Causality Criteria from Stability Analysis at Ultra-High Boost

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Outline

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- Frame-invariant Stability from Ultra-High Boost
- Causality Criteria from Stability Criteria
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Motivation

- **Hydrodynamics:** A low-energy effective theory to describe the evolution of a system in terms of its conserved quantities and their derivatives.
- **Filtering out unphysical theories:** Setting bounds on transport coefficients using physicality arguments like entropy positivity, stability, causality etc.
- **Causality:** Perturbations do not exit the light cone; no superluminal perturbations.
- **Stability:** Perturbations around equilibrium decay down with time.
- **Pathology free:** Both stable as well as causal.

Motivation

- Traditionally, stability is analysed at Low-k limit and causality at High-k limit (Asymptotic Causality) (Fox, Kuper, Lipson - 1970)
- Causal theory $\Rightarrow$ $v_g \equiv \lim_{k \to \infty} \left| \frac{\partial \, \mathrm{Re}(\omega)}{\partial k} \right|$, $v_g \leq 1$
- Conceptual disagreement: Low energy effective theory, but causality analysis at high energy; possibly outside the region of validity of the theory!
- Can we understand the causality of a theory without departing from the low-k regime, which is the valid regime for hydrodynamics as a low-energy effective theory?

Motivation

- **Hint:** Frame-invariant stability.
- **Gavassino 2022-23:** A dissipative theory stable in one frame is causal iff it is stable in all reference frames.
- Can we utilise frame-invariant stability to identify the causal parameter space of a theory?
- **Note:** Since stability analysis is performed in the low-k region, this method of causality analysis stays well within the region of validity of the theory.

Setup: General Conventions

- Metric: 4-D Minkowski

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$

- Linearised fluctuations around global equilibrium

$$\psi(t, \vec{x}) = \psi_0 + \delta\psi(t, \vec{x})$$

$$\delta\psi = \exp[i(kx - \omega t)]$$

- Conventions:
  - k and Boost: along x - $k^\mu = (\omega, k, 0, 0)$, $u_0^\mu = \gamma(1, v, 0, 0)$
  - Velocity fluctuation: Shear ch. along y, Sound ch. along x
- Conformal and Uncharged $\left(T^\mu{}_\mu = 0 ,\ J^\mu = 0 \right)$

Setup: MIS Theory

- Stable and Causal theory of hydrodynamics with corrections beyond first-order in gradient expansion. (Israel, Stewart, Muller - 1967-1979)
- Viscous contributions are new degrees of freedom.
- Stress tensor: $T_{\mu\nu} = \epsilon u_\mu u_\nu + p P_{\mu\nu} + \Pi_{\mu\nu}$
- Viscous degrees of freedom:

$$\Pi_{\mu\nu} = -2\eta\sigma_{\mu\nu} - \tau_\pi P^{\mu\nu}_{\alpha\beta}\, u\cdot\partial\,\Pi^{\alpha\beta}, \quad \Pi_{\mu\nu} = 0 + \delta\Pi_{\mu\nu}$$

- Dispersion polynomial: conservation of stress tensor

$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow f(\omega, k) = 0$$

Setup: BDNK Theory

- First-order stable-causal theory of hydrodynamics.
  (Bemfica, Disconzi, Noronha, Kovtun - 2018-2019)

- Independent degrees of freedom are given by derivatives acting on fluid-variables u_μ , ϵ .

- Stress tensor:

$$T^{\mu\nu} = (\epsilon + \epsilon_1) u^\mu u^\nu + (p + p_1) P^{\mu\nu} + 2 W^{(\mu} u^{\nu)} + \pi^{\mu\nu}$$

$$\epsilon_1 = \mathcal{E} \left(\frac{u \cdot \partial \epsilon}{\epsilon_0 + p_0} + \partial \cdot u \right) . \quad p_1 = \frac{1}{3} \epsilon_1 , \quad \pi^{\mu\nu} = -2 \eta \sigma^{\mu\nu}$$

$$W^\mu = \Theta \left(- \frac{\partial^\nu T}{T} + u \cdot \partial u^\nu \right) P_\nu{}^\mu .$$

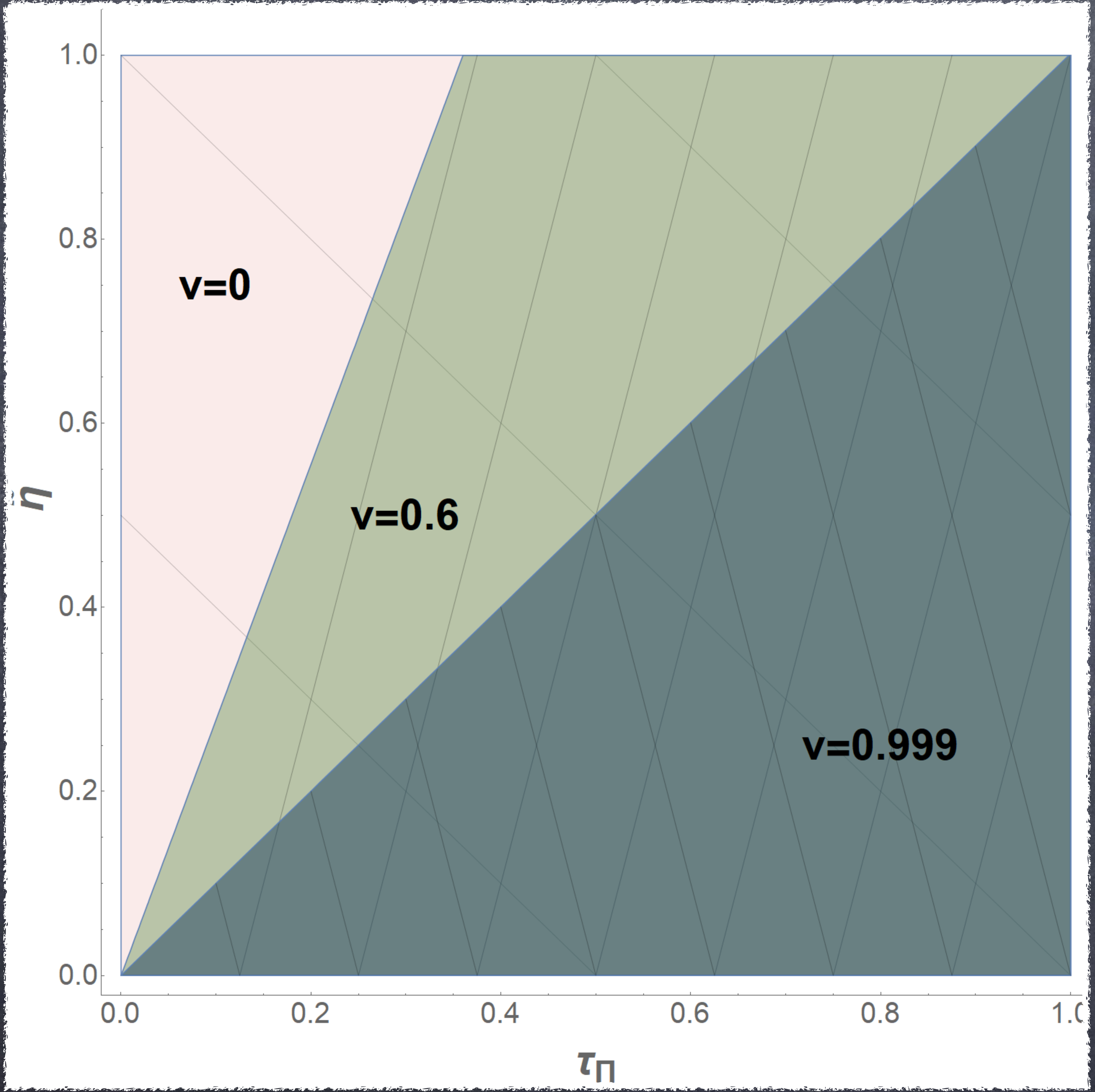
Algorithm

- Using linearised fluctuations and conservation of stress tensor, obtain dispersion relation at arbitrary boosted frame.
- Solve for ω in terms of transport coefficients and boost velocity in leading order in k.
- Perform stability analysis and explore the behaviour of stability criteria at different boost velocity v.
- The region of parameter space which remains stable at all velocities will be the causal region of the parameter space.

Stability Criteria: Routh-Hurwitz Test

- Since fluctuations are of the form $\exp[i(kx - \omega t)]$, Stability is achieved if $\mathrm{Im}(\omega) < 0$.
- Using Routh-Hurwitz Stability Criteria, stability of roots can be checked from the polynomial coefficients, without actually extracting the roots.
- Routh array can be constructed from the coefficients of the polynomial. Relative signs between the terms of the array indicate the number of stable roots.

Stability: MIS Shear

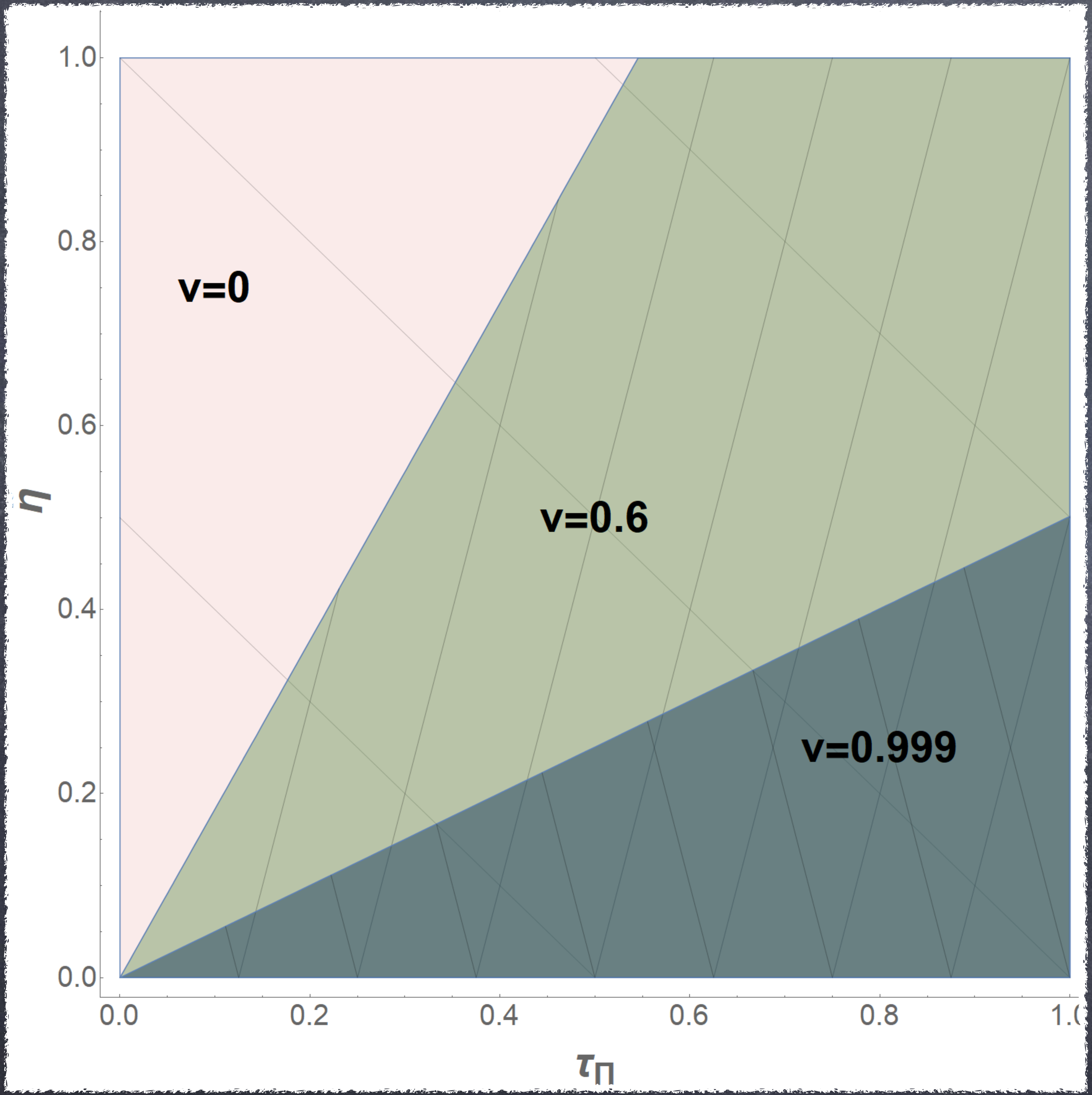


- Condition for stability:

$$\frac{\tau_\pi}{\tilde{\eta}} > v^2 , \quad \tilde{\eta} \equiv \frac{\eta}{\varepsilon_0 + p_0}$$

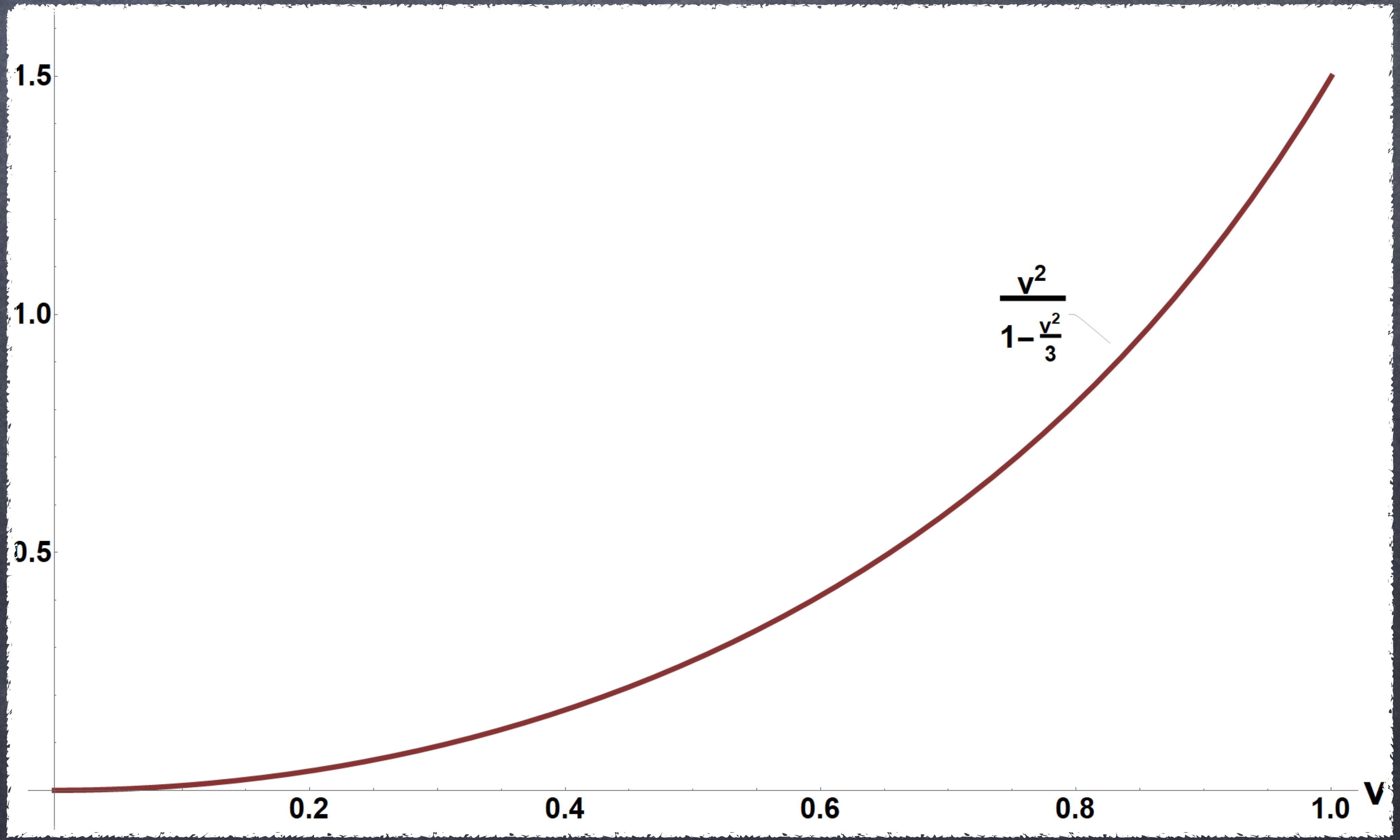
- R.H.S. is monotonic function of v with maximum at $v \to 1$.
- Bound gets stricter with increasing boost, strictest at full boost.
- Bound of every velocity is a subset of all lower velocities.

Stability: MIS Sound



- Condition for stability:

$$\frac{\tau_\pi}{\tilde{\eta}} > \frac{4}{3}\,\frac{v^2}{\left(1-\frac{v^2}{3}\right)}$$



Stability: BDNK Sound

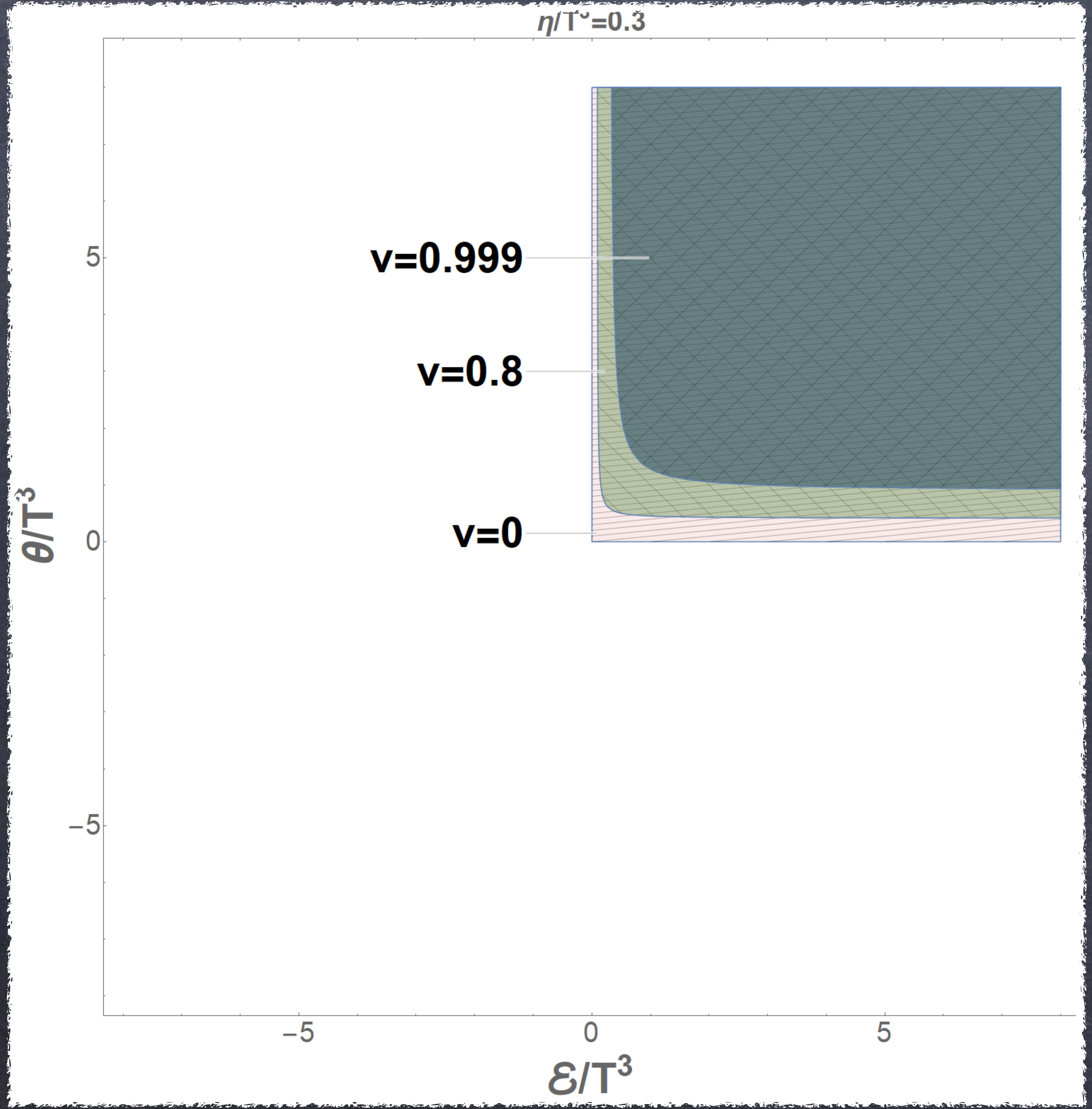
- Conditions for stability:

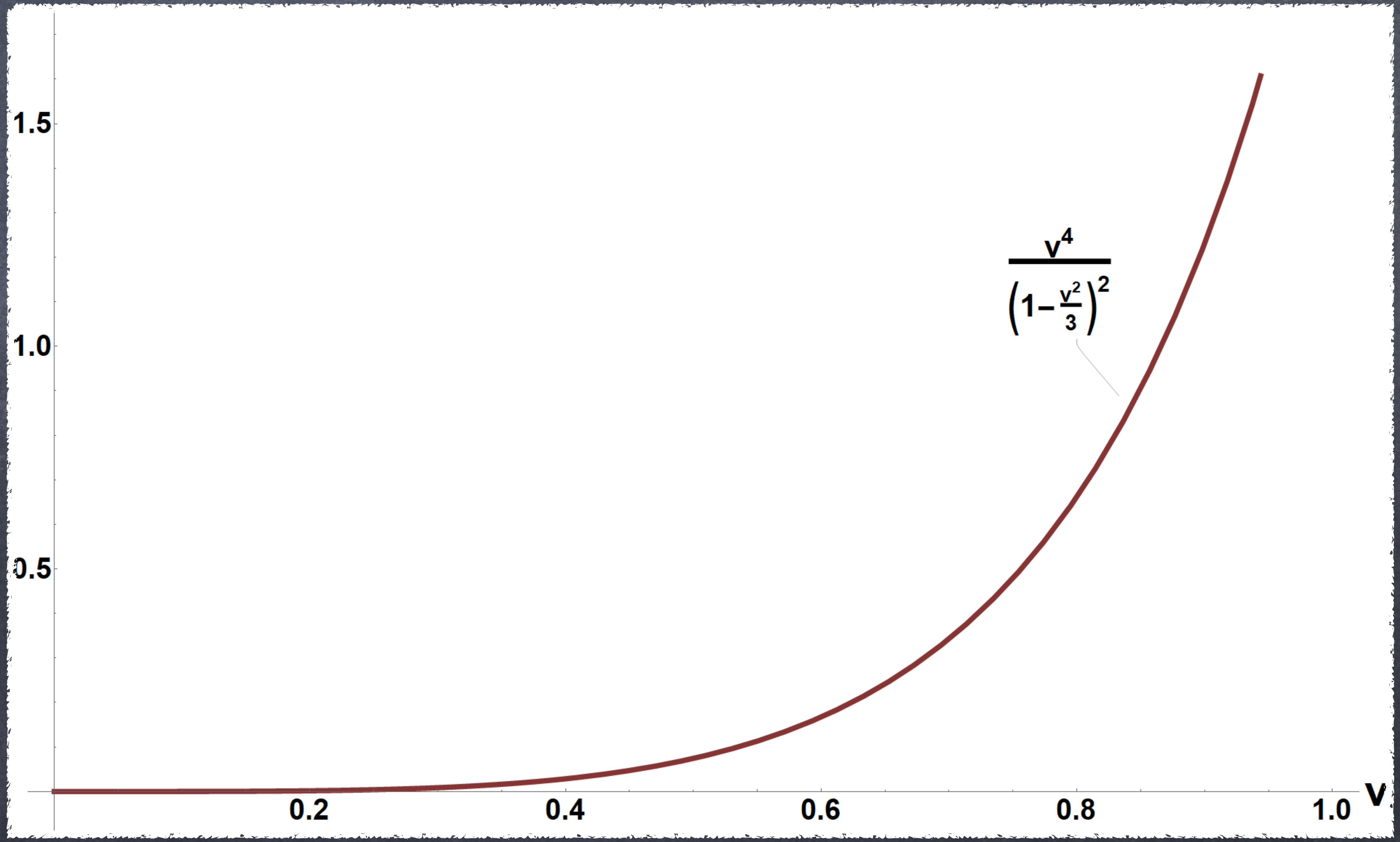
$$\mathcal{E}\theta\left(1-\frac{v^2}{3}\right)^2-\frac{4}{3}\eta v^2\left(\mathcal{E}+\frac{v^2}{3}\theta\right)>0$$

$$(\mathcal{E}+\theta)\left(1-\frac{v^2}{3}\right)-\frac{4}{3}\eta v^2>0$$

- Allowed parameter space (asymptotes of hyperbola):

$$\frac{\theta}{\eta}>\frac{4}{3}\frac{v^2}{\left(1-\frac{v^2}{3}\right)^2},\quad \frac{\mathcal{E}}{\eta}>\frac{4}{9}\left(\frac{v^2}{1-\frac{v^2}{3}}\right)^2$$





Causality from Stability

- Following the argument that there is a one-to-one relationship between the frame-invariantly stable parameter space and the causal parameter space of the theory, we can identify the causal parameter space by identifying the frame-invariantly stable parameter space.
- For all the above cases, the inequalities lead to progressively tighter bounds with increasing boost.
- Parameter space of higher boost is always enclosed within that of the lower boosts, so parameter space of highest boost should be enclosed within that of all other boosts.

Causality from Stability

- The parameter space at $v \to 1$ is the smallest and the frame-invariant subspace of stability and hence, stability at $v \to 1$ is necessary and sufficient condition for stability at $k \to 0$ limit.

- Conclusion: The stable parameter space at the highest boost i.e. $v \to 1$ is the frame-invariant stable parameter space and hence, the causal parameter space.

- Stability conditions at $v \to 1$ = Causality conditions

Stability Conditions at $v \to 1$

- MIS Shear: $\dfrac{\tau_\pi}{\tilde{\eta}} > 1 \Rightarrow \dfrac{\tilde{\eta}}{\tau_\pi} < 1$

- MIS Sound: $\dfrac{\tau_\pi}{2\tilde{\eta}} > 1 \Rightarrow \dfrac{2\tilde{\eta}}{\tau_\pi} < 1$

- BDNK Sound: $\varepsilon\theta - \eta(3\varepsilon + \theta) > 0.$

$$\varepsilon + \theta - 2\eta > 0.$$

- These criteria happen to be identical to those obtained from asymptotic causality analysis, but these have been obtained without departing from the small-k regime

Asymptotic Causality and v_g

- MIS Shear:

$$v_g^2 = \frac{\tilde{\eta}}{\tau_\pi}, \quad 0 < v_g^2 < 1 \Rightarrow \frac{\tilde{\eta}}{\tau_\pi} < 1.$$

- MIS Sound:

$$v_g^2 = \frac{4}{3}\frac{\tilde{\eta}}{\tau_\pi} + \frac{1}{3}, \quad 0 < v_g^2 < 1 \Rightarrow 2\frac{\tilde{\eta}}{\tau_\pi} < 1.$$

- BDNK Sound:

$$3v_g^4 \varepsilon\theta - 2v_g^2 \varepsilon(2\eta + \theta) + \frac{1}{3}(\varepsilon - 4\eta)\theta = 0$$

It can be shown that to have $v_g^2 < 1$, the first stability inequality must be satisfied. The second inequality further confines the parameter space to have only stable modes.

Asymptotic Causality: Schur Polynomials

- For general theories, polynomials can be higher-order in ω, k and finding v_g by root extraction can be difficult.
- Schur Polynomials can be used to find the existence of subluminal roots without solving the polynomial.
- Schur Stability: All the roots lie within a unit disc around the origin in the complex plane.
- Schur stability can be checked by applying a particular Möbius transformation on the original polynomial and then checking its RH stability.

Asymptotic Causality: Schur Polynomials

- Checking Schur Stability:

For the polynomial $P(z)$ of degree 'd',

Möbius Transform: $w = \dfrac{z+1}{z-1}$

$\rightarrow$ Maps the unit disc to left Half Plane.

$$Q(w) = (w-1)^d \, P\left(\frac{w+1}{w-1}\right)$$

Schur Stability of $P(z) \Leftrightarrow$ RH Stability of $Q(w)$.

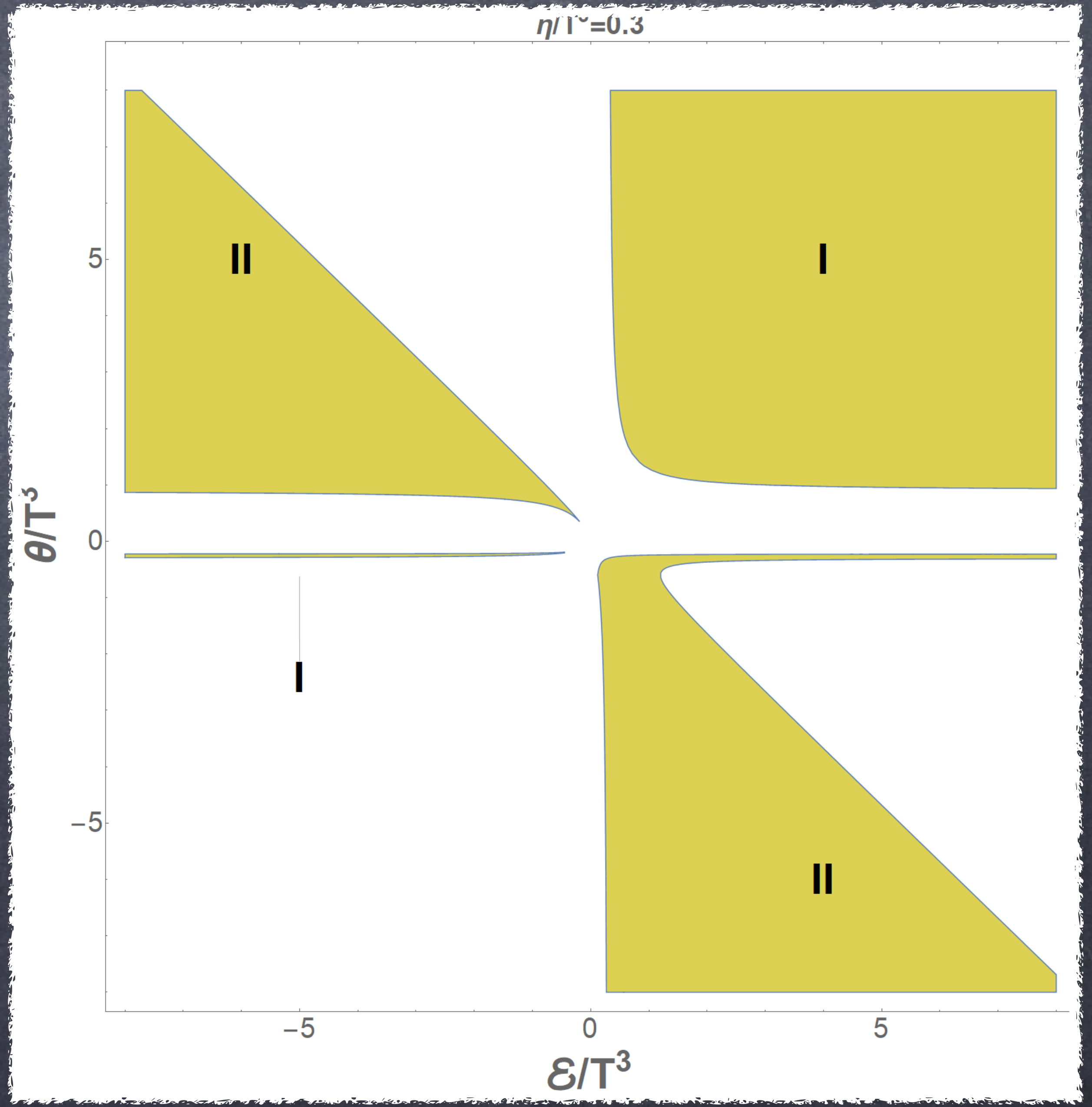
- For MIS Shear and Sound, and for BDNK Shear, Schur stability results easily match with the earlier obtained ones. BDNK Sound channel gives rise to non-trivialities due to multiple non-hydro modes.

The Curious Case of BDNK Sound Channel

Polynomial of v_g^2 : $P(z) = \varepsilon\theta z^2 - \frac{2}{3}\varepsilon(\theta+2\eta)z + \frac{1}{9}\theta(\varepsilon-4\eta) = 0$

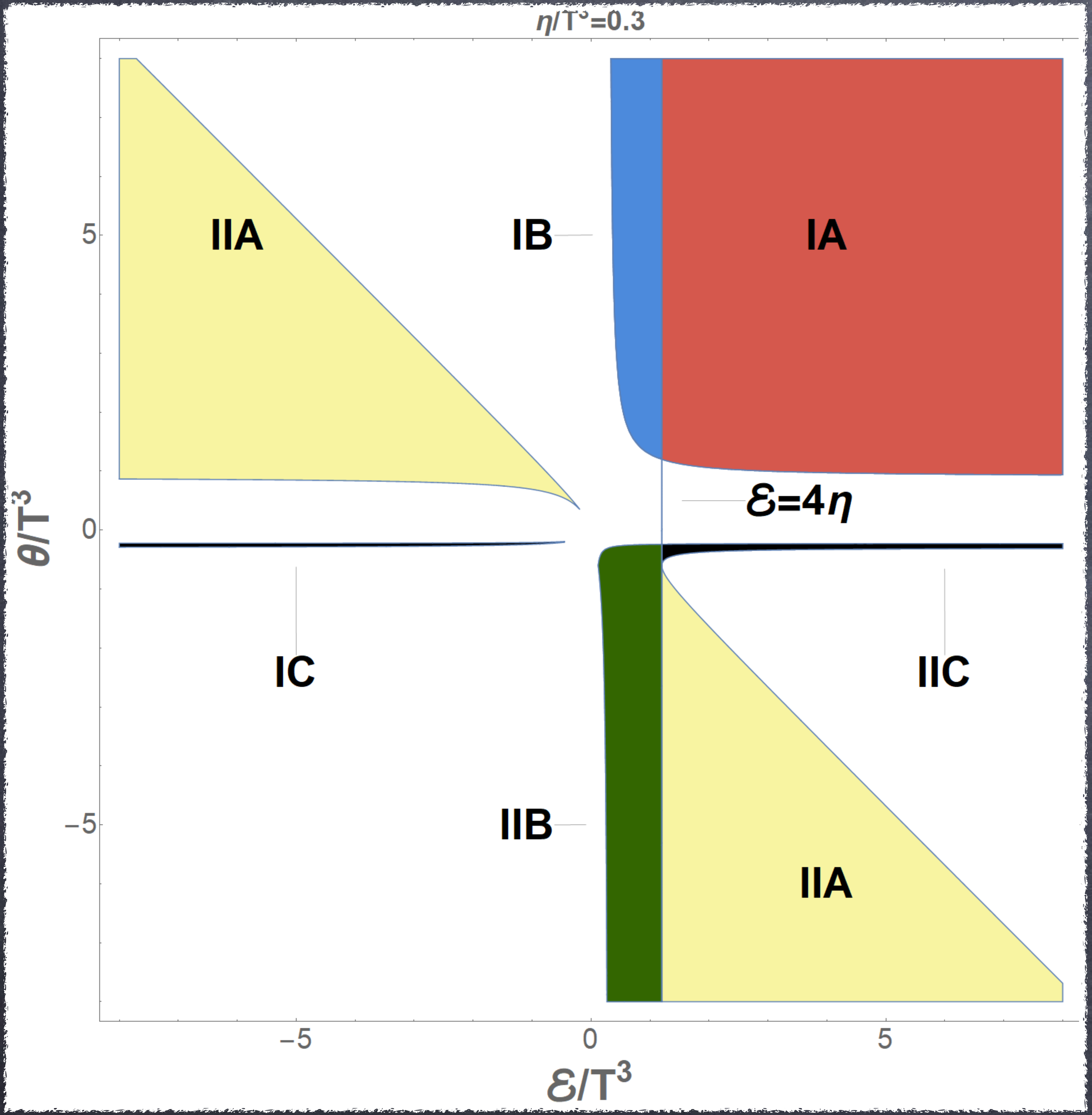
$$Q(w) = \underbrace{\left(\frac{\varepsilon\theta}{3} - \varepsilon\eta - \frac{\eta\theta}{3}\right)}_{T_1} w^2 + \frac{2}{3}\underbrace{\theta(\eta+2\varepsilon)}_{T_2} w + \underbrace{\left(\frac{4}{3}\varepsilon\theta + \varepsilon\eta - \frac{\eta\theta}{3}\right)}_{T_3} = 0$$

Possibility - 1	Possibility - 2
(i) $\frac{\varepsilon\theta}{3} - \varepsilon\eta - \frac{\eta\theta}{3} > 0$	$\frac{\varepsilon\theta}{3} - \varepsilon\eta - \frac{\eta\theta}{3} < 0$
(ii) $\theta(\eta+2\varepsilon) > 0$	$\theta(\eta+2\varepsilon) < 0$
(iii) $\frac{4}{3}\varepsilon\theta + \varepsilon\eta - \frac{\eta\theta}{3} > 0$	$\frac{4}{3}\varepsilon\theta + \varepsilon\eta - \frac{\eta\theta}{3} < 0$



Together, the total region from both possibilities give full causal space.

The Curious Case of BDNK Sound Channel



IC, IIC: $v_g^2 \in [-1,0]$: No propagating mode.

IA, IIA: $v_g^2 \in [0,1]$: 4 propagating modes.

IB, IIB: $1\times v_g^2 \in [0,1]$, $1\times v_g^2 \in [-1,0]$

$\Rightarrow$ 2 propagating, 2 non-propagating modes.

(i) = $V \rightarrow 1$ Stability Constraint

(ii) + (iii) + Real Roots = $\mathcal{E} > 0$. $\theta > 0$.

$\Rightarrow$ Define the stable parameter space.

Conclusion and Future Directions

- Stable parameter space at $v \to 1$ is the necessary and sufficient region for frame-invariant stability at the spatially homogeneous limit.
- Stability analysis at $v \to 1$ limit identifies the causal parameter space given by asymptotic causality criteria, without going out of the small-k regime. Hence, more suitable for low-energy effective theory like hydrodynamics.
- Stability criteria at $v \to 1$ give us the region of parameter space which is stable as well as causal in all reference frames.
- Schur stability check can be a very efficient tool.
- Non-zero k analysis? Nonlinear causality analysis?
- Generalisation to other stable-causal hydrodynamic models?

Thank You

Backup Slides

Dispersion Polynomials at Local Rest Frame

- MIS Shear: $4\tau_\pi \omega^2 + 4i\omega - \eta k^2 = 0$
- MIS Sound:

$$\tau_\pi \omega^3 + i\omega^2 - \frac{1}{3}(\eta + \tau_\pi)\omega k^2 - \frac{1}{3} i k^2 = 0$$

- BDNK Shear: $\theta \omega^2 + 4i\omega - \eta k^2 = 0$
- BDNK Sound:

$$3\varepsilon\theta\omega^4 + 12i(\varepsilon+\theta)\omega^3 - 2\{24 + k^2\varepsilon(2\eta+\theta)\}\omega^2$$

$$-4i(\varepsilon + 4\eta + \theta)k^2\omega + \frac{1}{3}k^2\{48 + k^2\theta(\varepsilon - 4\eta)\} = 0$$

Asymptotic Causality and v_g

- Special case of BDNK Sound:

$$\mathcal{E}\Theta\left(1-\frac{v^2}{3}\right)^2 - \frac{4}{3}\eta v^2\left(\mathcal{E}+\frac{v^2}{3}\Theta\right) > 0$$

$$= \left(\frac{1}{v^2}-x_1\right)\left(\frac{1}{v^2}-x_2\right) > 0 \Rightarrow \frac{1}{v^2} > x_1, x_2 \text{ or } \frac{1}{v^2} < x_1, x_2$$

where, x_1, x_2 are roots of eq^n. for v_g^2

$$\mathcal{E}\Theta x^2 - \frac{2}{3}\mathcal{E}(\Theta+2\eta)x + \frac{1}{9}\Theta(\mathcal{E}-4\eta) = 0$$

$v \in [0,1) \Rightarrow \frac{1}{v^2} \in (1,\infty) \Rightarrow \frac{1}{v^2} < x_1, x_2$ is unphysical.

Tightest Bound : $x_1, x_2 < 1$.

$\mathcal{E}+\Theta > 2\eta$ restricts parameter space to $\Theta > 0, \mathcal{E} > 0$.

$\rightarrow$ Both roots real, at least one root +ve.