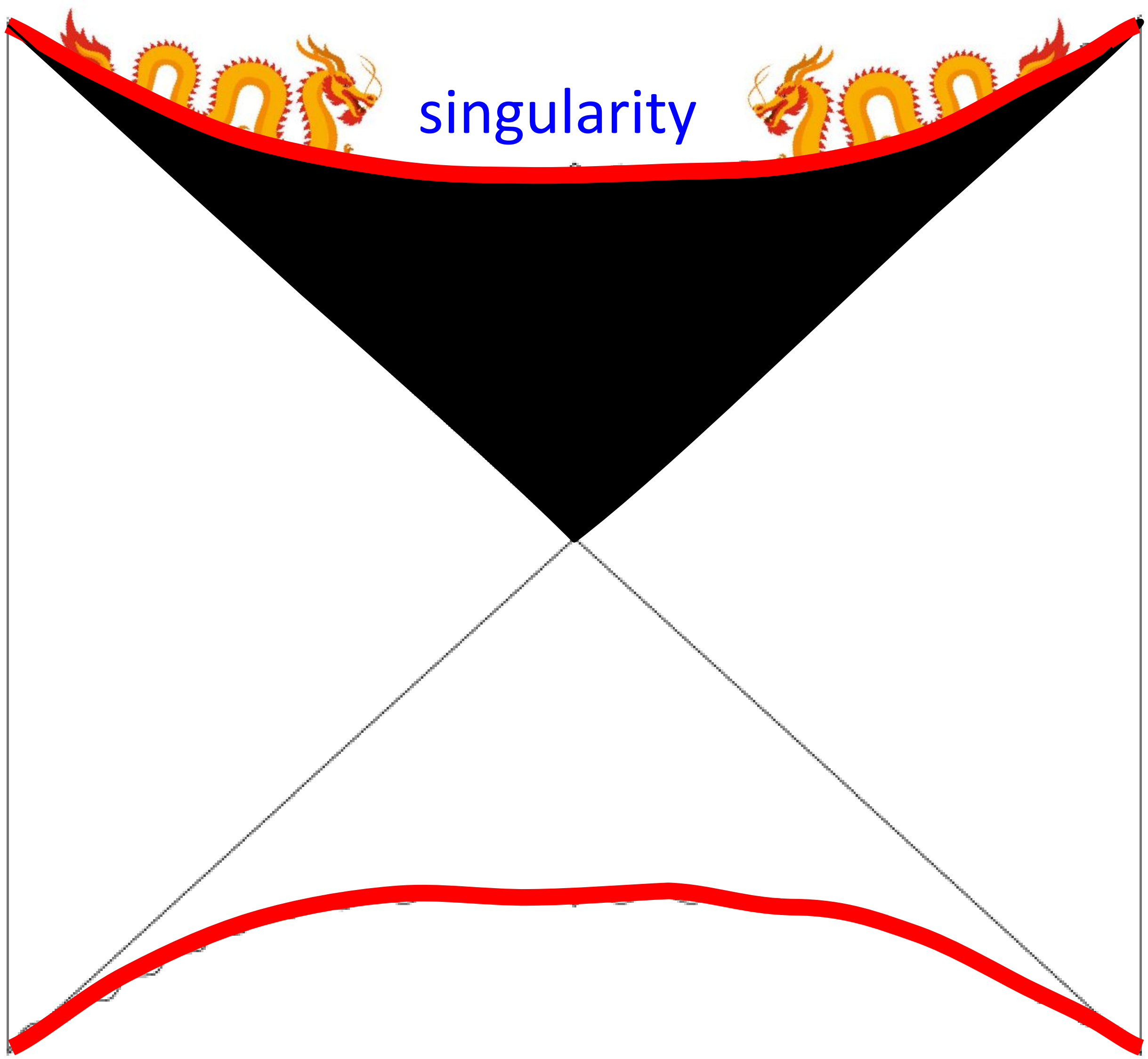

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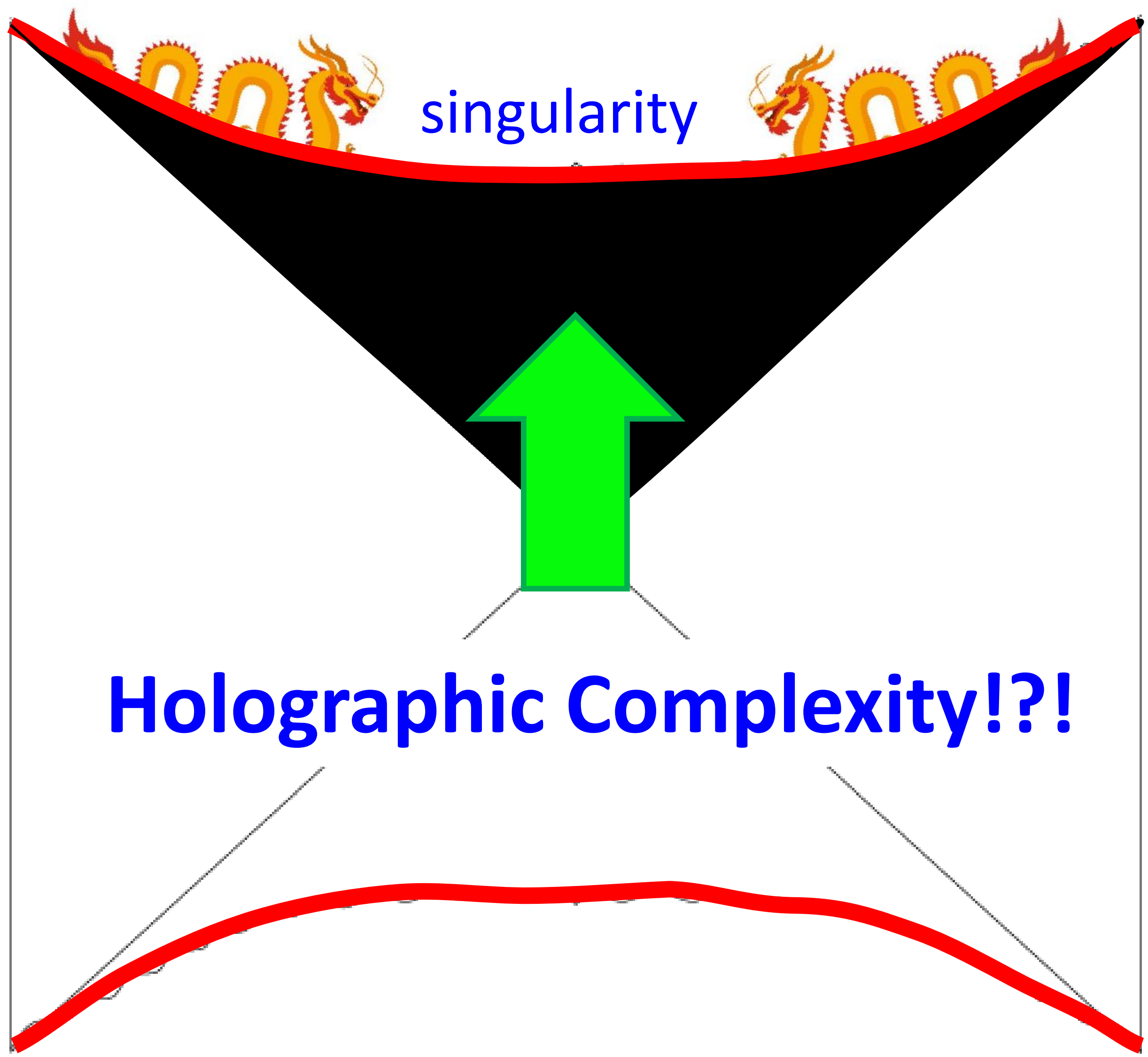

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eg, $|00000\cdots0\rangle$

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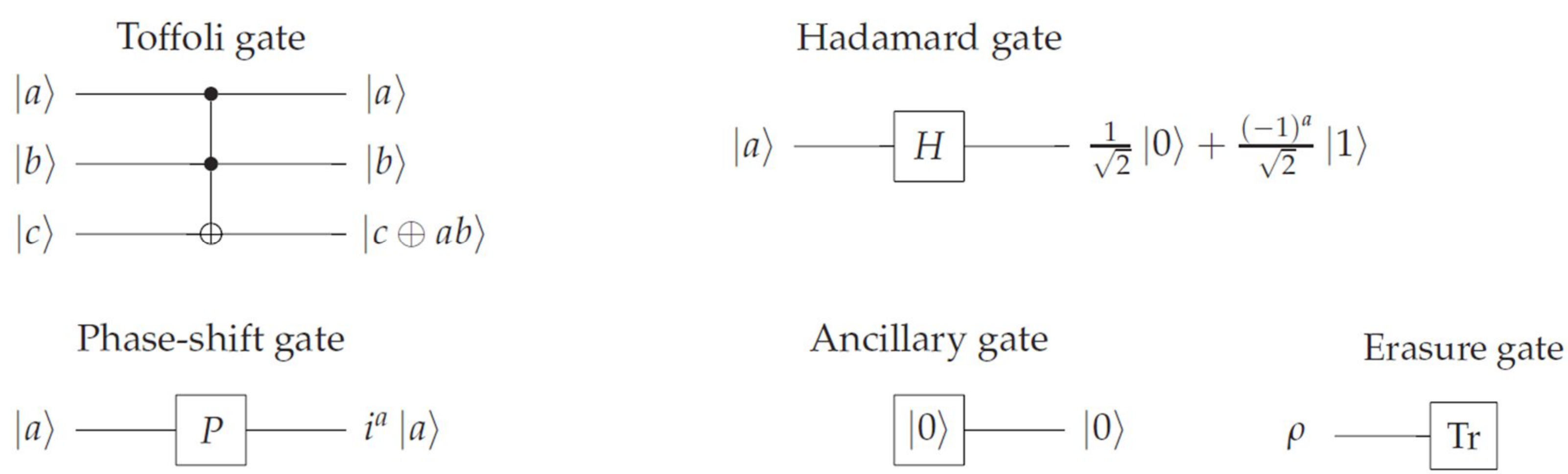

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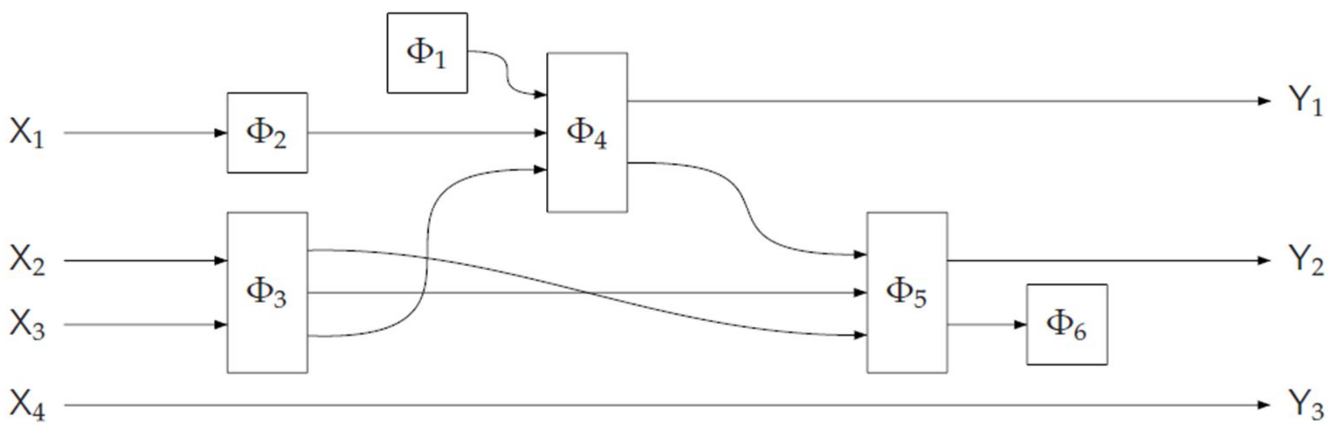

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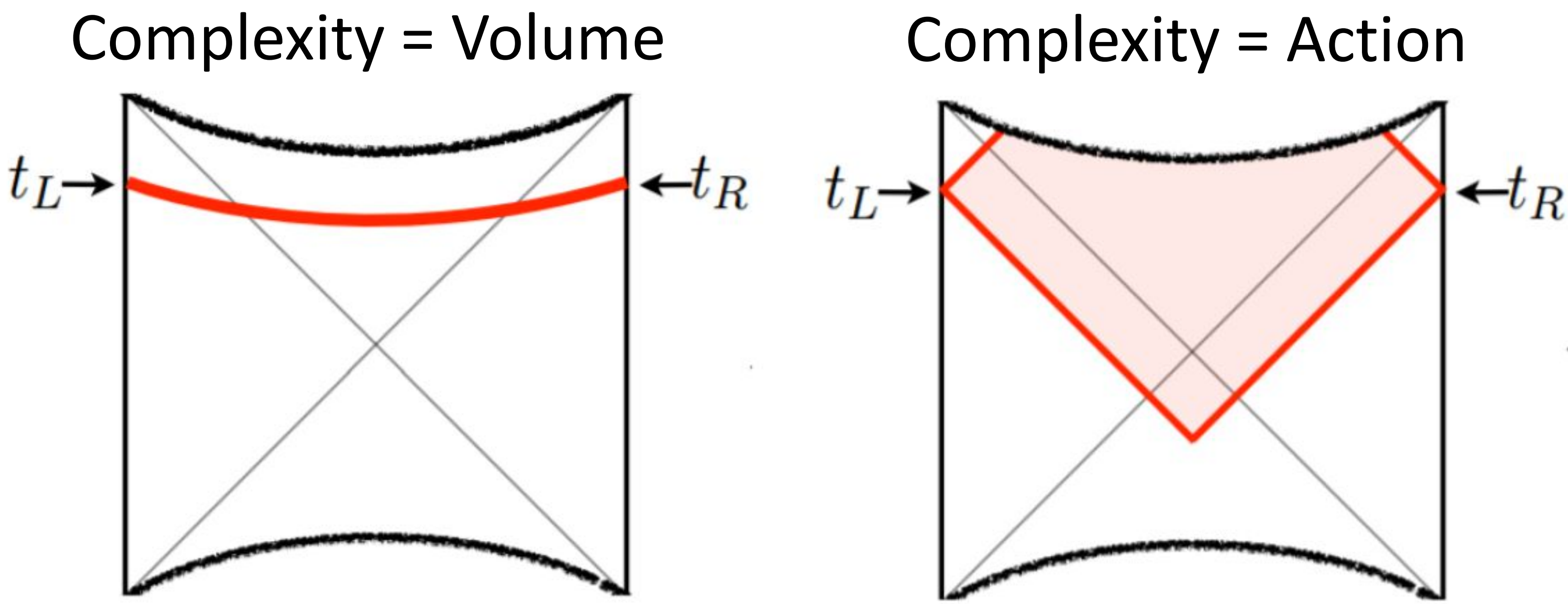
- does the answer depend on the choices?? **YES!!**

Holographic Complexity:

- complexity=volume: evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE)

(Stanford & Susskind)



- complexity=action: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT (Brown, Roberts, Swingle, Susskind & Zhao)

- both of these gravitational "observables" probe the black hole interior (at arbitrarily late times on boundary)

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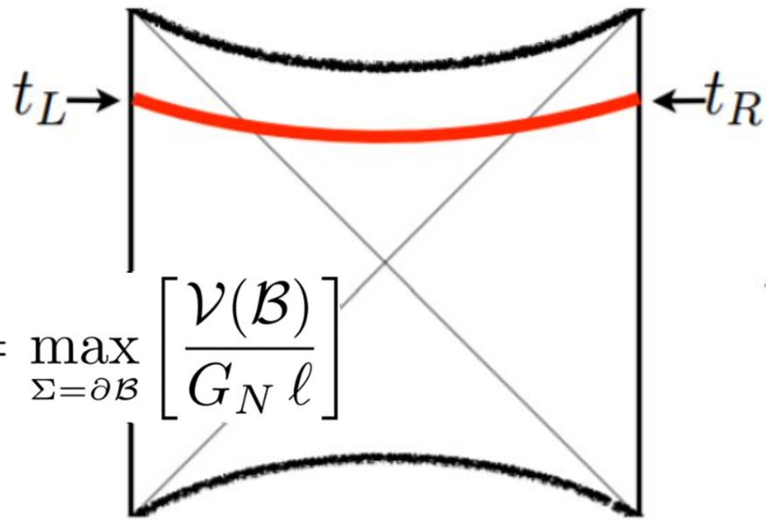

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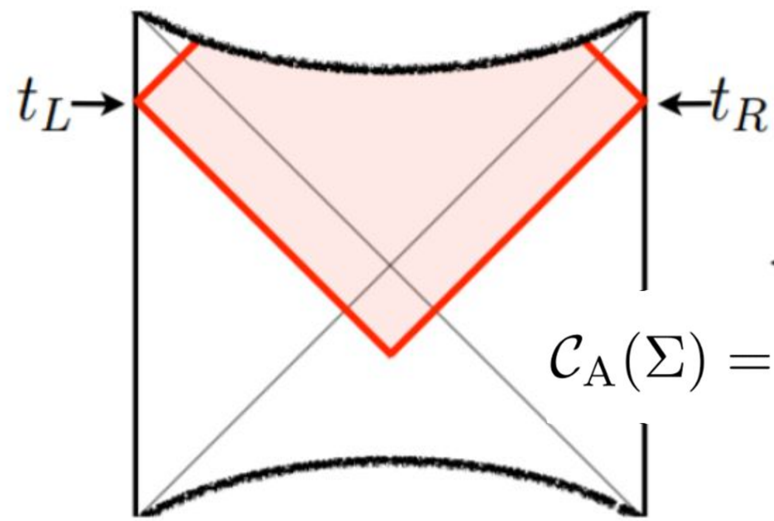

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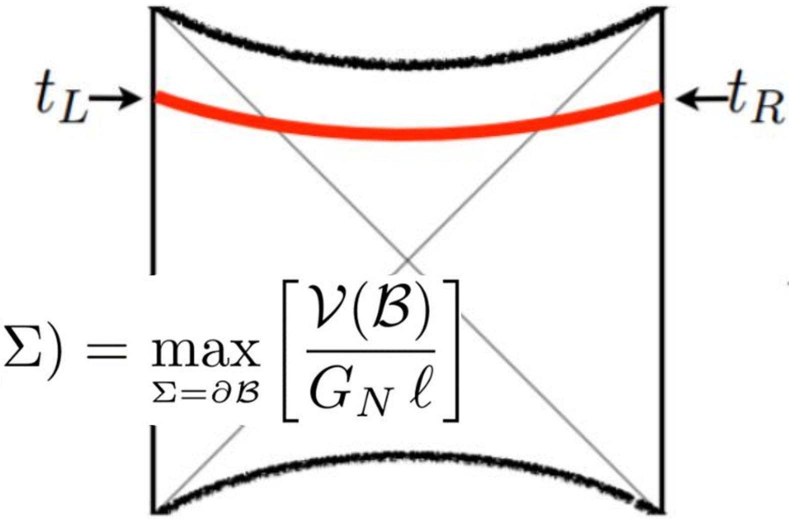

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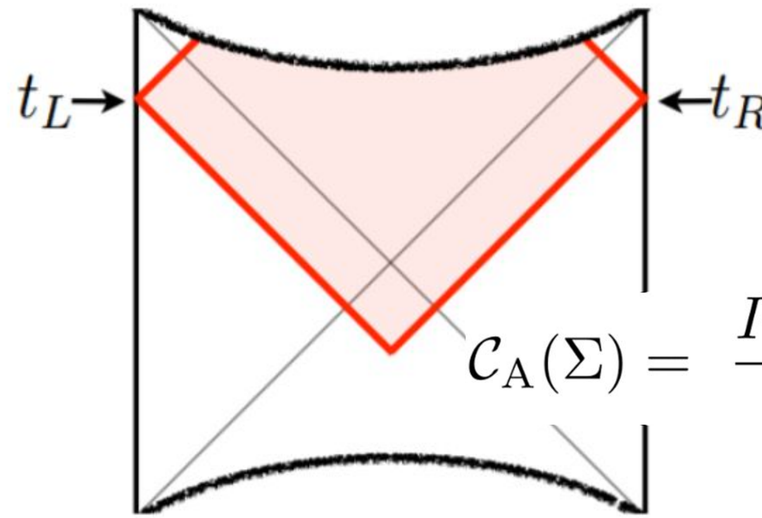

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WHY COMPLEXITY??

- linear growth (at late times) (d = boundary dimension)

$$\left.\frac{d\mathcal{C}_{\mathrm{V}}}{dt}\right|_{t\to\infty} = \frac{8\pi}{d-1}\,M \quad \text{(planar)} \qquad\qquad \left.\frac{d\mathcal{C}_{\mathrm{A}}}{dt}\right|_{t\to\infty} = \frac{2M}{\pi}$$

Susskind, Brown, . . .

- "switchback" effect: complexity $\propto \Sigma\,|t_i - t_{i+1}| - \underline{2\,n\,t_*}$

$\longrightarrow$ probe black holes with shock waves

Stanford, Susskind, . . .

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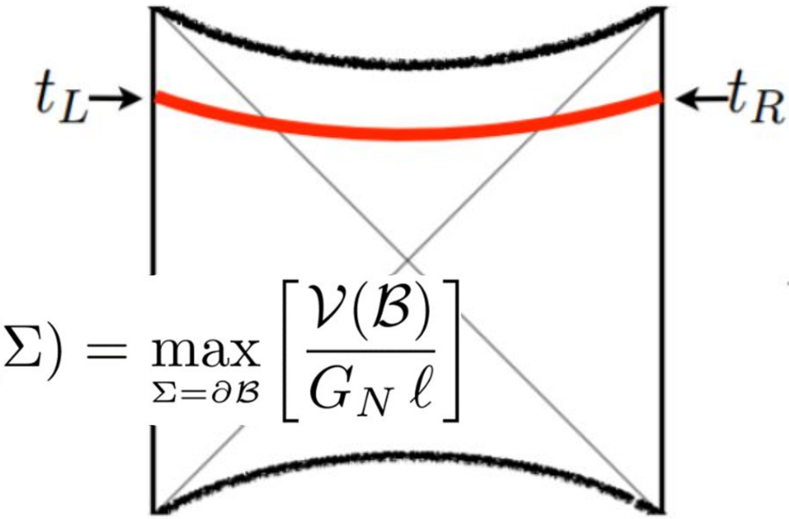

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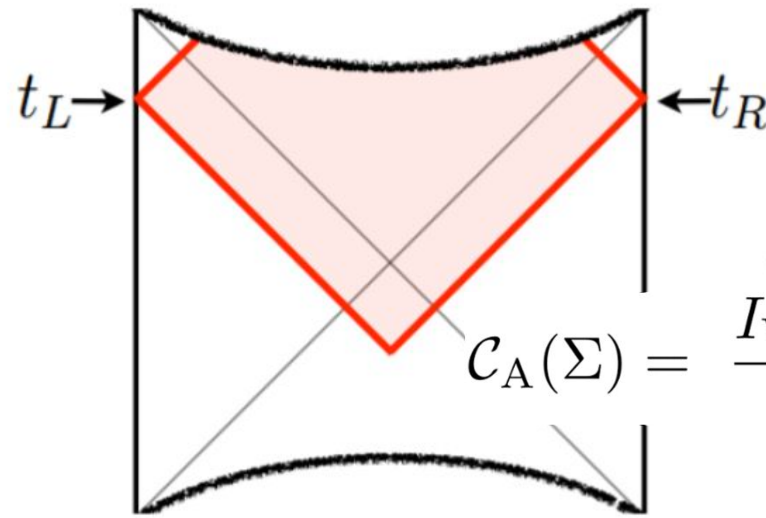

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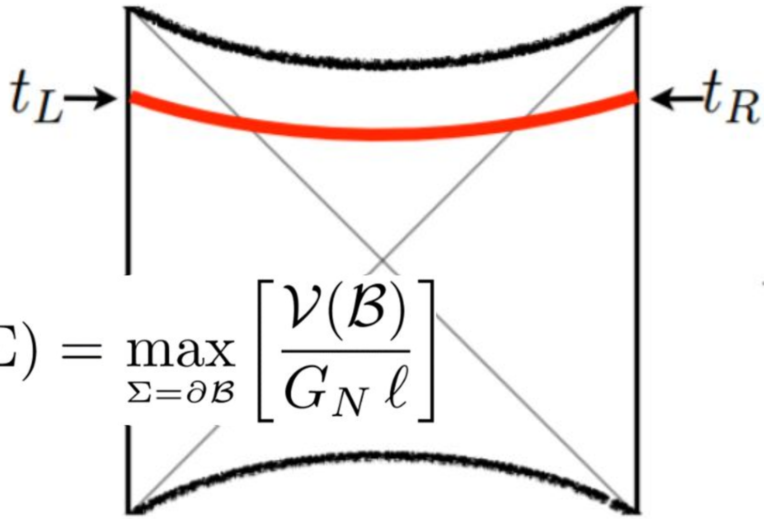

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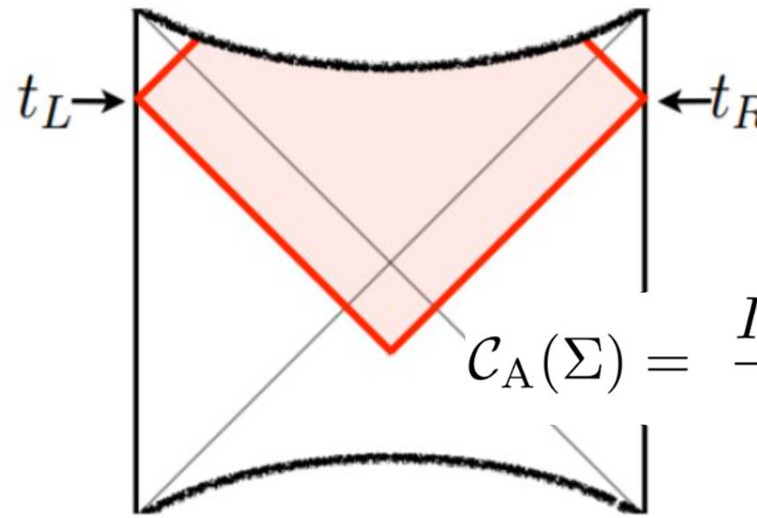

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Ambiguities in defining complexity?

- reference state? gate set? weighting of gates? . . . .

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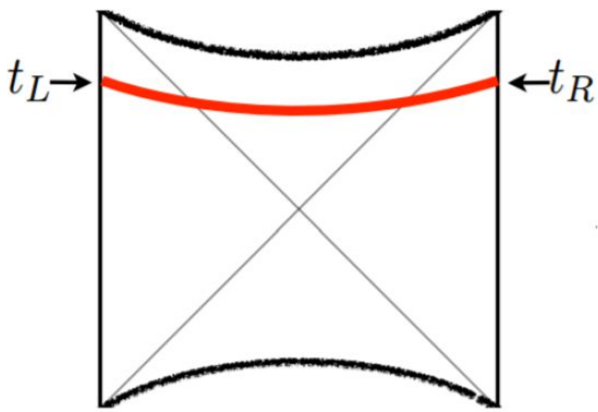

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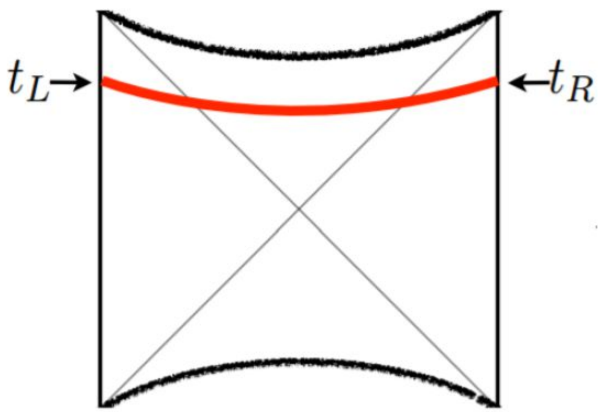

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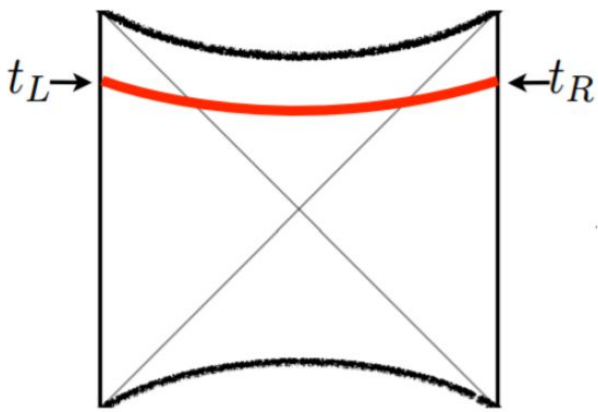

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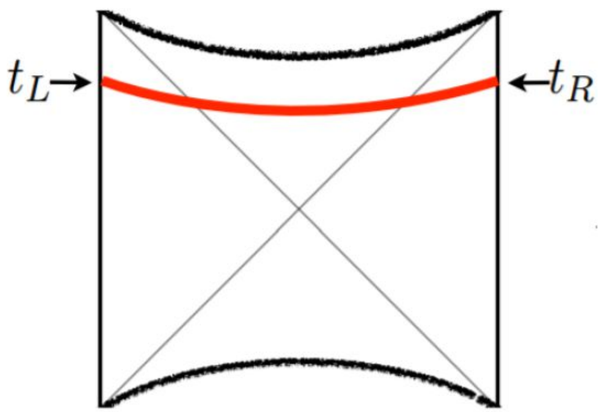

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- yields "nice" diffeomorphism invariant observable

Generalize two step procedure:

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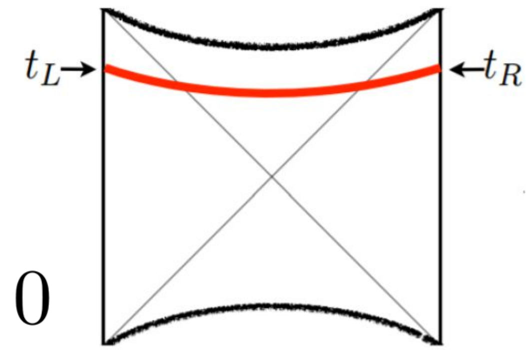

$$\delta_X \left(\int_\Sigma d^d\sigma \sqrt{h}\; F_2(g_{\mu\nu}; X^\mu) \right) = 0$$

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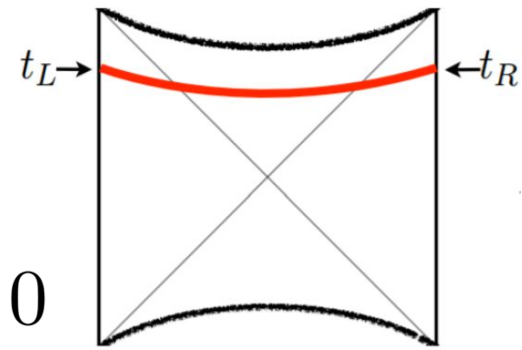

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$$O_{F_1,\Sigma_{F_2}}(\Sigma_{CFT}) = \frac{1}{G_N\, L} \int_{\Sigma_{F_2}} d^d\sigma\, \sqrt{h}\; F_1(g_{\mu\nu}; X^\mu)$$

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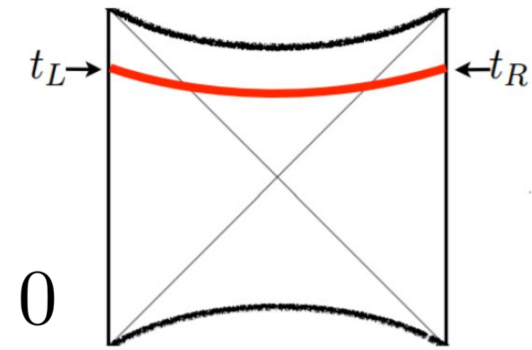

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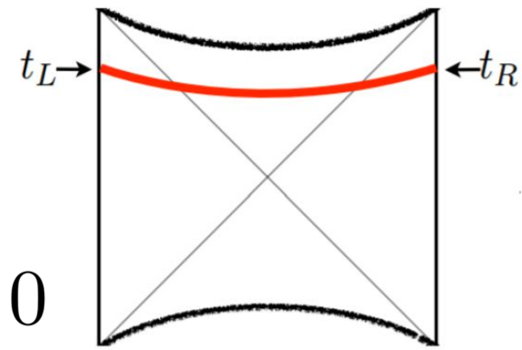

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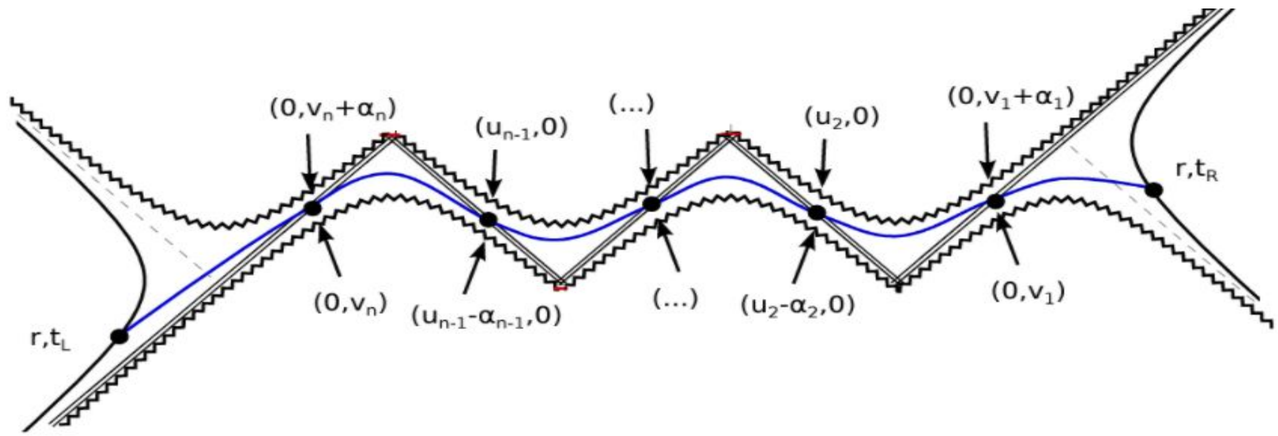
$$\lim_{\tau \to \infty} O_{F_1, \Sigma_{F_2}}(\tau) \sim P_\infty \, \tau$$

where in large *T* limit, the constant $P_\infty \propto$ mass

2) Observables exhibit “switchback effect”, ie, universal time delay in response to shock waves falling into the dual black hole



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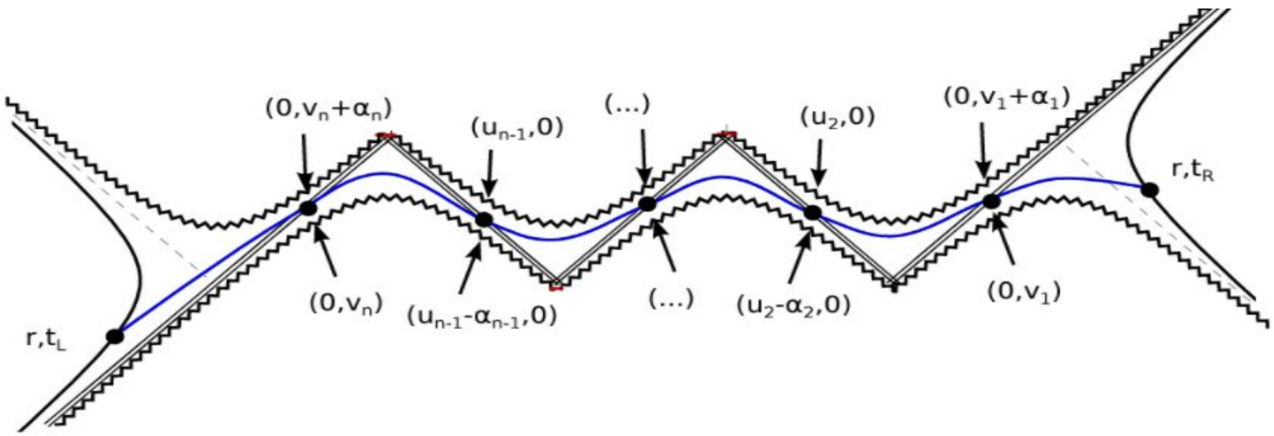

- **universality displayed by observables suggests that all of them are equally viable candidates for holographic complexity!!**

Simple Example: $F_1 = F_2 = 1 + \lambda L^4 \, C_{abcd}C^{abcd}$

- generalized "volume": $\mathcal{C}_{\rm gen} = \frac{V_x}{G_N L}\int_\Sigma d\sigma \left(\frac{r}{L}\right)^{d-1}\sqrt{-f(r)\dot{v}^2 + 2\dot{v}\,\dot{r}}\; a(r)$

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planar AdS black hole

$\tilde{\lambda} = d(d-1)^2(d-2)\,\lambda$

- "gauge fix" worldvolume coordinate: $\sqrt{-f(r)\dot{v}^2 + 2\dot{v}\,\dot{r}} = a(r)\left(\frac{r}{L}\right)^{d-1}$

- conserved "momentum": $P_v = \dot{r} - f(r)\,\dot{v} \quad \Rightarrow \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} = \frac{1}{2}P_v$

- profile determined by classical mechanics problem

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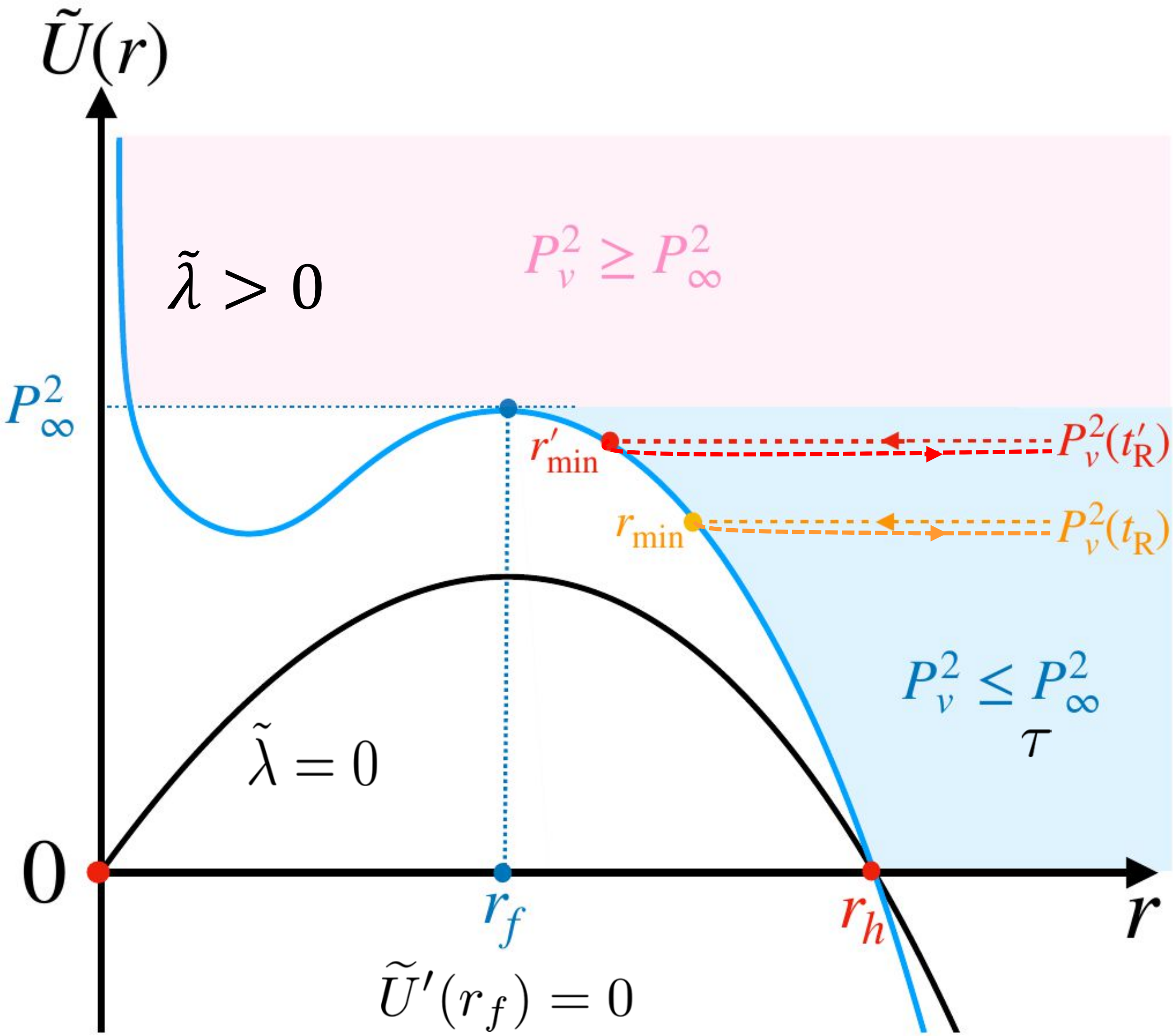

- turning point:

$$P_v^2 = \widetilde{U}(r_{min})$$

Simple Example: F_1

- profile determined by class

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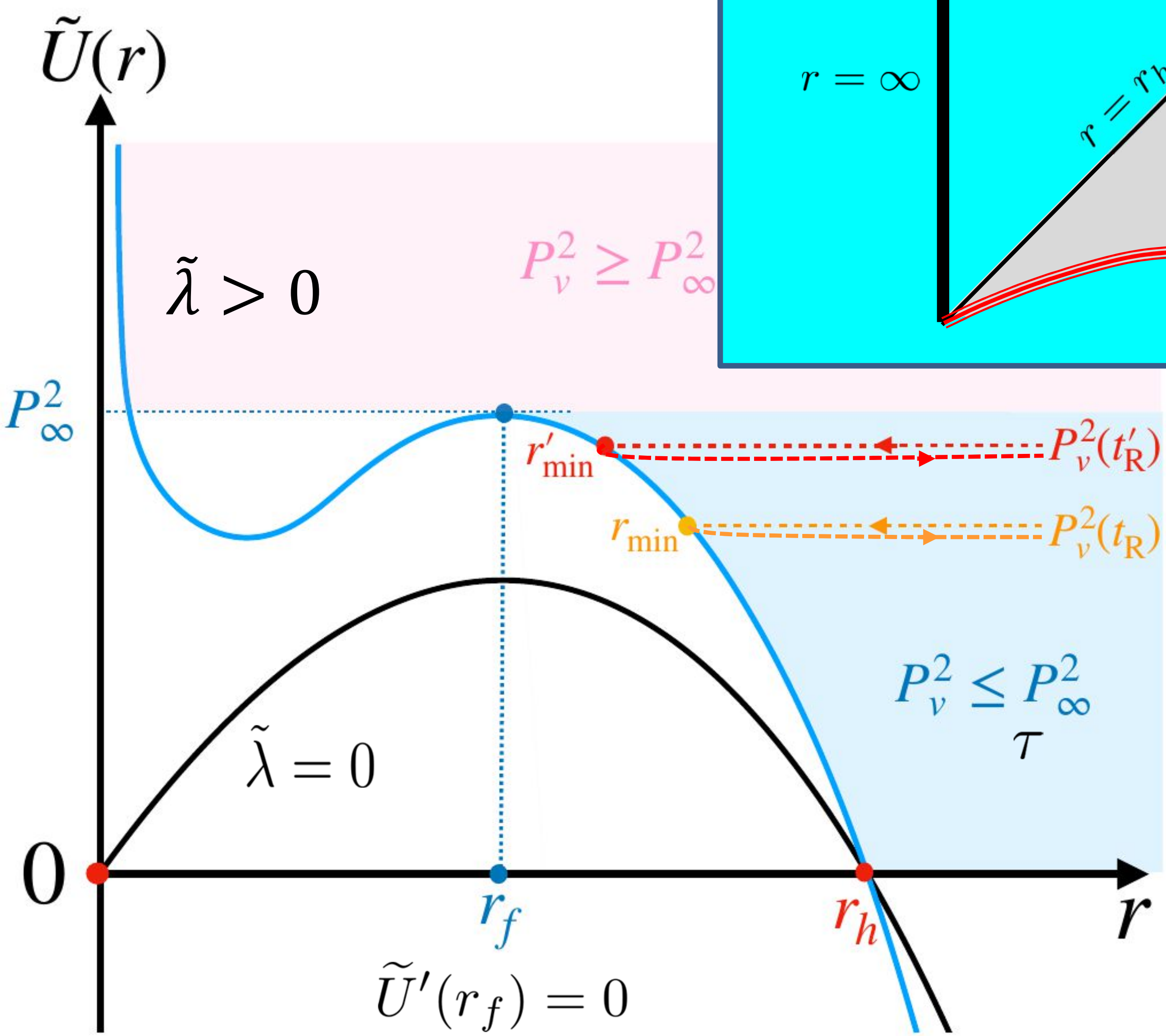

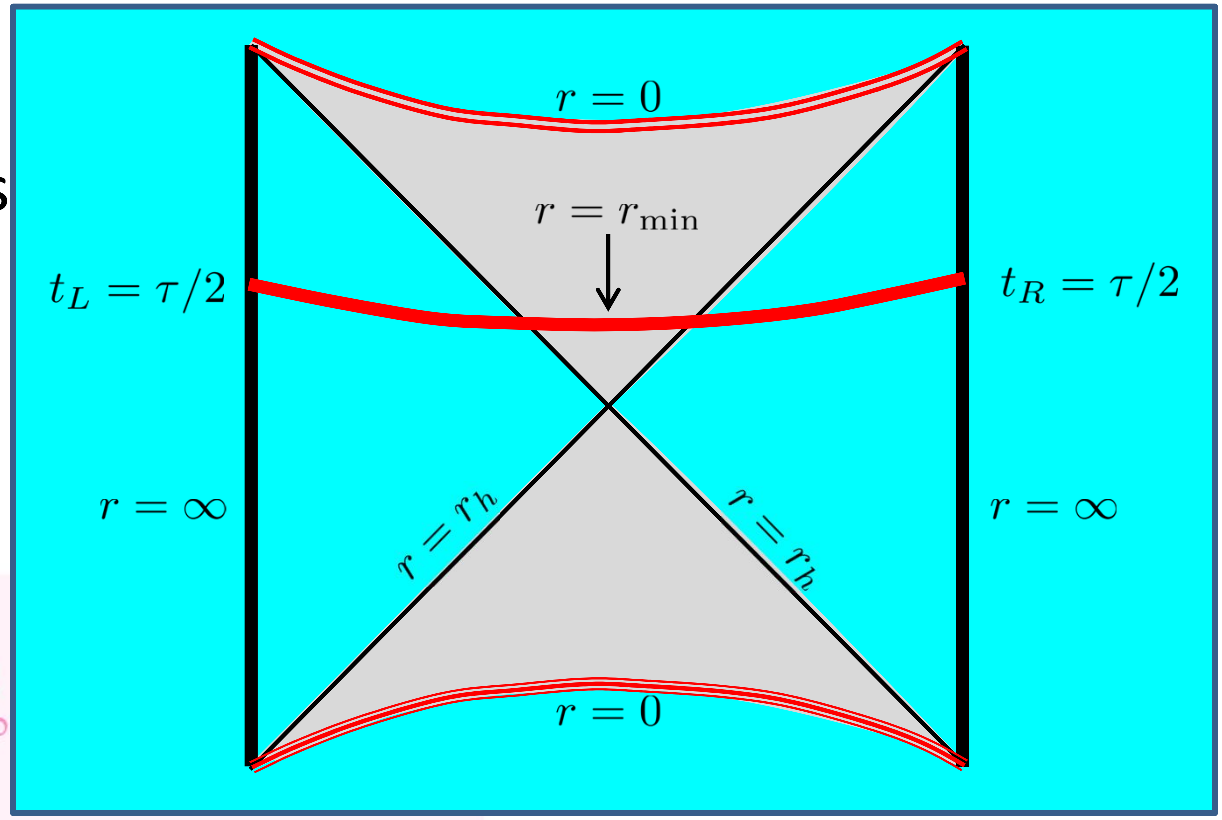

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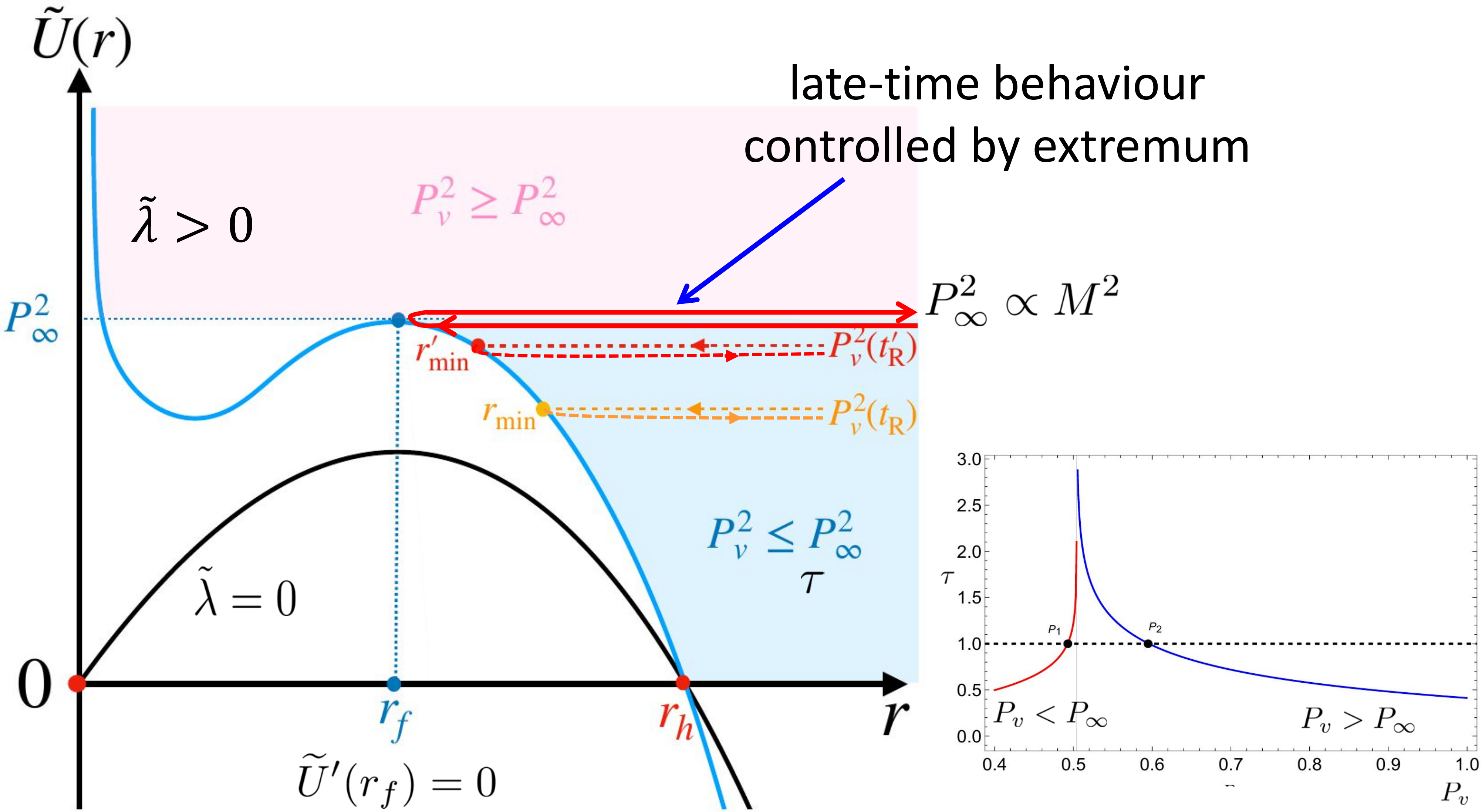

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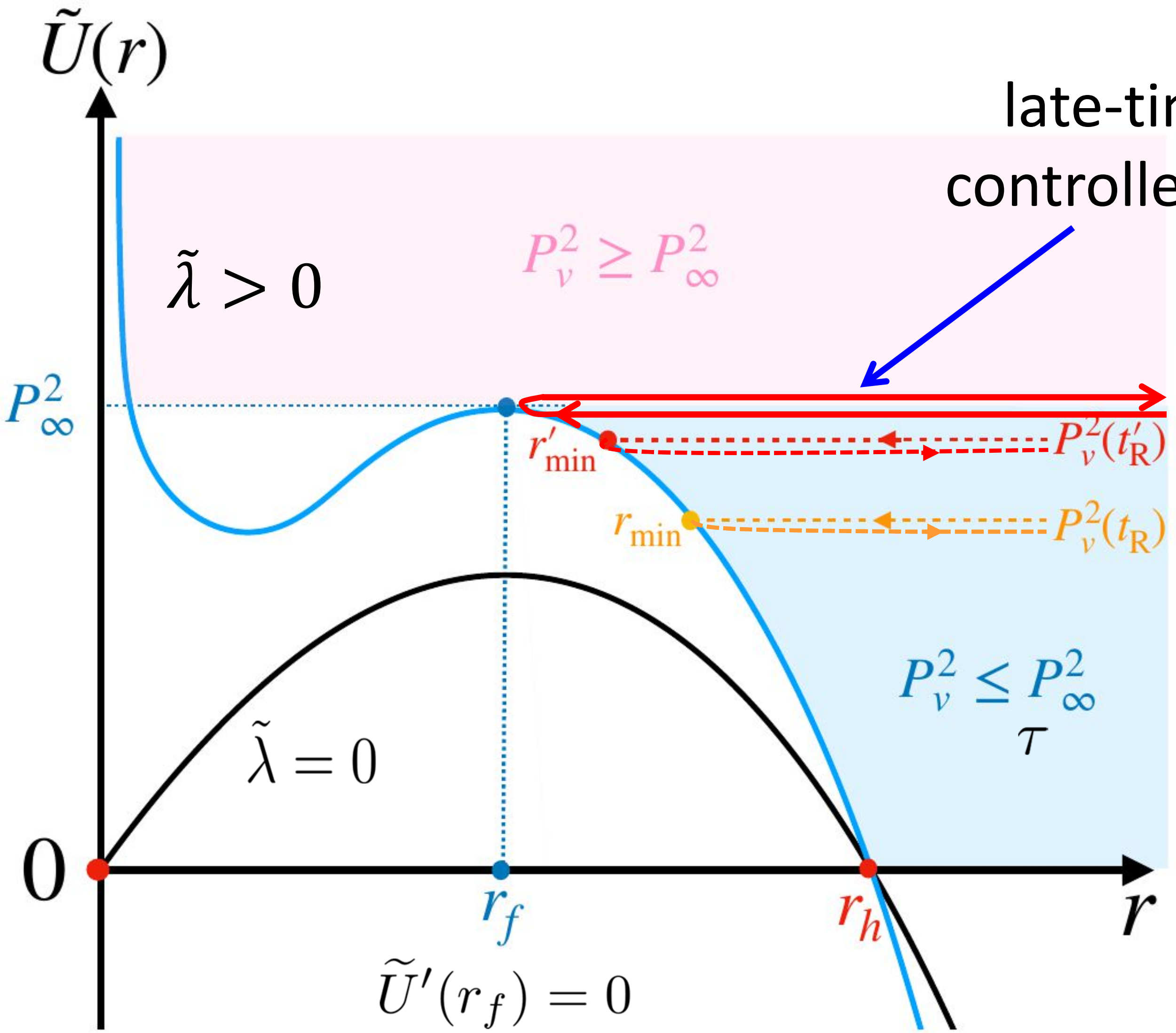
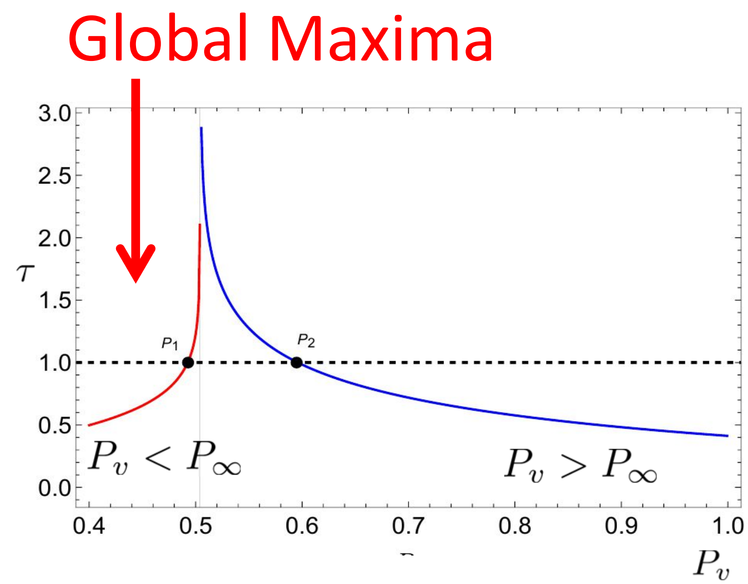
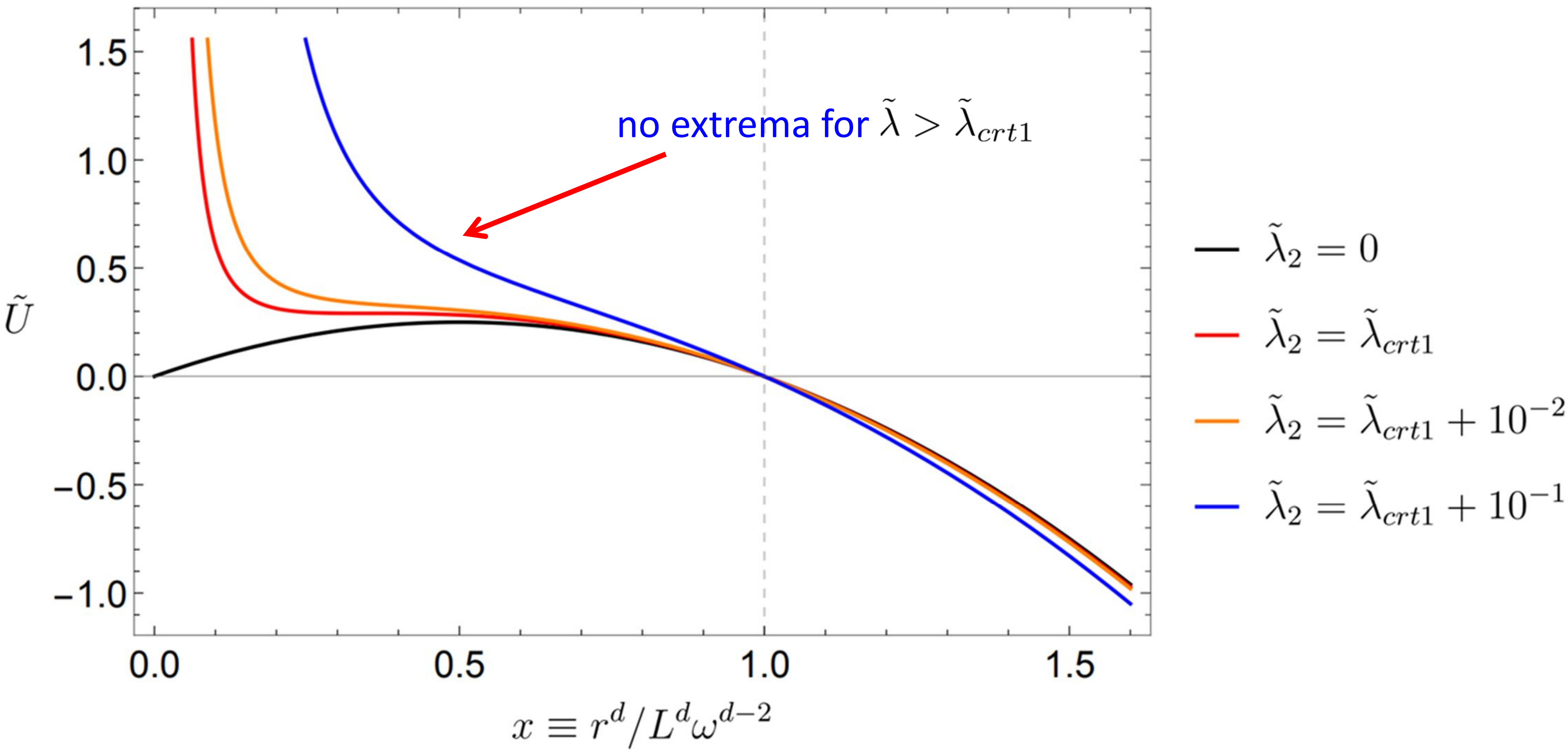




Complexity = Almost Anything:

- recall simple example: $F_1 = F_2 = 1 + \lambda\, L^4\, C_{abcd} C^{abcd}$
- coupling cannot be "too large", ie, $\tilde{\lambda} = d(d-1)^2(d-2)\,\lambda$

$$-1 < \tilde{\lambda} < \tilde{\lambda}_{crit} = \frac{47 - 13\sqrt{13}}{8} \simeq .016$$



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- coupling for curvature invariants should not be too large

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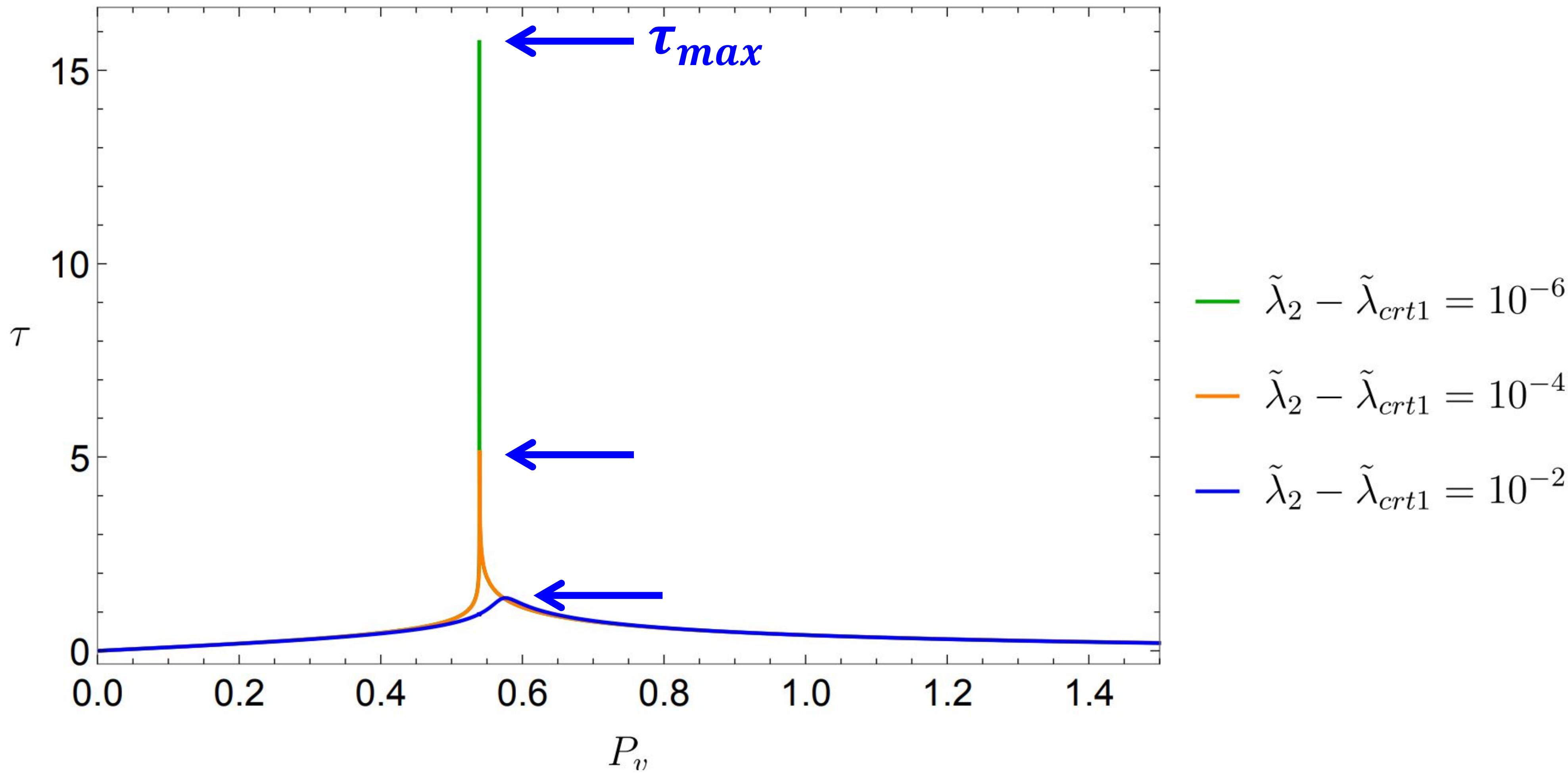

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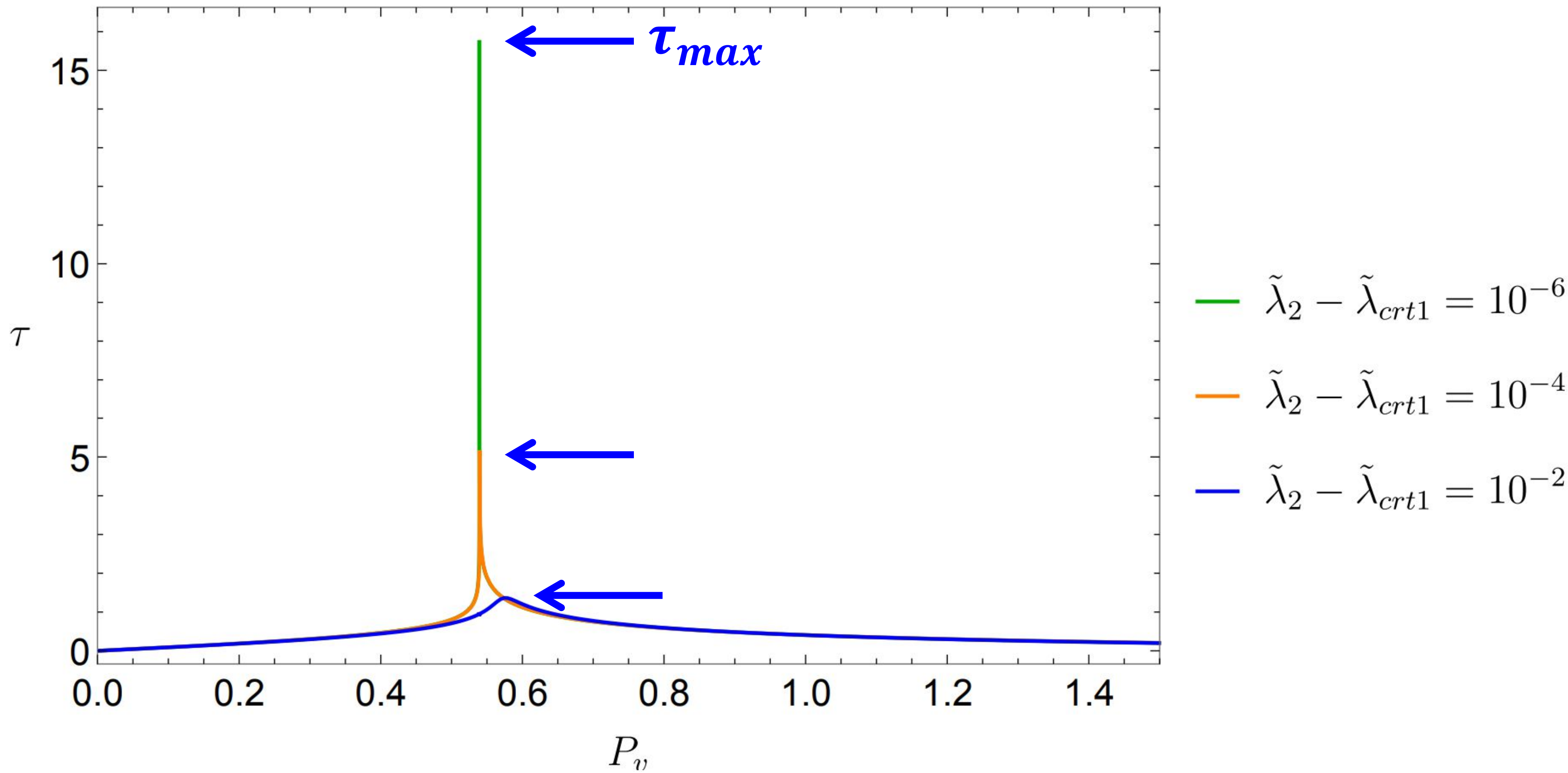
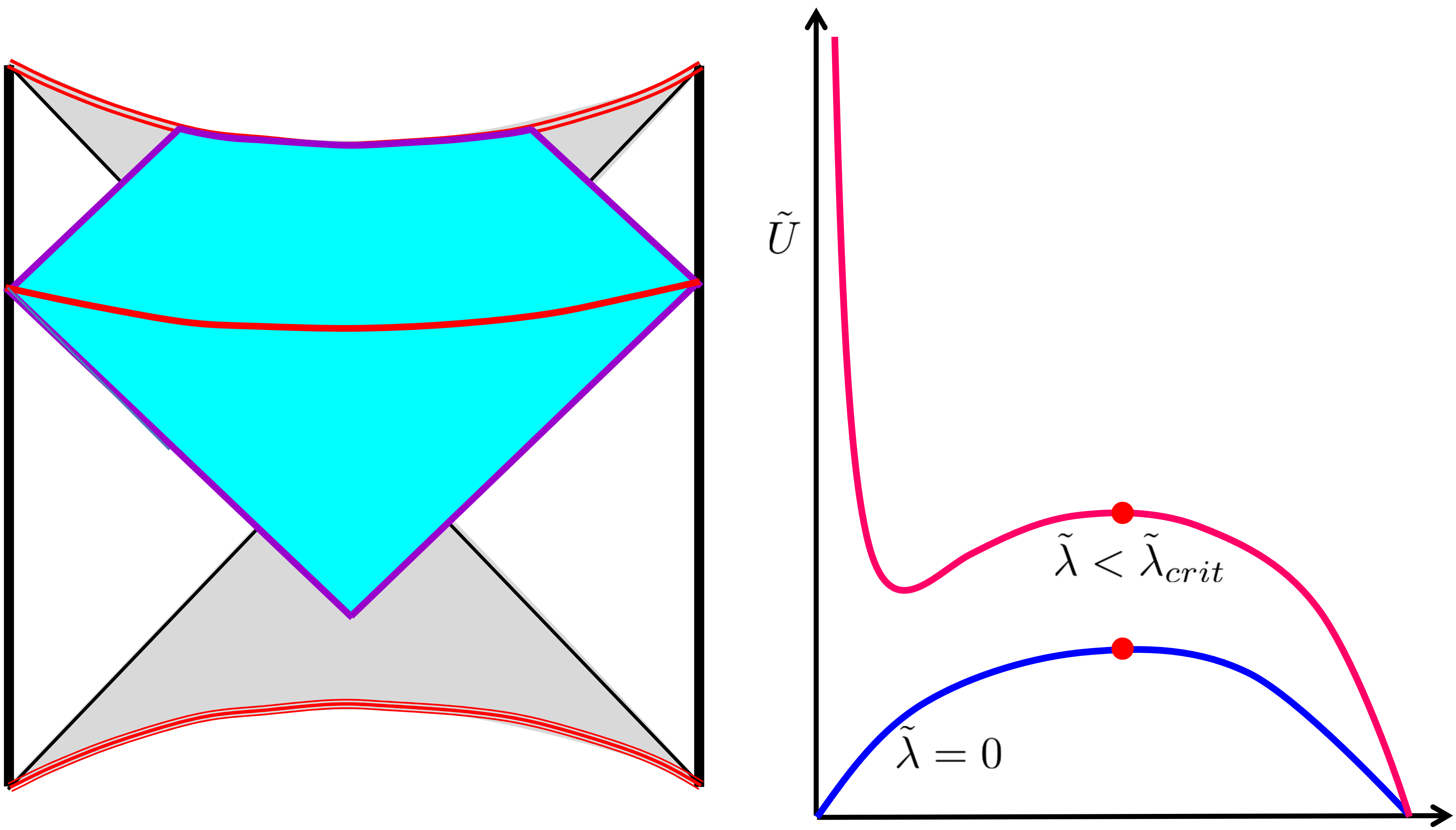


only reach some maximum $\boldsymbol{\tau_{max}}$ as P_v is scanned

but what happens for $\boldsymbol{\tau > \tau_{max}}$???

Complexity = Almost Anything:

- where is surface yielding maximal value of C_{gen} beyond $\boldsymbol{\tau_{max}}$??



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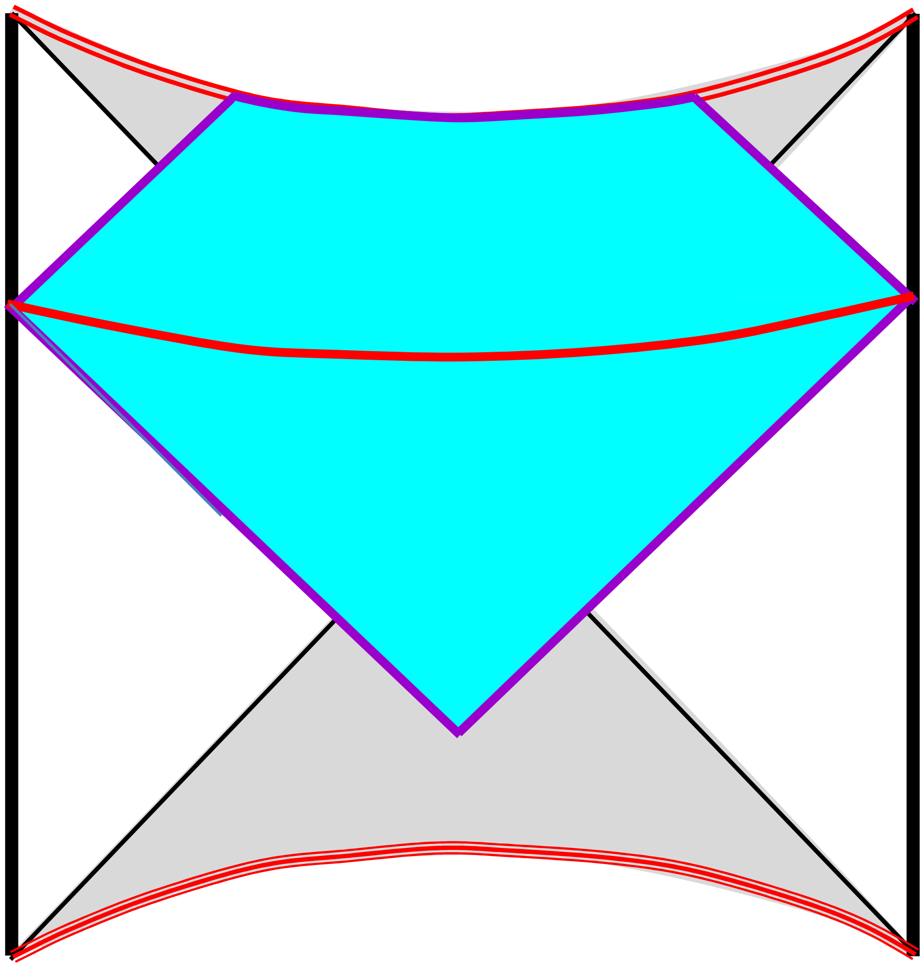

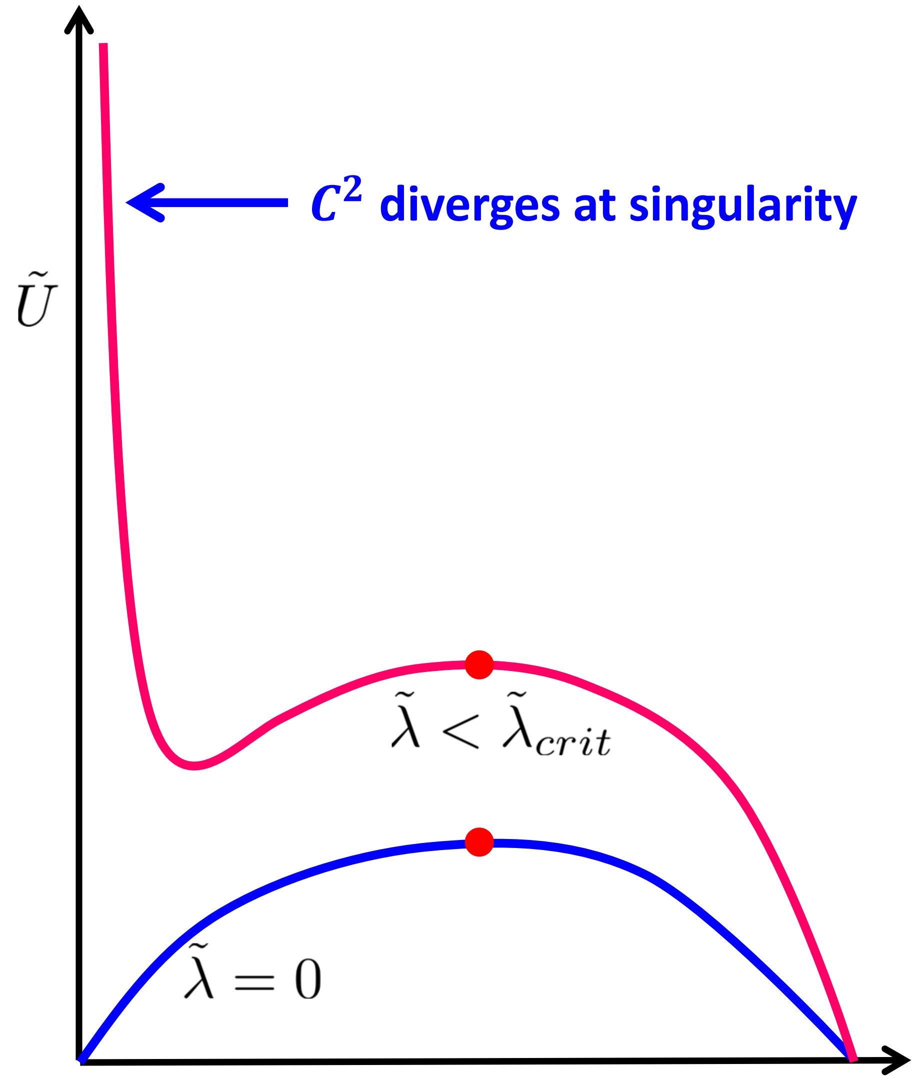

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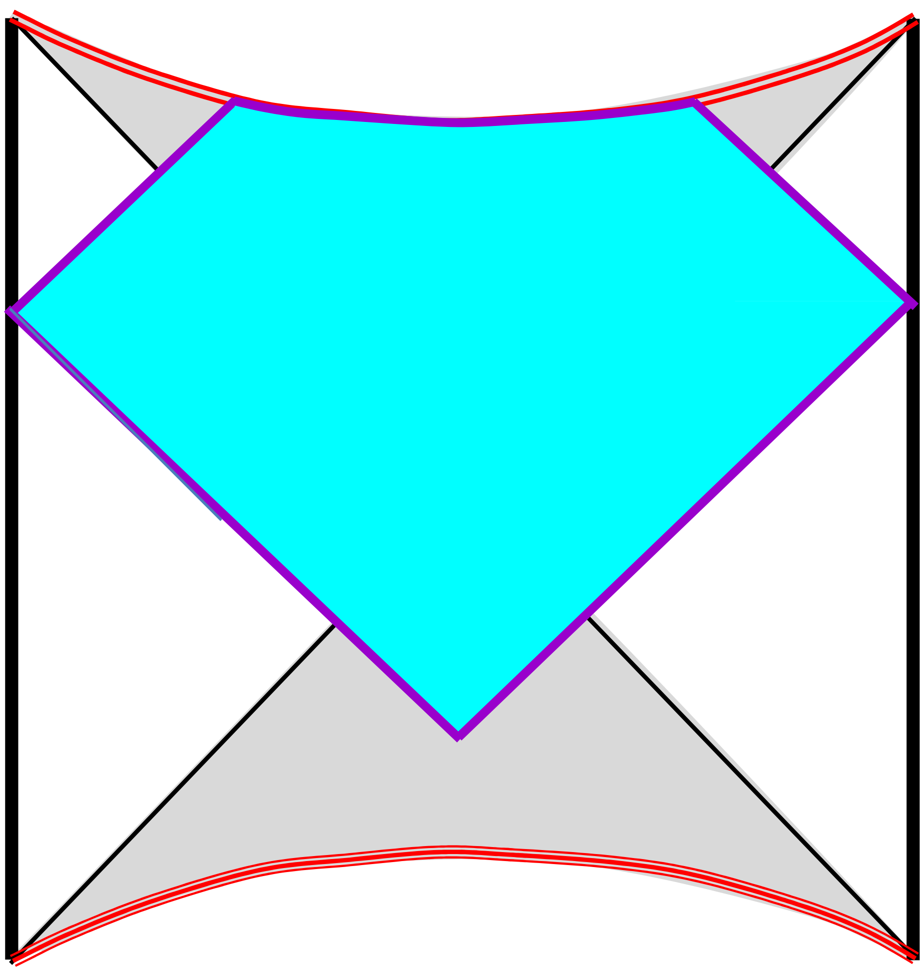
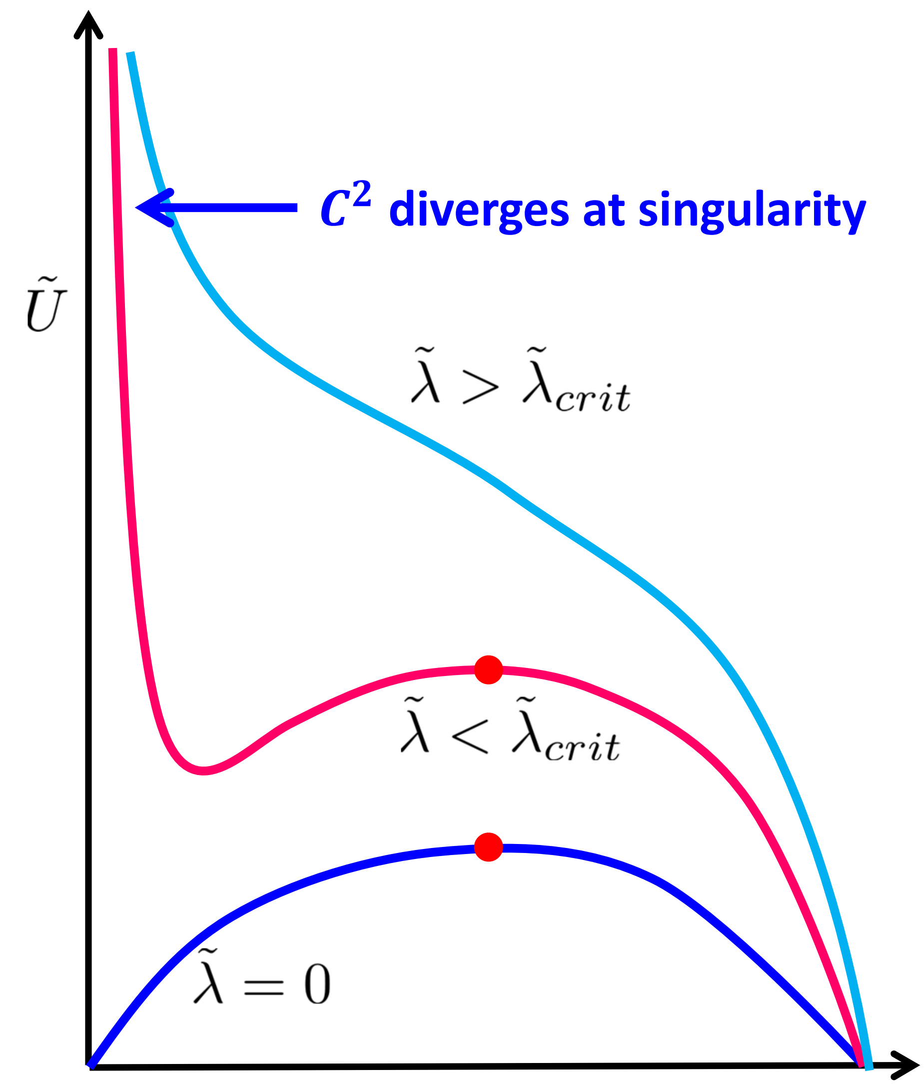
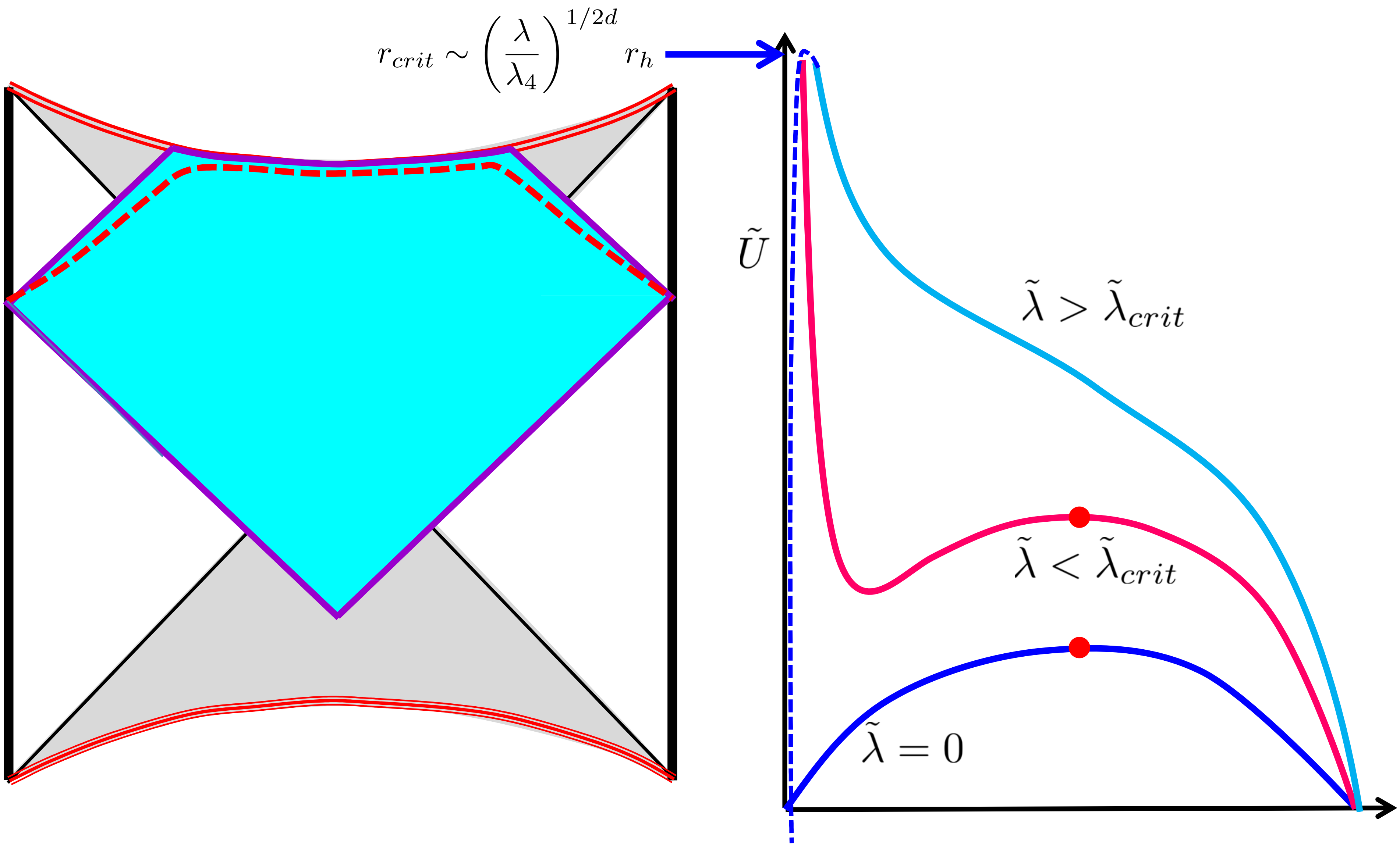




Complexity = Almost Anything:

"regulate": $F_1 = F_2 = 1 + \lambda L^4 C_{abcd}C^{abcd} \underline{- \lambda_4 \, L^8 (C_{abcd}C^{abcd})^2}$



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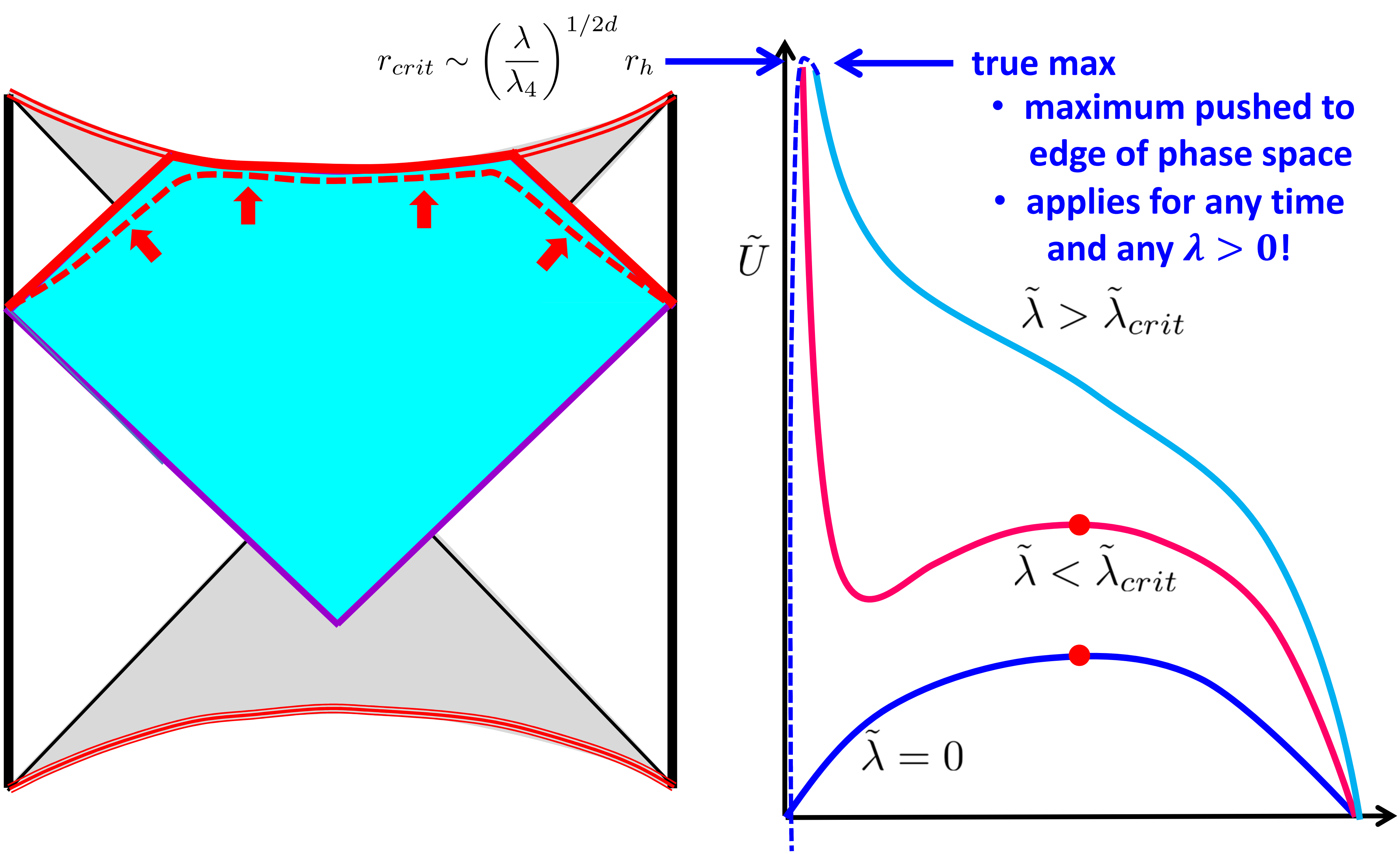

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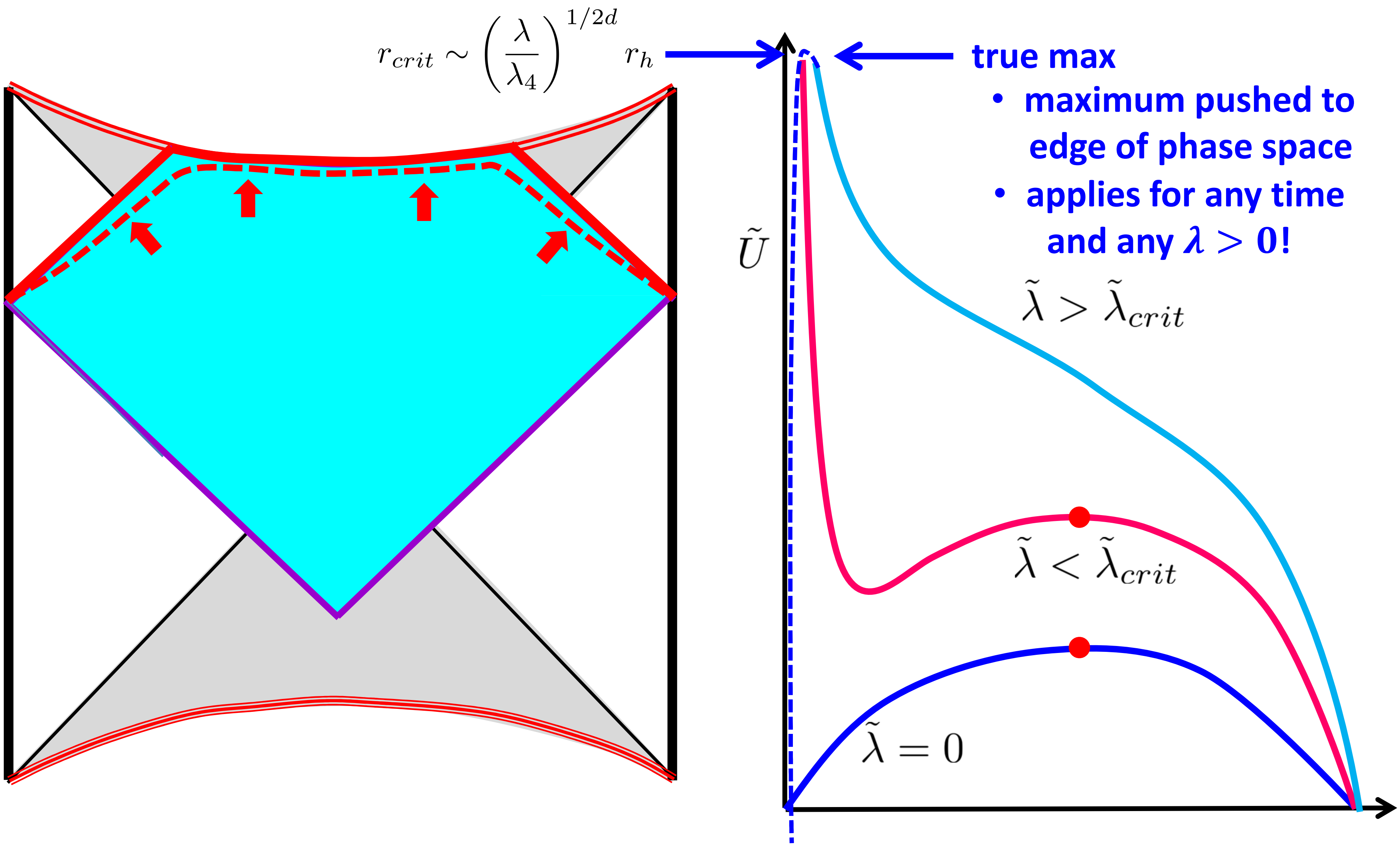

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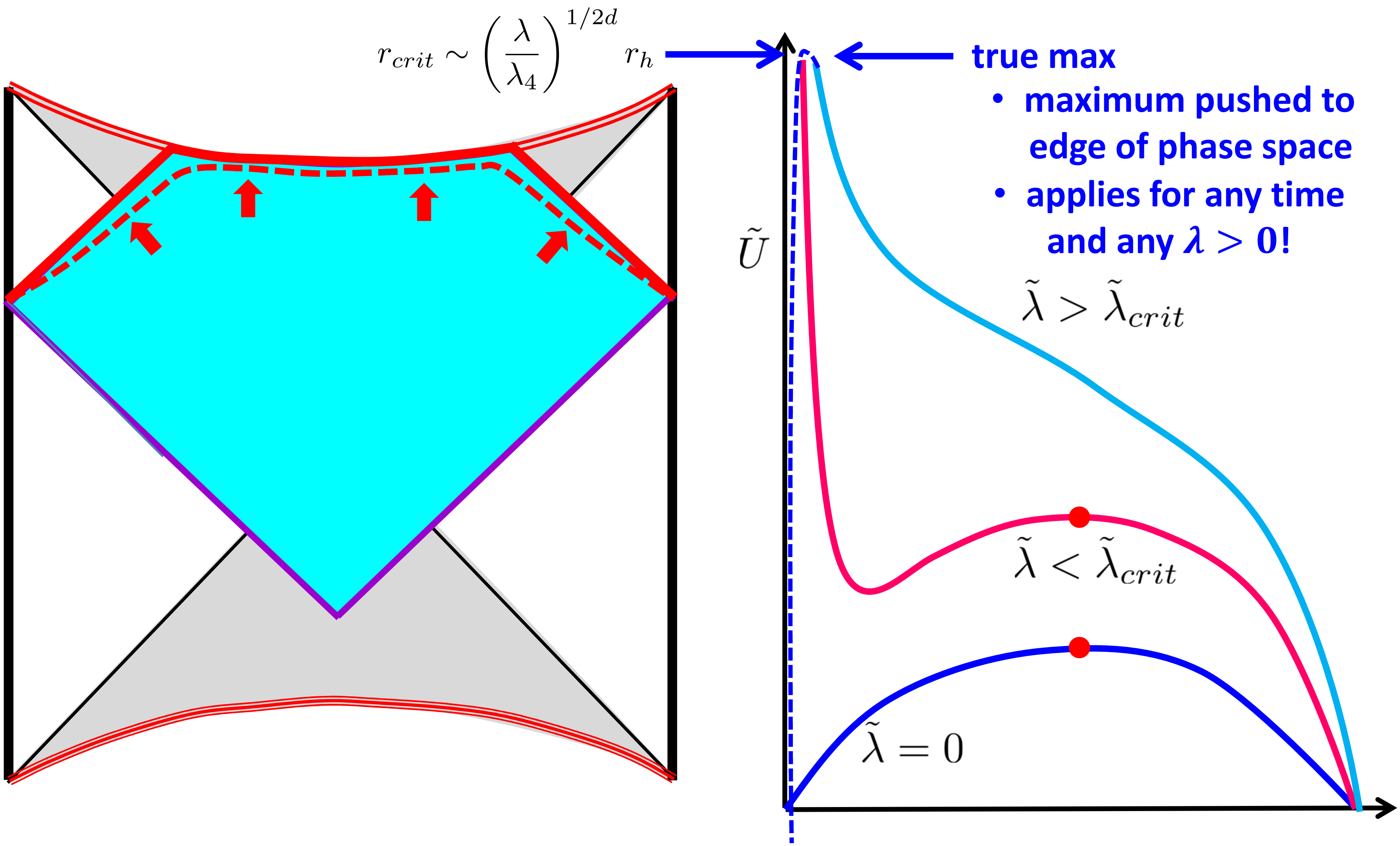

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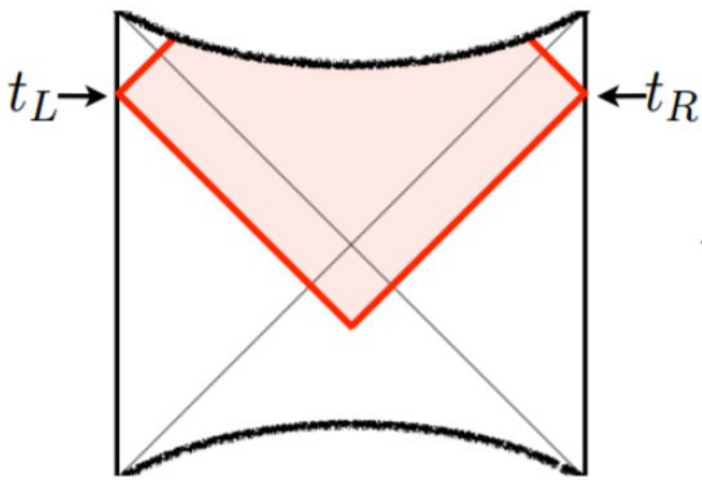
(another interesting story for $\lambda < -1$)

Generalizing CA:

- complexity=action: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT (Brown, Roberts, Swingle, Susskind & Zhao)



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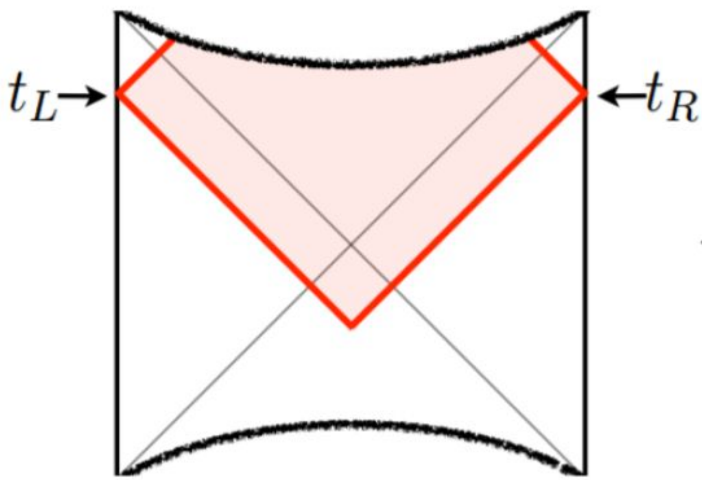
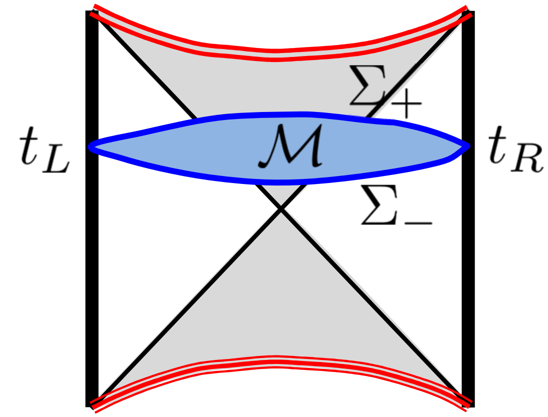


$$\mathcal{C}_{\rm A}(\Sigma) = \frac{I_{\rm WDW}}{\pi\,\hbar}$$

Two steps: 1) find a special surfaces bounding codim.-0 region

2) evaluate geometric feature of codim.-0 region (& boundary surfaces)

Generalizing CA:



1) find a bounding surfaces $\Sigma_{\pm}$:

$$\delta_{\{X_-,X_+\}}\Big(\int_{\mathcal{M}} d^{d+1}\sigma\,\sqrt{-g}\;F_6(g_{\mu\nu})\;+$$
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- F_4 and F_5 are scalar functions of bkgd metric $g_{\mu\nu}$ and embeddings $X^\mu_\pm(\sigma)$ respectively, while F_6 is scalar function of bkgd metric $g_{\mu\nu}$

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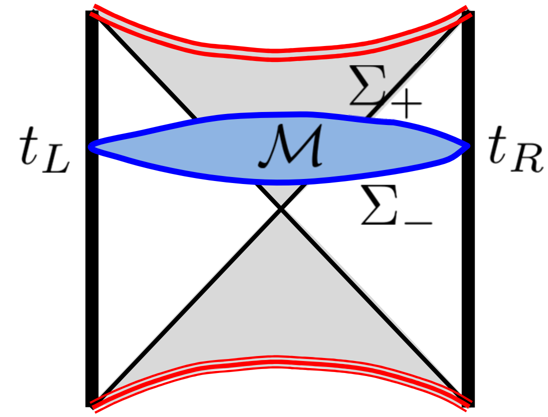

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$$O(\Sigma_{CFT})=\frac{1}{G_N\,L^2}\int_{\mathcal{M}} d^{d+1}\sigma\,\sqrt{-g}\;F_3(g_{\mu\nu})\,+\frac{1}{G_N\,L}\int_{\Sigma_+} d^d\sigma\,\sqrt{h}\;F_1(g_{\mu\nu};X^\mu_+)+\frac{1}{G_N\,L}\int_{\Sigma_-} d^d\sigma\,\sqrt{h}\;F_2(g_{\mu\nu};X^\mu_-)$$

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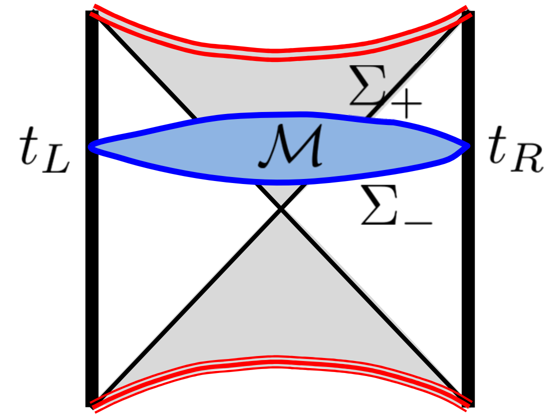

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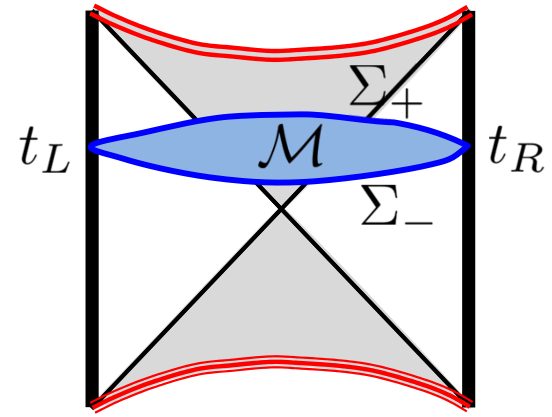
- yields “nice” diffeomorphism invariant observables, which exhibit linear growth at late times as well as the switchback effect!!

Simplest Example:



- extremize the functional

$$O(\Sigma_{CFT}) = \frac{\alpha_+}{G_N\, L}\int_{\Sigma_+} d^d\sigma\,\sqrt{h} + \frac{\alpha_-}{G_N\, L}\int_{\Sigma_-} d^d\sigma\,\sqrt{h} + \frac{1}{G_N\, L^2}\int_{\mathcal{M}} d^{d+1}\sigma\,\sqrt{-g}$$

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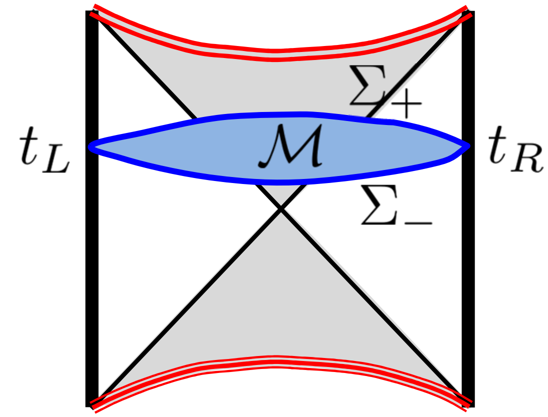

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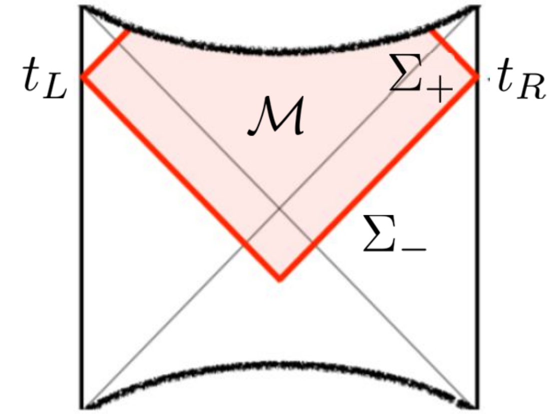

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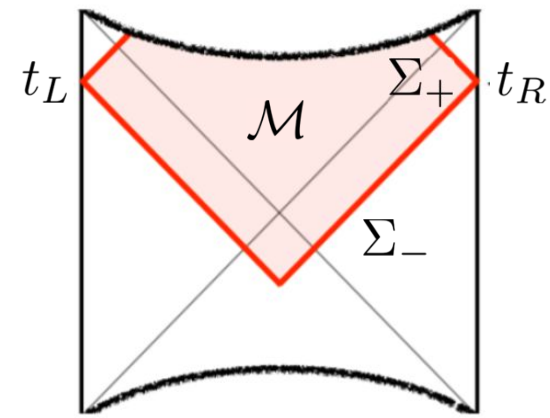

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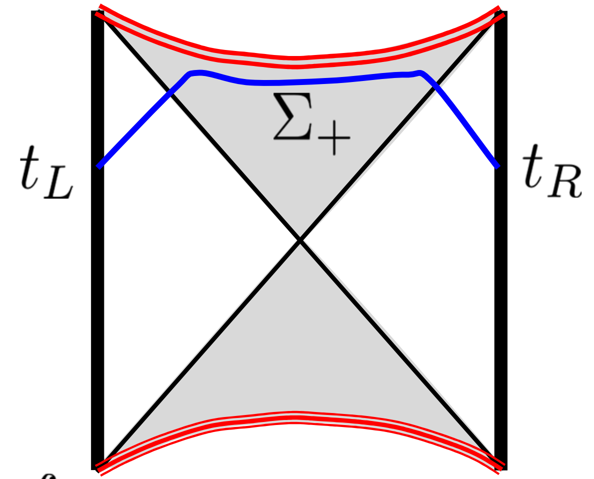
- evaluate action (including bdy terms) $\longrightarrow$ complexity=action
- evaluate volume (same functional) $\longrightarrow$ complexity = volume2.0

(Couch, Fischler & Nguyen)

Probe the singularity with simple example:



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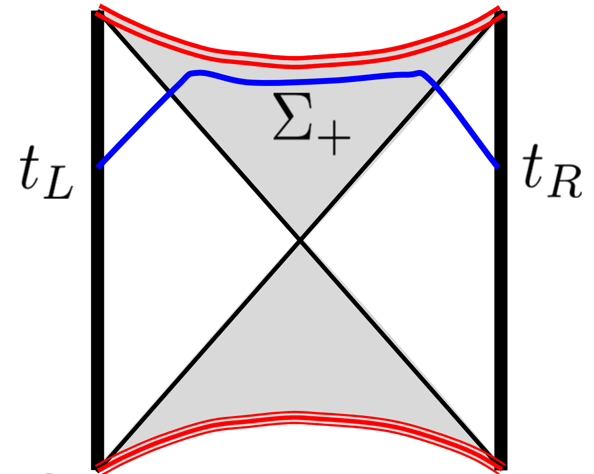

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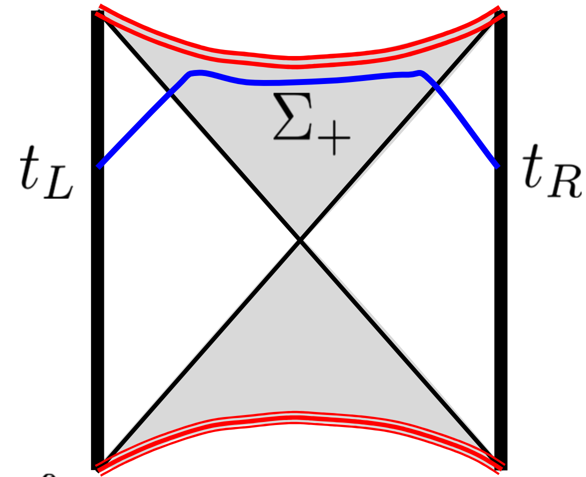

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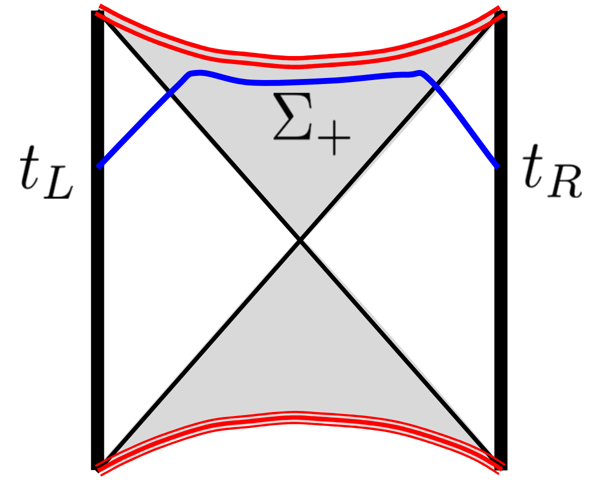

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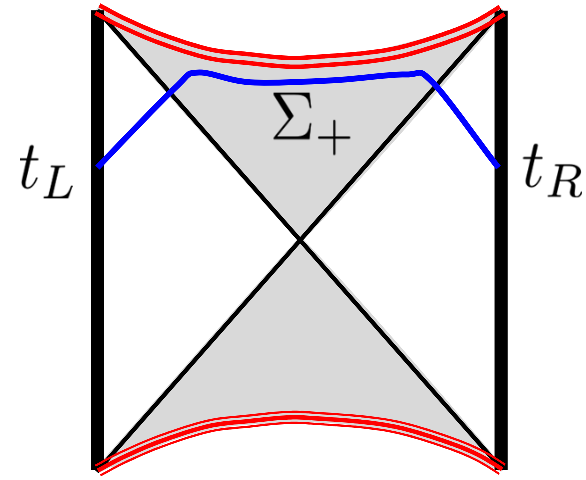

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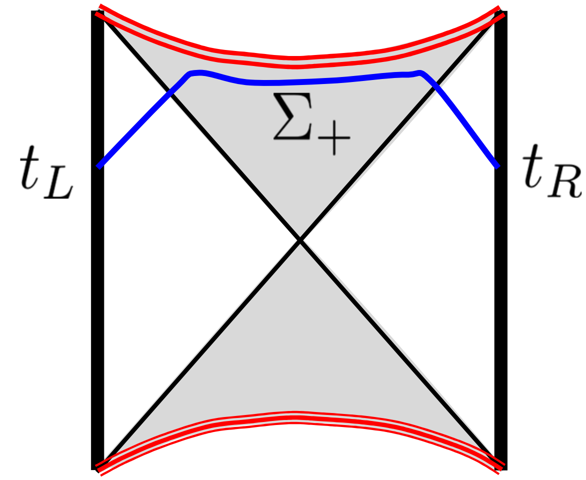

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Reminder: $M = \frac{d-1}{d} S\, T$

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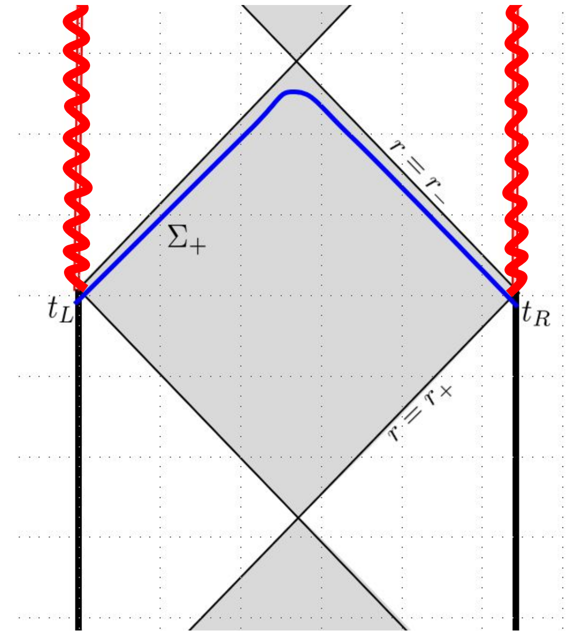

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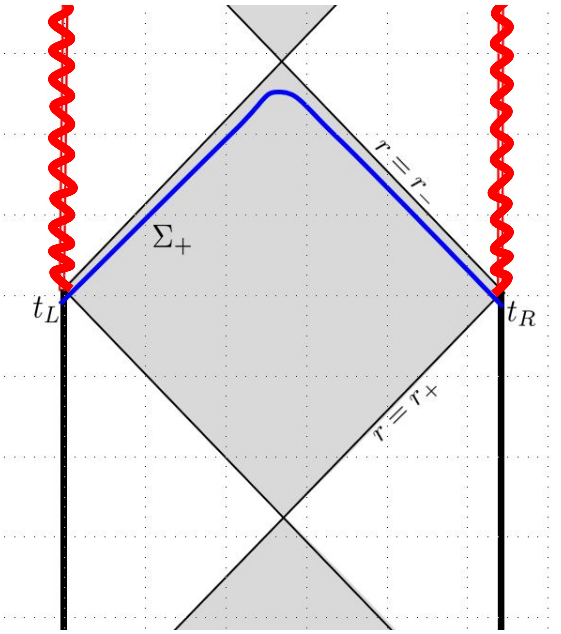

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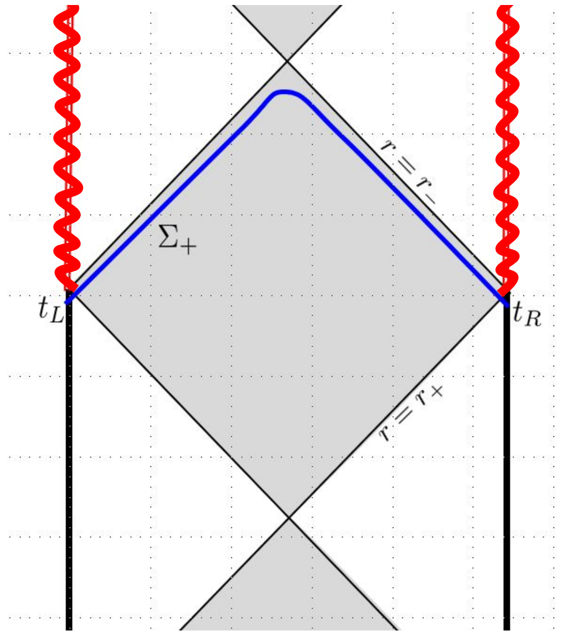
Probe the singularity with simple example:



- consider only Σ_+ with $K = -{}^{1}/_{\alpha_+ L}$ and $\alpha_+ \ll 1$

$$O(\Sigma_{CFT}) = \frac{1}{G_N\, L} \int_{\Sigma_+} d^d\sigma\,\sqrt{h}\; F_1(g_{\mu\nu}; X_+^\mu)$$

- compare to charged black hole

- late time growth dominated by surface $r_f \simeq r_- + 4\pi L^2 T_-\,\alpha_+^2$

only probe up to Cauchy horizon

$$F_1 = 1 \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq 16\pi\, S_- T_-\,\alpha_+ \to 0$$

$$F_1 = -L\,K \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq 16\pi\, S_- T_-$$

entropy and temperature of inner horizon

$$F_1 = L^4\,C^2 \quad : \quad \frac{d\mathcal{C}_{\rm gen}}{d\tau} \simeq \text{“mess”}\;\; S_- T_-\,\alpha_+ \to 0$$

Comments:

- have found an infinite class of gravitational observables which exhibit complexity-like behaviour

$\longrightarrow$ **feature** of holographic complexity! (not a bug!)

$\longrightarrow$ families of observables allow for systematic study of physics beyond the horizon

$\longrightarrow$ in boundary, different realizations of complexity make different features of the spacetime geometry manifest

$\longrightarrow$ key challenge: find interpretation of gravitational observables in boundary QFT?

$\longrightarrow$ may be related to ambiguities in defining complexity?

- reference state? gate set? weighting of gates? . . . .

Conclusions/Questions/Outlook:

- simple example but "classical mechanics" analysis readily extends to $F_1(g_{\mu\nu}, \mathcal{R}_{\mu\nu\rho\sigma}, \nabla_\mu)$ and to observables where $F_1 \neq F_2$
- couplings for curvature invariants should not be too large
- similar behaviour appears to hold for functionals including dependence on extrinsic curvature
- **infinite class of holographic observables equally viable candidates for gravitational dual of complexity!!**
- can freedom in constructing gravitational observables be related to freedom in constructing complexity model in boundary QFT
- is there something that singles out maximal volume?
- what is role of extremal solutions which are not global maxima and probe very near to singularity?
- add matter contributions to new observables (eg, CA proposal)
- **precise interpretation of gravitational observables in boundary QFT**

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Lots to explore!