Thermal CFTs, KMS and Tauberian theorems

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Based on [E.Marchetto, AM, E. Pomoni; to appear]
and [E.Marchetto, AM, E. Pomoni; 2306.12417]

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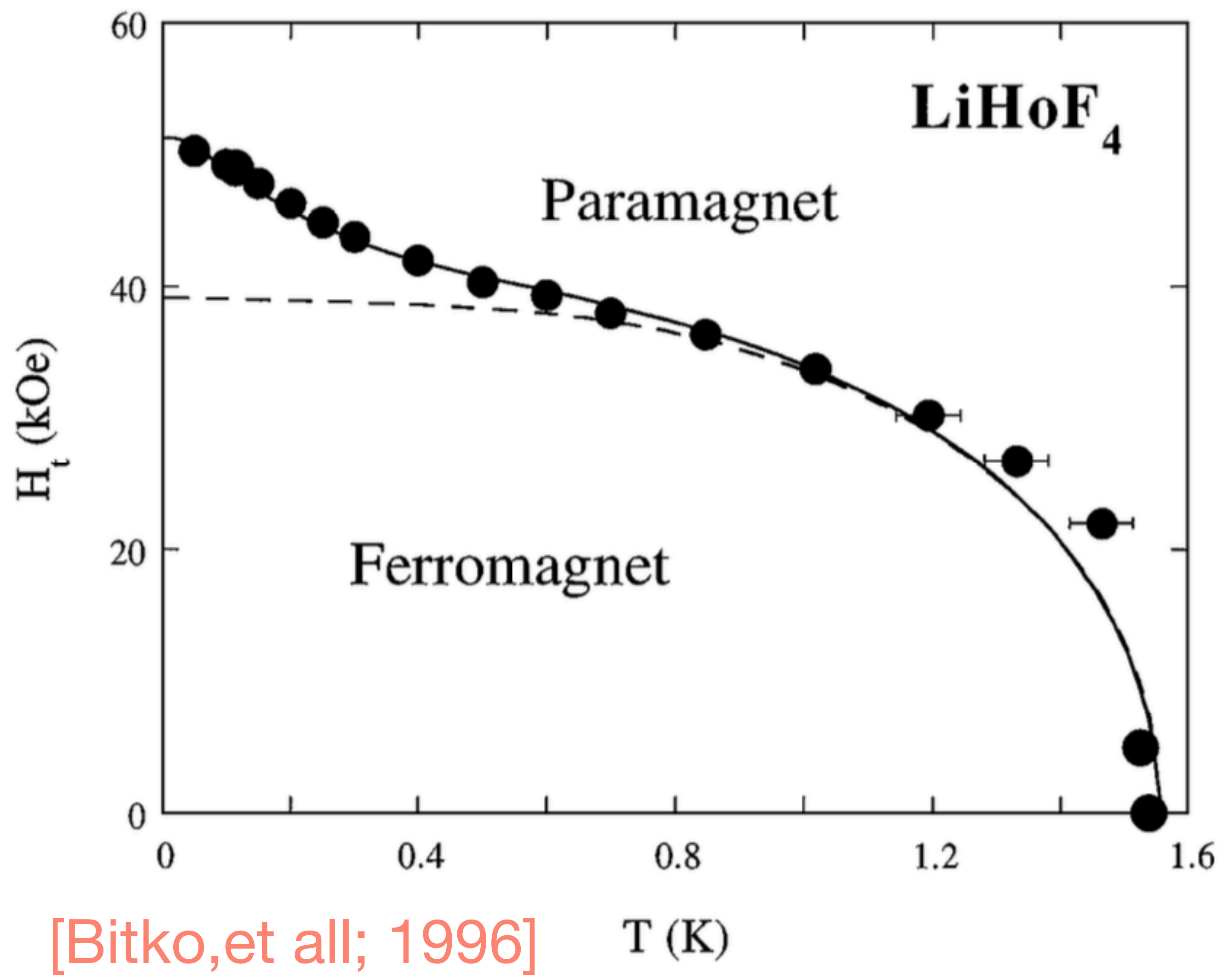
DESY Theory Workshop: String Theory/Mathematical Physics Parallel Sessions

Motivations : Why thermal CFTs?

- *Thermal effects in QFT*: the effect of the temperature on QFTs are relevant in labs;



[Bitko,et all; 1996]

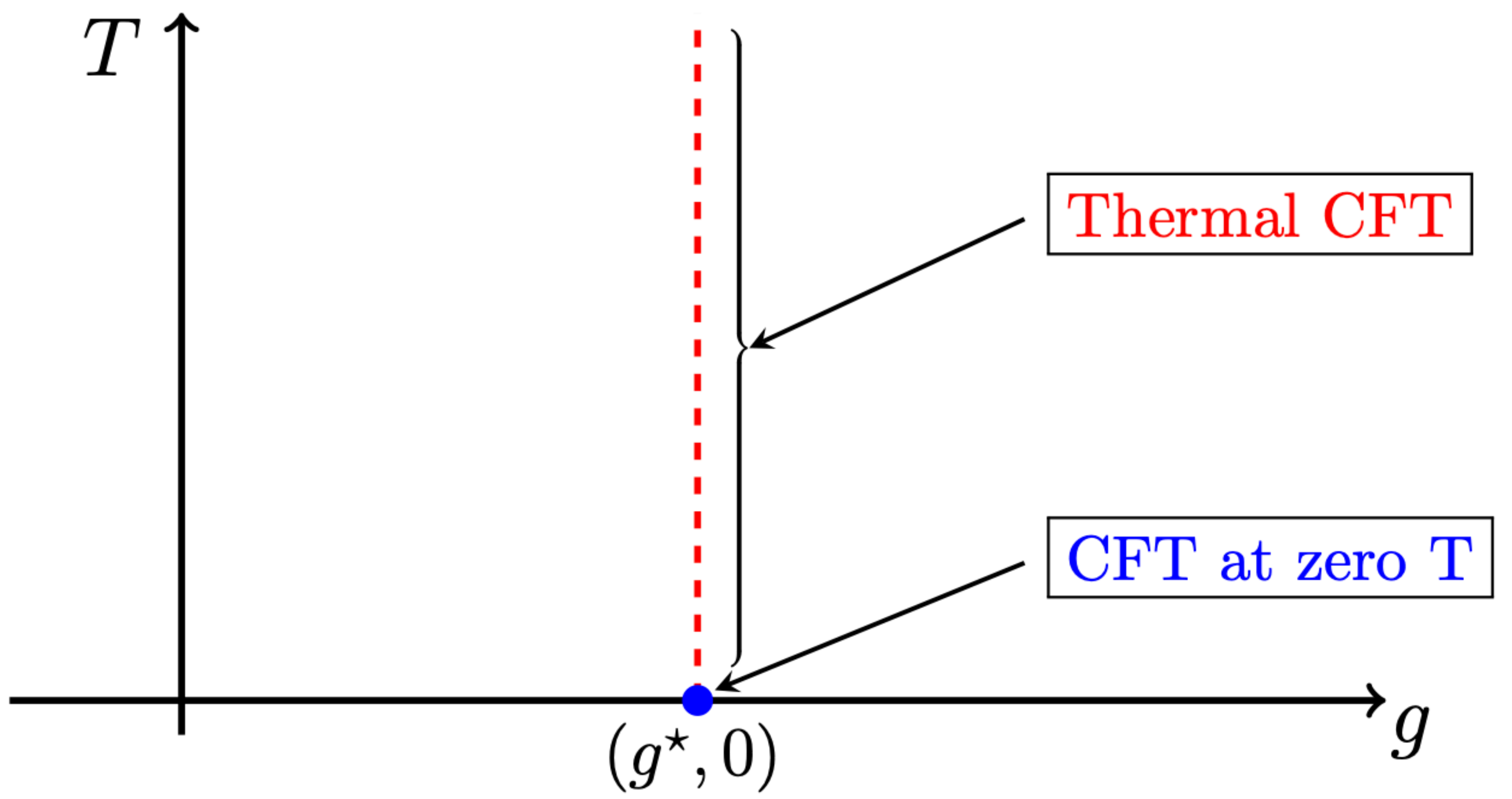
- *Black Holes and AdS/CFT*: Thermal CFTs are fundamental in AdS/CFT since they are duals to a black hole in AdS;

- *CFTs on non-trivial manifolds*: Thermal effects on a QFTs can be studied by compactifying the time direction.

Thermal CFTs: the setup

A thermal CFT can be thought as a CFT placed on the manifold

$$S^1_\beta \times \mathbb{R}^{d-1} \qquad \left(\beta = \frac{1}{T}\right)$$



Many structures of the CFT are preserved:

$$\Delta_{\mathcal{O}}\ , \qquad \mathcal{O}_1(x) \times \mathcal{O}_2(0) = \sum_{\mathcal{O}} f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}} |x|^{\Delta_{\mathcal{O}}-\Delta_{\mathcal{O}_1}-\Delta_{\mathcal{O}_2}-J} x_{\mu_1}\ldots x_{\mu_J}\ \mathcal{O}^{\mu_1\ldots\mu_J}(0)$$

$$|x| < \beta$$

But new observable are available:

$$\boxed{\langle \mathcal{O}(x)\rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}}$$

Thermal CFTs: the setup

A two-point function between scalars at finite temperature is

$$\langle \phi(\tau, r) \times \phi(0) \rangle_\beta = \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} \left(\sqrt{r^2 + \tau^2} \right)^{\Delta_{\mathcal{O}} - 2\Delta_\phi - J} x_{\mu_1} \ldots x_{\mu_J} \, \langle \mathcal{O}^{\mu_1 \ldots \mu_J} \rangle_\beta$$

Thermal CFTs: the setup

A two-point function between scalars at finite temperature is

$$\langle \phi(\tau, r) \times \phi(0) \rangle_\beta = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \left(\sqrt{r^2+\tau^2}\right)^{\Delta_{\mathcal{O}}-2\Delta_\phi} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2+r^2}}\right)$$

$$\boxed{a_{\mathcal{O}} = b_{\mathcal{O}} f_{\mathcal{O}\phi\phi} \frac{J!}{2^J (\nu)_J}}$$

$$\nu = \frac{d-2}{2}$$

From symmetries (e.g. broken Ward identities)

[Iliesiu, Kologlu, et all; 2018]

[Marchetto,AM, Pomoni; 2023]

Setting the problem…

Zero Temperature	Finite Temperature
$\Delta_{\mathcal{O}}$	$b_{\mathcal{O}}$
$f_{\mathcal{O}\phi\phi}$	

What can be said about thermal two-point functions and one-point functions?

The Plan

- Equations from KMS -> Light operators;
- A thermal Tauberian theorem -> heavy operators;
- A different prospective: the Källén-Lehmann spectral representation and form factors;

Light operators and KMS

- Equations from KMS -> Light operators;
- A thermal Tauberian theorem -> heavy operators;
- A different prospective: the Källén-Lehmann spectral representation and form factors;

KMS: a set of equations for OPE coefficients

Consider for simplicity r = 0, i.e. zero spatial coordinates:

Kubo–Martin–Schwinger condition (KMS):

[Kubo; 1957]
[Martin,Schwinger; 1959]

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \langle \phi(\tau+\beta)\phi(0)\rangle_\beta$$

Parity:

[Iliesiu, Kologlu, et all; 2018]

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \langle \phi(-\tau)\phi(0)\rangle_\beta$$

$$\left\langle \phi\left(\frac{\beta}{2}+\tau\right)\phi(0)\right\rangle_\beta = \left\langle \phi\left(\frac{\beta}{2}-\tau\right)\phi(0)\right\rangle_\beta$$

Crossing-like equation

[El-Showk,Papadodimas; 2011]

KMS: a set of equations for OPE coefficients

Consider for simplicity r = 0, i.e. zero spatial coordinates:

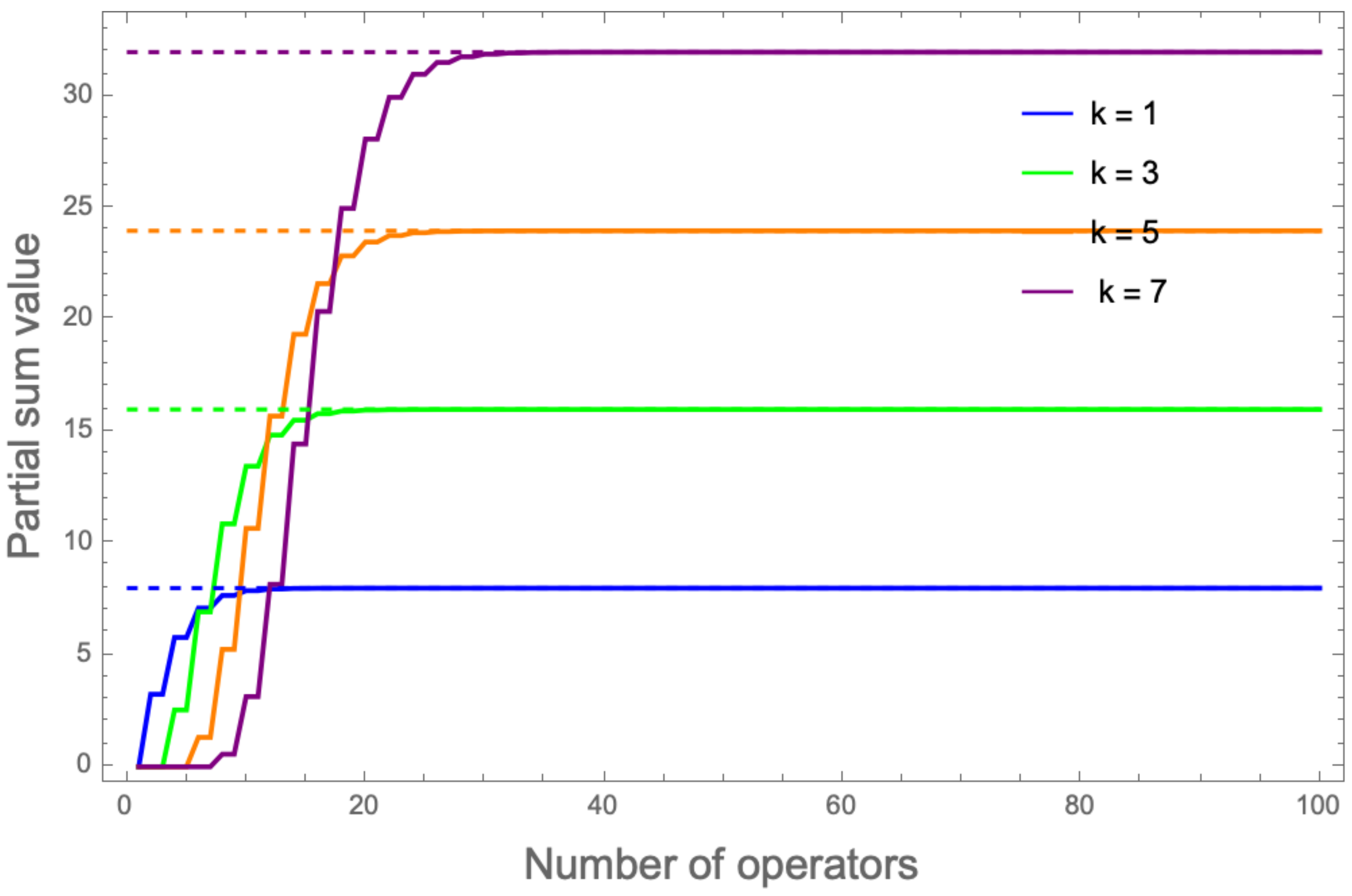
$$\boxed{\frac{\Gamma\left(2\Delta_\phi + k\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta \neq 0} \frac{a_\Delta}{2^\Delta} \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{\Gamma\left(\Delta - 2\Delta_\phi - k + 1\right)}}$$

$k \in 2\mathbb{N} + 1$

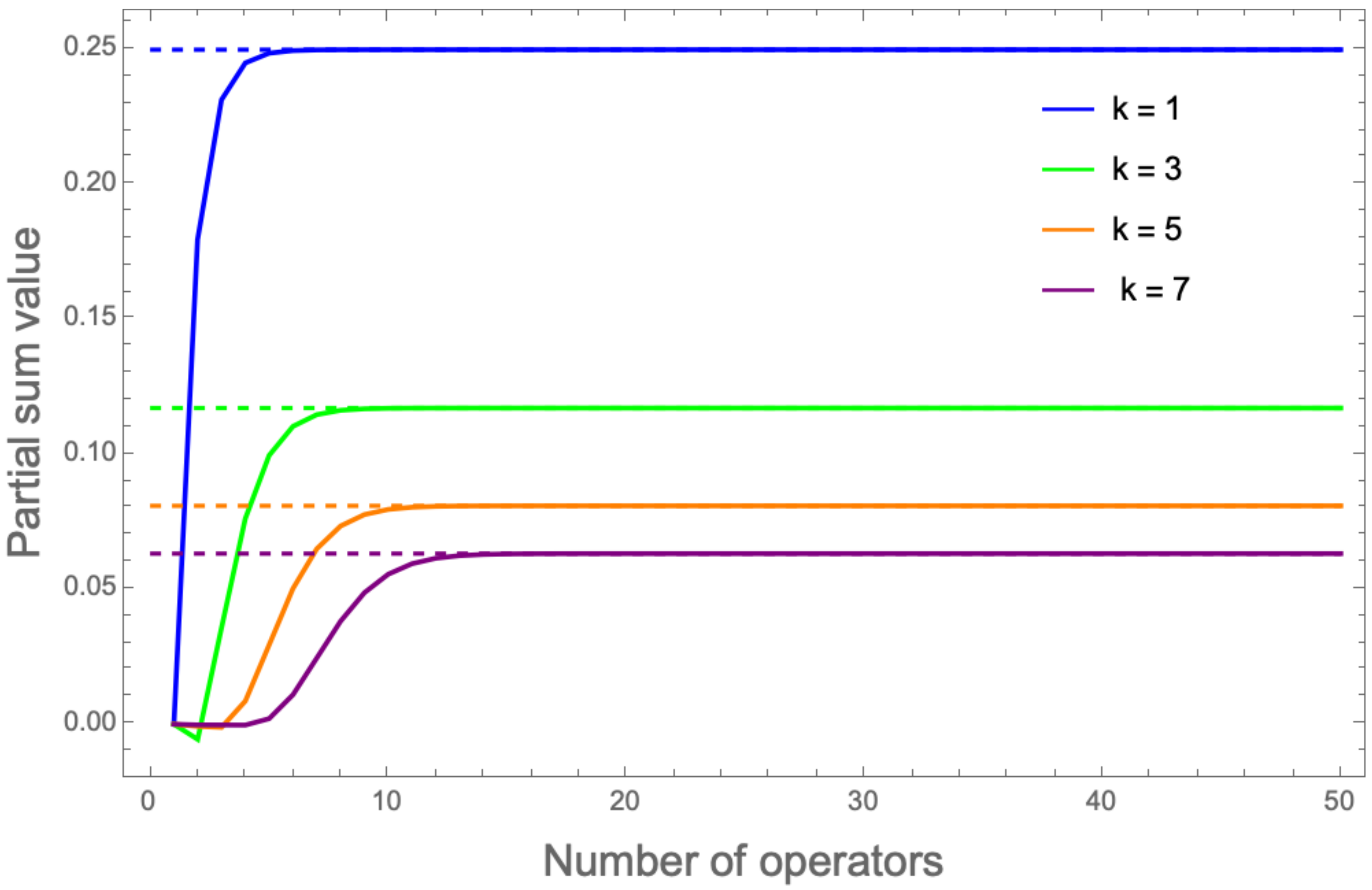
Infinite set of equations…

KMS set of equations: some examples

4-dimensional free theory



$\langle\sigma(\tau)\sigma(0)\rangle_\beta$ *correlator of 2d Ising*



Observation: for <u>small k light operators contributes</u>. The bigger is k the more operators we have to insert. We can justify this (later).

KMS: prediction for large gapped theories

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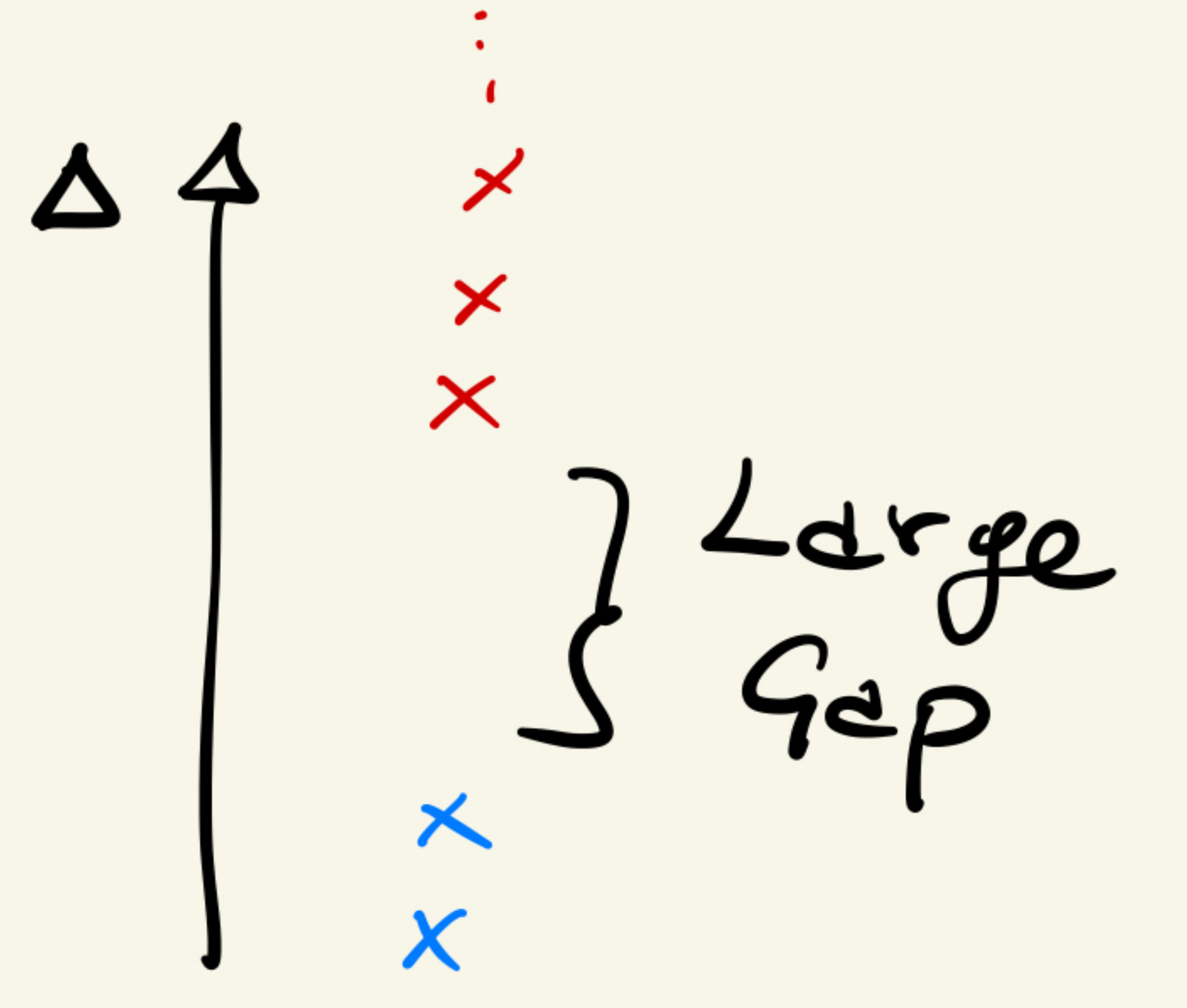
- Consider large gapped theories with few light operators (e.g. 1,2,…)
- Solve a finite number (1,2,…) of KMS equations in terms of the OPE coefficients and one-point functions.



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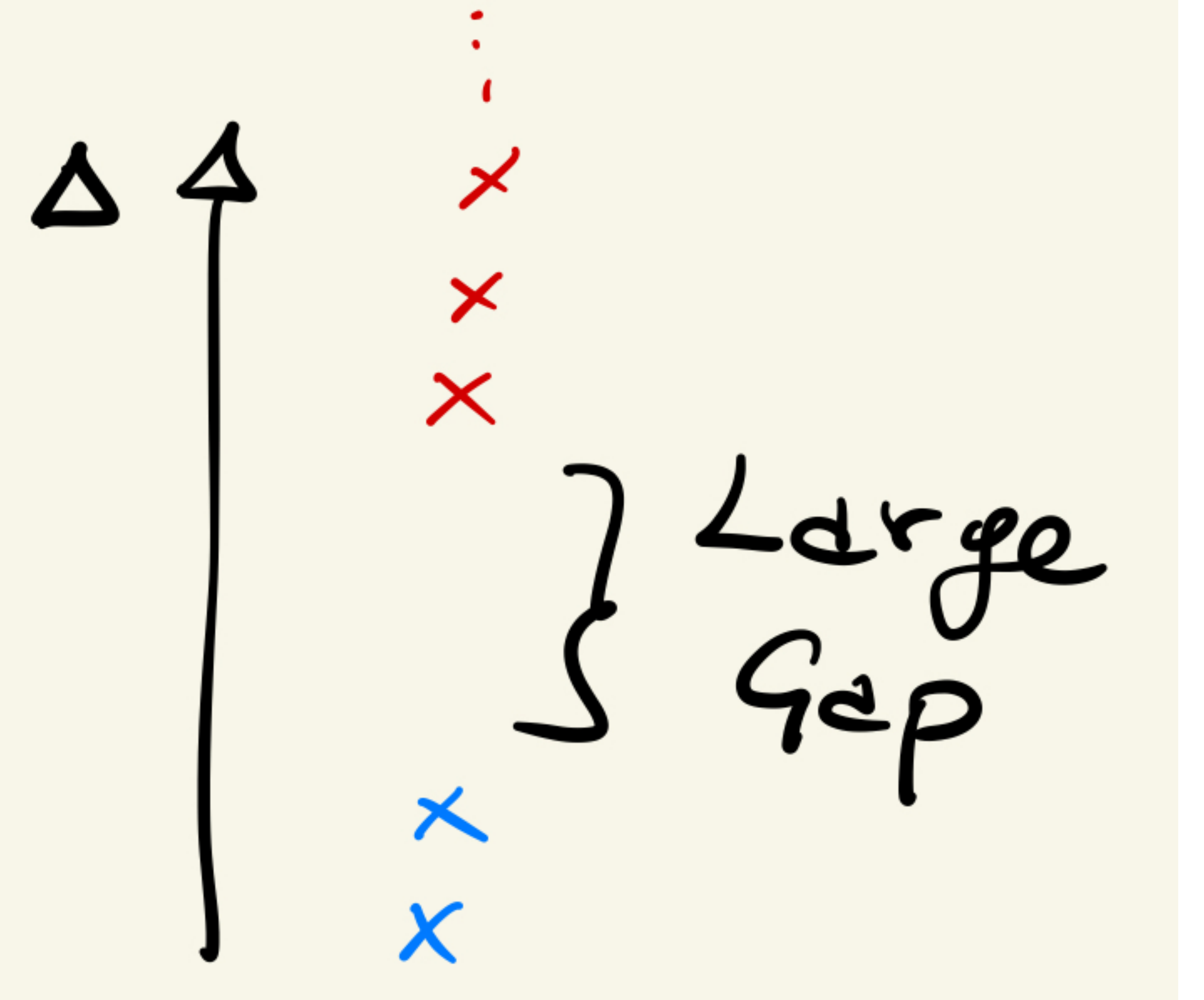

One light operator:

$$b_{\mathcal{O}_L} = 2^{\Delta_{\mathcal{O}_L}} \frac{2\Delta_\phi}{\Delta_{\mathcal{O}_L} - 2\Delta_\phi} \frac{1}{f_{\phi\phi\mathcal{O}_L}}$$

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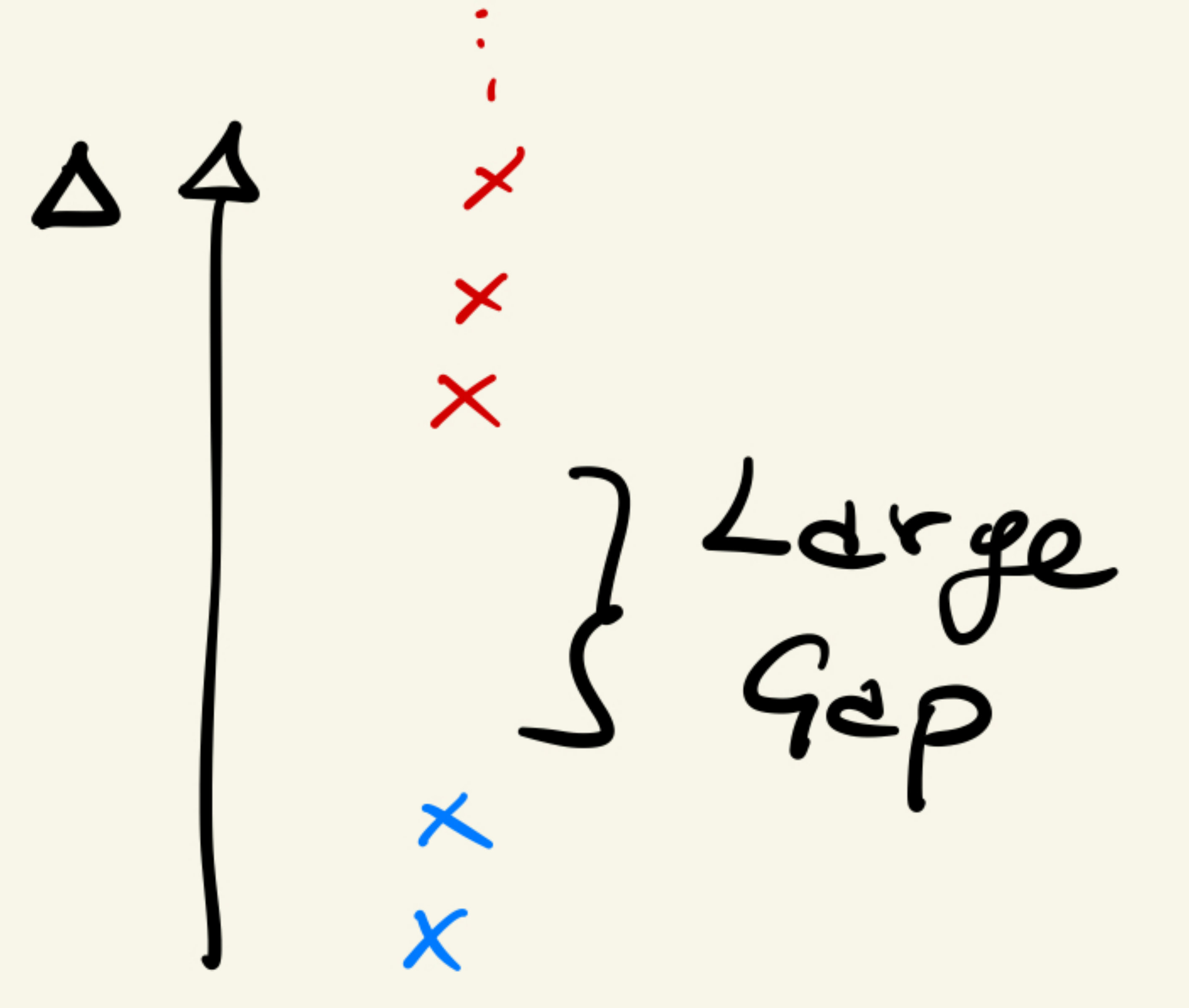

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Two light operators:

$$b_{\mathcal{O}_{1L}} = \frac{(\Delta_1 - 2\Delta_\phi)^{-1}\Big(2\Delta_\phi(1+2\Delta_\phi-\Delta_2)(2+2\Delta_\phi-\Delta_2) - 4\Delta_\phi(1+\Delta_\phi)(1+2\Delta_\phi)\Big)}{f_{\phi\phi\mathcal{O}_{1L}} 2^{\Delta_1}\Big((1+2\Delta_\phi-\Delta_2)(2+2\Delta_\phi-2\Delta_2) - (1+2\Delta_\phi-\Delta_1)(2+2\Delta_\phi-\Delta_1)\Big)}$$

$$b_{\mathcal{O}_{2L}} = \frac{(\Delta_2 - 2\Delta_\phi)^{-1}\Big(2\Delta_\phi(1+2\Delta_\phi-\Delta_1)(2+2\Delta_\phi-\Delta_1) - 4\Delta_\phi(1+\Delta_\phi)(1+2\Delta_\phi)\Big)}{f_{\phi\phi\mathcal{O}_{2L}} 2^{\Delta_2}\Big((1+2\Delta_\phi-\Delta_1)(2+2\Delta_\phi-2\Delta_1) - (1+2\Delta_\phi-\Delta_2)(2+2\Delta_\phi-\Delta_2)\Big)}$$

Heavy operators

- Equations from KMS -> Light operators;
- A thermal Tauberian theorem -> heavy operators;
- A different prospective: the Källén-Lehmann spectral representation and form factors;

"Thermal" OPE density: heuristic derivation

Consider r = 0, i.e. zero spatial coordinates: the two-point function is

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \int_0^\infty \rho(\Delta)\,\frac{\tau^{\Delta-2\Delta_\phi}}{\beta^\Delta} \overset{\tau\to\beta}{\sim} \left(\beta-\tau\right)^{-2\Delta_\phi}$$

Where $\rho(\Delta) = \sum_{\Delta'} \delta(\Delta'-\Delta)a_\Delta$

Try the following **ansatz**: $\rho(\Delta) \sim A\Delta^{-\alpha}$

$$\Rightarrow \quad \rho(\Delta) \simeq \frac{1}{\Gamma(2\Delta_\phi)}\Delta^{2\Delta_\phi-1} \quad \Rightarrow \quad \int_0^{\tilde{\Delta}} \rho(\Delta)\, d\Delta \sim \frac{\tilde{\Delta}^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)}$$

To justify why the density has to grow as a power-law…

“Thermal” OPE density

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Where $\rho(\Delta) = \sum_{\Delta'} \delta(\Delta'-\Delta)a_\Delta$

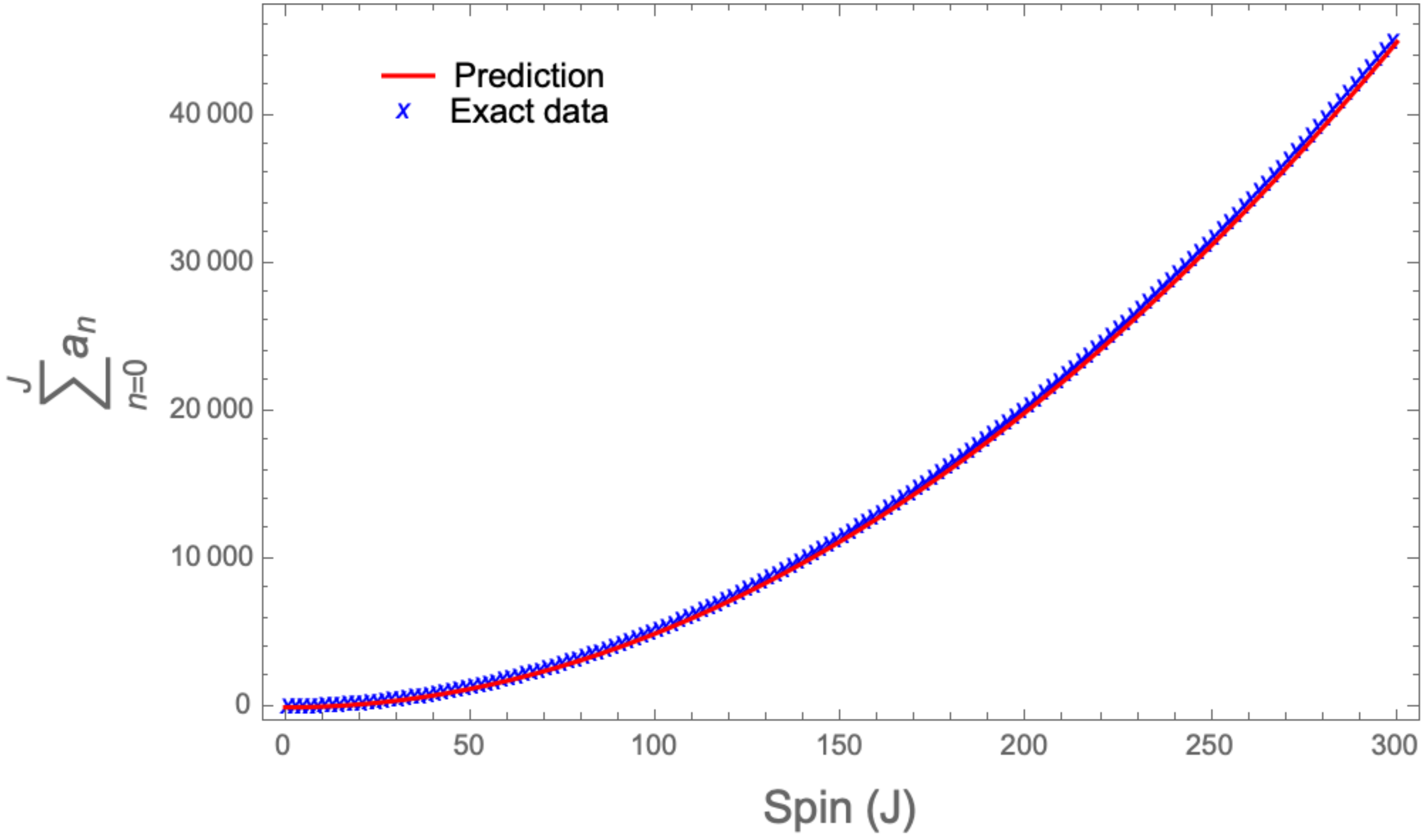
$$\Rightarrow \int_0^{\tilde{\Delta}} \rho(\Delta)\, d\Delta \sim \frac{\tilde{\Delta}^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)} \Rightarrow \boxed{a_{\tilde{\Delta}} \sim \frac{\tilde{\Delta}^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\delta\tilde{\Delta}}$$

For large $\tilde{\Delta}$

“Thermal” OPE density: free theory in 4d/2d Ising

Free theory in 4d which is equivalent $\langle \epsilon(\tau)\epsilon(0)\rangle_\beta$ *in 2d Ising :*

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \left(\frac{\pi}{\beta}\right)^2 \csc^2\left(\frac{\pi}{\beta}\tau\right)$$



“Thermal” OPE density: 3d O(N) model at large N

Lagrangian description :

Hubbard-Stratanovich field

$$\mathscr{L} = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}\sigma\phi_i^2 - \frac{\sigma}{4\lambda}$$

The critical point is $\lambda \to \infty$

$$\langle \phi_i(\tau, r)\phi_j(0,0)\rangle_\beta = \delta_{ij}\sum_{m=-\infty}^{\infty}\int \frac{d^2k}{(2\pi)^2}\frac{e^{-i\vec{k}\cdot\vec{x}-i\omega_m\tau}}{\omega_n^2+\vec{k}^2+m_{th}^2} = \delta_{ij}\sum_{m=-\infty}^{\infty}\frac{e^{-m_{th}\sqrt{(\tau+m\beta)^2+r^2}}}{\sqrt{(\tau+m\beta)^2+r^2}}$$

$$\langle\sigma\rangle_\beta = m_{th}^2 = \frac{4}{\beta^2}\log^2\left(\frac{1+\sqrt{5}}{2}\right)$$

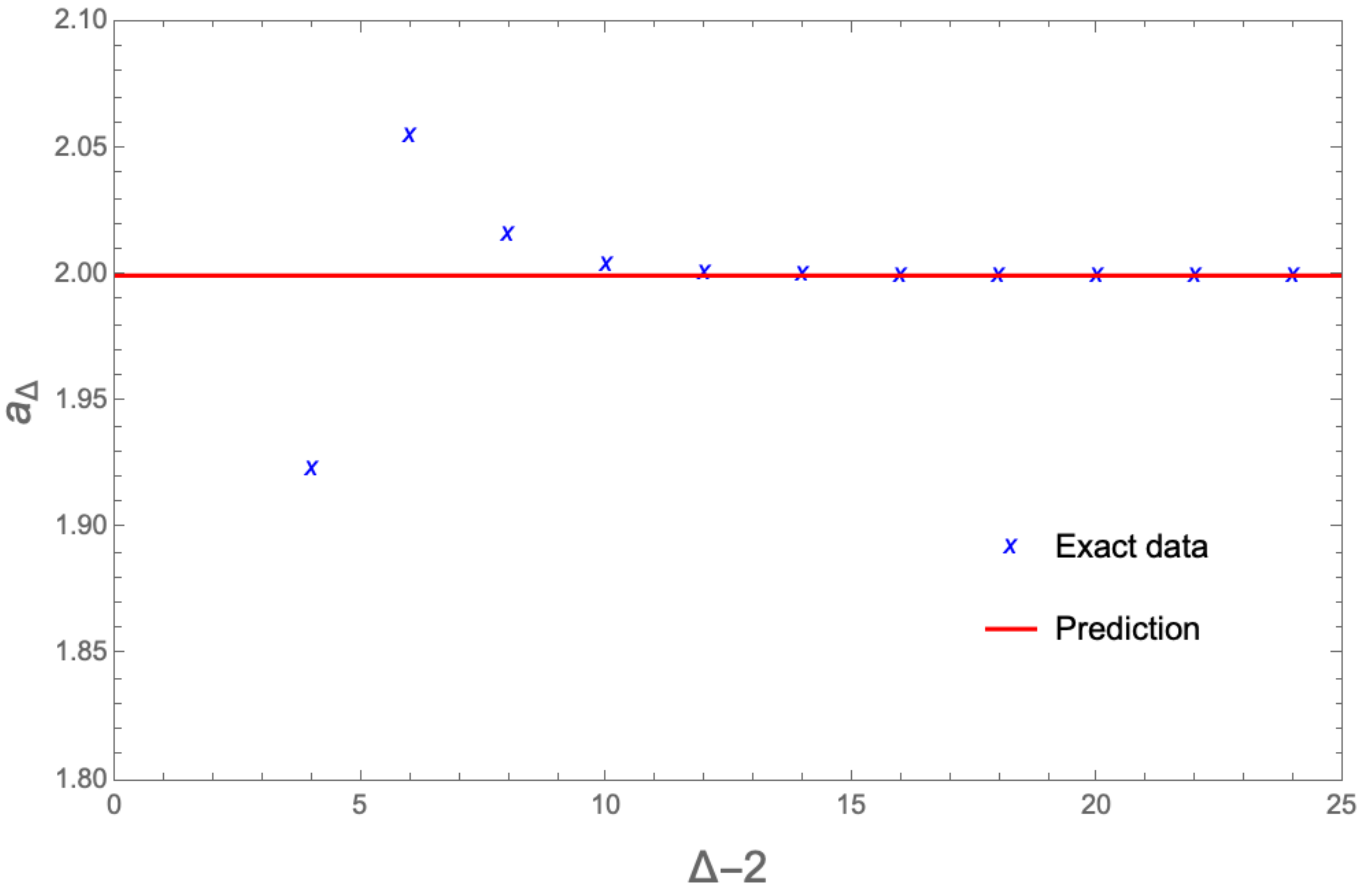
[Sachdev, Ye; 1992]

[Iliesiu, Kologlu, et all; 2018]

OPE density: 3d O(N) model at large N

The two-point function for zero spatial coordinates is

$$\langle \phi_i(\tau)\phi_j(0)\rangle_\beta = \delta_{ij}\left(\frac{e^{m_{th}(\tau-\beta)}}{\beta-\tau}\ {}_2F_1\left(\left\{\begin{matrix} 1\ ,\frac{\beta-\tau}{\beta} \\ \frac{2\beta-\tau}{\beta}\end{matrix}\right\}\middle|\ e^{-m_{th}\beta}\right)+ \right.$$
$$\left. +\frac{e^{-m_{th}\tau}}{\tau}\ {}_2F_1\left(\left\{\begin{matrix} 1\ ,\frac{\tau}{\beta} \\ \frac{\beta+\tau}{\beta}\end{matrix}\right\}\middle|\ e^{-m_{th}\beta}\right)\right)$$



OPE density: the precise derivation

Tauberian theorems:

[Hardy] [Littelhood][Karamata,…] [Korevaar; 2004] (review)
[Vladimirov, Zavyalov; 1981] [Pappadopulo, Rychkov, Espin, Rattazzi;2012]
[Qiao, Rychkov; 2017] [Mukhametzhanov, Zhiboedov; 2019]

$$\int_0^\infty d\Delta\ \rho(\Delta)(1-x)^\Delta \overset{x\to 0}{\sim} x^{-2\Delta_\phi} \quad \Rightarrow \quad \int_0^{\tilde{\Delta}} \rho(\Delta)\ d\Delta \sim \frac{\tilde{\Delta}^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)}$$

Under the condition: $\rho(\Delta) \geq -C$

OPE density: the precise derivation

To prove: $\rho(\Delta) \geq -C$ *Work in progress…*

But verified in different cases:

- (Generalized) free theories;
- Two dimensional conformal field theories; (Unitarity plays a role…)
- Large N O(N) model in 3d;
- Holographic theories for heavy operators;

[Rodriguez-Gomez, Russo; 2021]

A Rigorous Tauberian theorem for thermal QFTs

- Equations from KMS -> Light operators;
- A thermal Tauberian theorem -> heavy operators;
- A different prospective: the Källén-Lehmann spectral representation and form factors;

Källén-Lehmann spectral density

Consider r = 0, i.e. zero spatial coordinates: the two-point function is

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \sum_{\mathscr{E}} e^{i\mathscr{E}\tau} \left| \langle 0 | \phi | \mathscr{E} \rangle_\beta \right|^2 = \int_0^\infty \rho_{KL}(\mathscr{E}) e^{i\tau\mathscr{E}} \, d\mathscr{E}$$

$$\rho_{KL}(\mathscr{E}) = \sum_{\tilde{\mathscr{E}}} \delta(\mathscr{E} - \tilde{\mathscr{E}}) \left| \langle 0 | \phi | \tilde{\mathscr{E}} \rangle_\beta \right|^2$$

Tauberian theorem:

$$\int_0^{\tilde{\mathscr{E}}} \rho(\mathscr{E}) d\mathscr{E} \sim \frac{(-)^{\Delta_\phi} \tilde{\mathscr{E}}^{2\Delta_\phi}}{\Gamma(2\Delta_\phi + 1)} \quad \Rightarrow \quad \boxed{\left| \langle 0 | \phi | \mathscr{E} \rangle_\beta \right|^2 \sim \frac{(-)^{\Delta_\phi} \mathscr{E}^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \delta\mathscr{E}}$$

True in any thermal QFT!!!

Combing with Broken Ward identities

$$2\pi\mathbb{D}\,\langle \mathcal{O}_1\ldots\mathcal{O}_n\rangle_\beta = \beta\int_{\mathbb{R}^{d-1}} d^{d-1}x\,\langle T^{00}(0,x)\mathcal{O}_1\ldots\mathcal{O}_n\rangle_\beta \quad\Longrightarrow\quad \boxed{\mathbb{H} = \frac{2\pi}{\beta}\mathbb{D}}$$

[Marchetto,AM, Pomoni; 2023]

Some predictions can be tested analytically:

Free theory in 4d:

$$\left|\langle 0|\phi(0)|\Delta\rangle\right|^2 = -\frac{2\pi}{\beta}n$$

Two dimensional primaries' two-point functions:

$$\left|\langle 0|\phi(0)|\Delta\rangle\right|^2 \overset{\Delta\to\infty}{\sim} (-)^{\Delta_\phi}\left(\frac{2\pi}{\beta}\right)^{2\Delta_\phi-1}\frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}$$

Conclusions:

- Infinite set of equation coming from KMS and one-point functions for light operators in large gapped theories:

$$\frac{\Gamma\left(2\Delta_\phi + k\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta\neq 0} \frac{a_\Delta}{2^\Delta} \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{\Gamma\left(\Delta - 2\Delta_\phi - k + 1\right)} \qquad k \in 2\mathbb{N} + 1$$

- *Prediction for heavy operators OPE coefficients*:

$$a_\Delta \sim \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \delta\Delta \qquad \bigstar$$

- *Prediction for high energy form factors*:

$$\left| \langle 0 | \phi | \mathscr{E} \rangle_\beta \right|^2 \sim \frac{(-)^{\Delta_\phi} \mathscr{E}^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \delta\mathscr{E}$$

Work in progress:

1. Prove $\rho(\Delta) \geq - C$; $\quad\star$
2. Test for high energy form factors in thermal QFT (not conformal);

Future work:

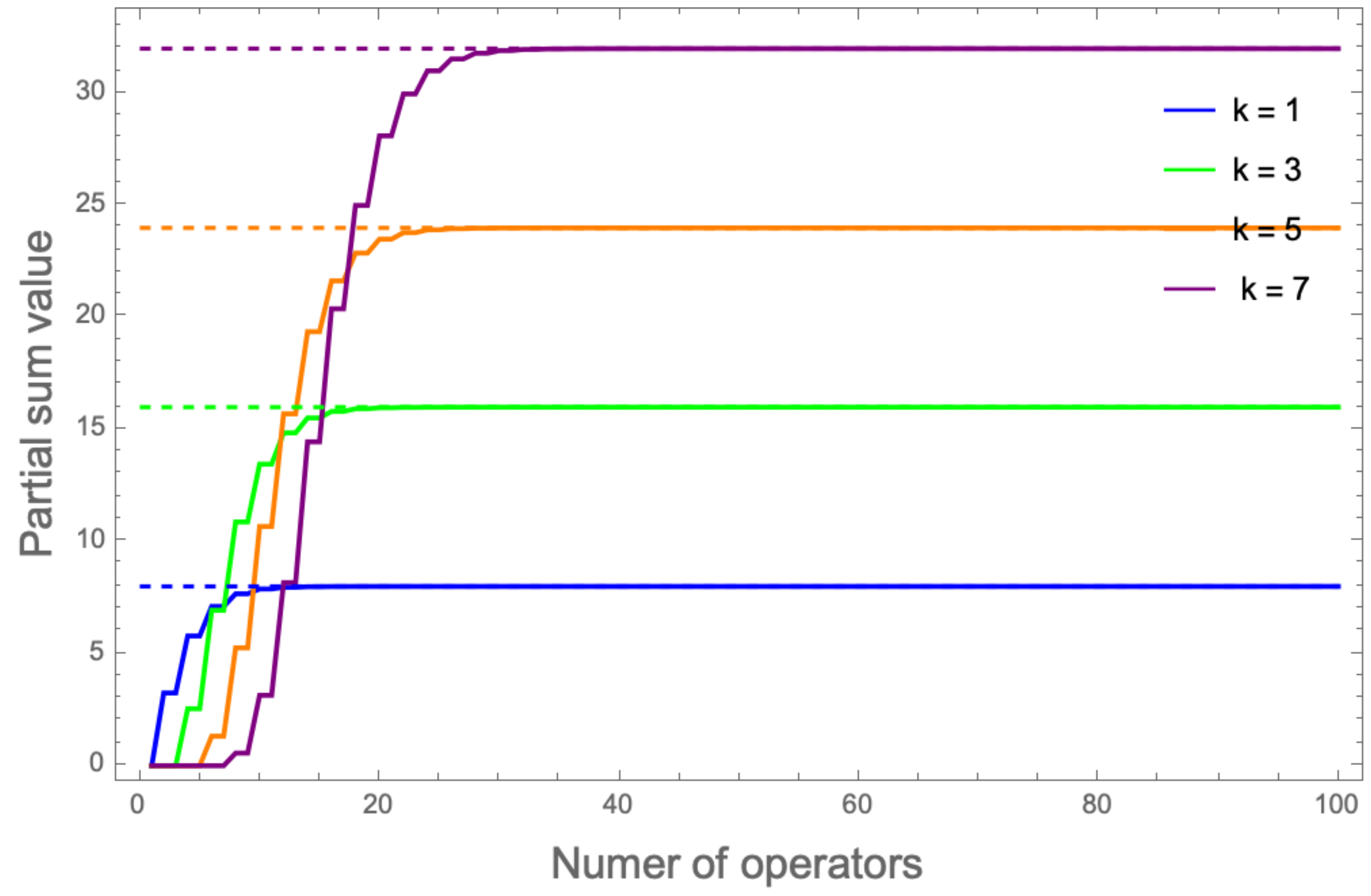
A. Find solutions and/or methods to solve the infinite set of equations for one-point functions (unicity of the problem?) for light operators.

B. Compute thermal two-point function and extract data about AdS back holes.

KMS: including the spin…

Consider for simplicity r = 0, i.e. zero spatial coordinates, but also let us now consider also the spin structure of the operators

$$\frac{\Gamma(2\Delta_\phi + n)}{\Gamma(2\Delta_\phi)} = \sum_{\mathcal{O}\neq 1} \frac{a_{\mathcal{O}}}{2^{\Delta_{\mathcal{O}}}} \frac{\Gamma(\Delta_{\mathcal{O}} - 2\Delta_\phi + 1)\Gamma(J_{\mathcal{O}} + 2\nu)}{J_{\mathcal{O}}!\Gamma(\Delta_{\mathcal{O}} - 2\Delta_\phi - n + 1)\Gamma(2\nu)}$$



4-dimensional free theory