Quantum information geometry of driven CFTs

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Outline

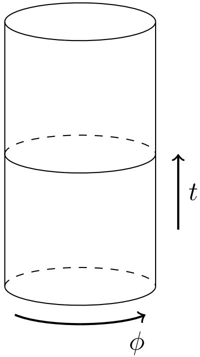
1. **2D CFT on a cylinder and the Virasoro group**
2. **Time evolution of the CFT in a background metric**
3. **Information geometry of Virasoro states**

2D conformal field theory on a cylinder

- Consider a 2D CFT on an infinite cylinder $\mathbb{R} \times S^1$ with coordinates (t, ϕ) with $\phi \sim \phi + 2\pi$
- Endow the cylinder with the flat Lorentzian metric

$$ds^2 = dx^- dx^+ \tag{1}$$

where $x^\pm = \phi \pm t$ are light-ray coordinates



The Lorentzian conformal group

- Under the diffeomorphism

$$x^- \to F(x^-), \quad x^+ \to \overline{F}(x^+) \tag{2}$$

  the metric transforms by a conformal factor

$$ds^2 = F'(x^-)\,\overline{F}'(x^+)\,dx^- dx^+ \tag{3}$$

- The Lorentzian conformal group is thus given by

$$(F, \overline{F}) \in \mathrm{Diff}_+ S^1 \times \mathrm{Diff}_+ S^1 \tag{4}$$

  where $\mathrm{Diff}_+ S^1$ is the diffeomorphism group of the circle (due to $F(\phi + 2\pi) = F(\phi) + 2\pi$)

[Kong–Runkel '09, Schottenloher '08]

Projective representations of the conformal group

- On the Hilbert space of the CFT, the conformal transformation $(F, \overline{F})$ is represented by the unitary operator

$$V_{F,\overline{F}} = V_F \otimes \overline{V}_{\overline{F}} \tag{5}$$

where V_F is a projective representation of $\mathrm{Diff}_+ S^1$:

$$V_{F_1} V_{F_2} = e^{icB(F_1,F_2)} V_{F_1 \circ F_2} \tag{6}$$

- The unitary V_F is an element of the Virasoro group which is the central extension of $\mathrm{Diff}_+ S^1$

[Fewster–Hollands '04, Oblak '16]

Elements of the Virasoro group

- The Virasoro group is generated by the Virasoro algebra

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c}{12}n^3\,\delta_{n,-m}, \qquad L_n^\dagger = L_{-n} \tag{7}$$

- Virasoro group elements V_F are exponentials of Virasoro generators (up to a phase)

$$V_F \propto \exp\left(-i\sum_{n=-\infty}^{\infty} u_n L_n\right) \tag{8}$$

where the coefficients u_n are fixed by F

Elements of the Virasoro group

- Consider a curve on the conformal group

$$(F_t, \overline{F}_t) \in \text{Diff}_+ S^1 \times \text{Diff}_+ S^1, \quad F_0 = \overline{F}_0 = \text{id} \tag{9}$$

- The corresponding Virasoro group element is explicitly

$$V_{F_t, \overline{F}_t} \propto \mathcal{T} \exp\left(i \int_0^t ds\, H_s \right) \tag{10}$$

where the Hamiltonian generator

$$H_t = -\int_0^{2\pi} dx^- \,(\dot{F}_t \circ F_t^{-1})(x^-)\, T_{--}(x^-) + (- \leftrightarrow +), \qquad T_{--}(x^-) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} L_n\, e^{inx^-} \tag{11}$$

Virasoro group elements from CFT time-evolution

$$V_{F_t,\overline{F}_t} \propto \mathcal{T} \exp\left(i\int_0^t ds\, H_s\right), \qquad H_t = -\int_0^{2\pi} dx^- \,(\dot{F}_t \circ F_t^{-1})(x^-)\, T_{--}(x^-) + (- \leftrightarrow +) \tag{12}$$

- Given a curve $(F_t, \overline{F}_t)$, can we realize $V_{F_t,\overline{F}_t}$ as time-evolution of the CFT_2 on the cylinder?
- Yes by coupling the CFT to a non-trivial background metric

[De Boer–Godet–JK–Keski-Vakkuri '23]

- Can also be interpreted as a time-dependent foliation of the cylinder

[Erdmenger–Flory–Gerbershagen–Heller–Weigel '21]

and Tim Schuhmann's talk

Coupling the CFT to a background metric

Cylinder with a general metric

- We will endow the cylinder $(t, \phi) \in \mathbb{R} \times S^1$ with a general (curved) metric g

$$ds^2 = e^{\omega}\,(d\phi + \nu dt)(d\phi + \bar{\nu} dt) = e^{\varphi}\, dx^- dx^+ \tag{13}$$

  with three arbitrary functions $\omega(t, \phi)$, $\nu(t, \phi)$ and $\bar{\nu}(t, \phi)$

- Given a curve $(F_t, \overline{F}_t)$, we set

$$\nu(t, \phi) \equiv \frac{\dot{F}_t(\phi)}{F_t'(\phi)}, \qquad \bar{\nu}(t, \phi) \equiv \frac{\dot{\overline{F}}_t(\phi)}{\overline{F}_t'(\phi)}. \tag{14}$$

- The CFT is coupled to this time-dependent metric

Time-dependent Hamiltonian

$$ds^2 = e^{\omega}\left(d\phi + \frac{\dot{F}_t(\phi)}{F'_t(\phi)}\,dt\right)\left(d\phi + \frac{\dot{\overline{F}}_t(\phi)}{\overline{F}'_t(\phi)}\,dt\right) \tag{15}$$

- The Hamiltonian of the CFT on a constant-t slice (the Noether charge of t-translations)

$$H_t = -\int_0^{2\pi} d\phi\sqrt{-g}\,T_t{}^{t}(t,\phi) \tag{16}$$

- CFT with a time dependent Hamiltonian $\equiv$ driven CFT
- By an explicit computation

$$H_t = -\int_0^{2\pi} dx^-\,(\dot{F}_t \circ F_t^{-1})(x^-)\,T_{--}(x^-) + \int_0^{2\pi} dx^+\,(\dot{\overline{F}}_t \circ \overline{F}_t^{-1})(x^+)\,T_{++}(x^+) \tag{17}$$

which exactly generates the Virasoro group element $V_{F_t,\overline{F}_t} = \mathcal{T}e^{i\int_0^t ds\,H_s}$

Time-evolution in the background metric

- Take the initial Hamiltonian to be (initially flat metric)

$$H_0 = L_0 + \bar{L}_0 \tag{18}$$

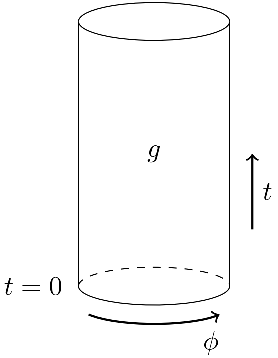
- Take the initial state to be a thermal state

$$\sigma_0 = \sigma_\beta \equiv \frac{e^{-\beta(L_0+\bar{L}_0)}}{\mathrm{Tr}\, e^{-\beta(L_0+\bar{L}_0)}} \tag{19}$$

- Unitary time evolution in the metric g produces the state

$$\sigma_t = V_{F_t,\bar{F}_t}\, \sigma_\beta\, V^\dagger_{F_t,\bar{F}_t} \tag{20}$$

  on a constant-t slice

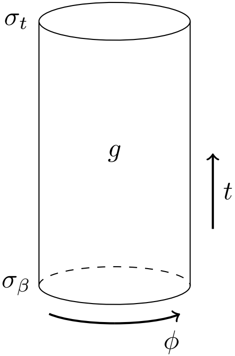


Operator formulation

VS

Path integral formulation

$$\sigma_t = V_{F_t,\overline{F}_t}\, \sigma_\beta\, V^\dagger_{F_t,\overline{F}_t}$$



Stress tensor in the operator formulation

- We can compute stress tensor expectation value using the Virasoro algebra

$$\mathrm{Tr}\left(\sigma_t\, T_{--}(x^-)\right) = f_t'(x^-)^2 \,\langle T\rangle_\beta - \frac{c}{24\pi}\{f_t(x^-), x^-\}, \tag{21}$$

where $f_t \equiv F_t^{-1}$, $\langle T\rangle_\beta \equiv \mathrm{Tr}\left(\sigma_\beta\, L_0\right)$ and $\{f_t(x^-), x^-\} = \frac{f_t'''(x^-)}{f_t'(x^-)} - \frac{3}{2}\left(\frac{f_t''(x^-)}{f_t'(x^-)}\right)^2$.

- Due to explicit time dependence of σ_t, this is not conserved in the metric g:

$$\nabla^b\, \mathrm{Tr}\left(\sigma_t\, T_{ab}\right) \neq 0 \tag{22}$$

- Hence the operator formulation breaks diffeomorphism invariance

[De Boer–Godet–JK–Keski-Vakkuri '23]

Also anticipated in [Oblak '17]

Stress tensor in the path integral formulation

- The same answer should be obtained from the path integral of the CFT

$$\mathrm{Tr}\,(\sigma_t\, T_{ab}) \overset{?}{=} \int [d\Phi]_g\, T_{ab}\, e^{iI_{\mathrm{CFT}}[g,\Phi]}, \tag{23}$$

  but the right-hand side is conserved due to diffeomorphism invariance of $I_{\mathrm{CFT}}[g,\Phi]$ and $[d\Phi]_g$

- One can break diffeomorphism invariance of the path integral to match with the operator formulation by a change of renormalization scheme

[De Boer–Godet–JK–Keski-Vakkuri '23]

Information geometry of Virasoro states

Definition of a Virasoro state

- The state at each time can be written as

$$\sigma_t = V_{F_t,\overline{F}_t}\,\sigma_\beta\,V^\dagger_{F_t,\overline{F}_t} = \frac{e^{-\beta H_{f_t,\bar{f}_t}}}{\mathrm{Tr}\,e^{-\beta H_{f_t,\bar{f}_t}}} \tag{24}$$

where in the exponent ($f_t = F_t^{-1}$, $\bar{f}_t = \overline{F}_t^{-1}$)

$$H_{f_t,\bar{f}_t} = \int_0^{2\pi} dx^- \,\frac{T_{--}(x^-)}{f_t'(x^-)} + \int_0^{2\pi} dx^+ \,\frac{T_{++}(x^+)}{\bar{f}_t'(x^+)} \tag{25}$$

Compare with [Cardy–Tonni '16]

- Hence we consider *Virasoro states* defined as

$$\sigma_f = \frac{e^{-\beta H_f}}{\mathrm{Tr}\,e^{-\beta H_f}}, \quad H_f = \int_0^{2\pi} dx\,\frac{T(x)}{f'(x)}, \quad T(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} L_n\,e^{inx} \tag{26}$$

which are in 1-to-1 correspondence with $\frac{\mathrm{Diff}_+ S^1}{U(1)}$

Relative entropy of Virasoro states

- Consider relative entropy between two such states

$$S(\sigma_{f_1} \| \sigma_{f_2}) = \mathrm{Tr}\left[\sigma_{f_1}(\log \sigma_{f_1} - \log \sigma_{f_2})\right] \tag{27}$$

- The result is

$$S(\sigma_{f_2} \| \sigma_{f_1}) = \frac{c\beta}{48\pi} \int_0^{2\pi} dx \left[\left(\frac{\mathcal{F}''(x)}{\mathcal{F}'(x)} \right)^2 + \frac{48\pi \langle T \rangle_\beta}{c} \left[\mathcal{F}'(x)^2 - 1 \right] \right] \tag{28}$$

where $\mathcal{F} = f_1 \circ f_2^{-1}$ and $\langle T \rangle_\beta \equiv \mathrm{Tr}\,(\sigma_\beta\, L_0)$

[De Boer–Godet–JK–Keski-Vakkuri '23]

Quantum information metric

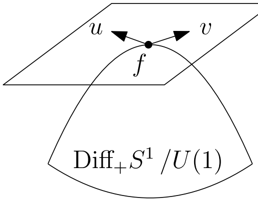
- Consider two diffeomorphisms f_1, f_2 that are perturbations of f:

$$f_1(x) = f(x) + \lambda_1 u(x), \quad f_2(x) = f(x) + \lambda_2 v(x) \tag{29}$$

- Relative entropy can be used to define a metric on the space of Virasoro states as

$$\mathcal{G}_f(u, v) = -\frac{\partial^2}{\partial\lambda_1 \partial\lambda_2} S(\sigma_{f+\lambda_1 u} \| \sigma_{f+\lambda_2 v}) \Big|_{\lambda_1=\lambda_2=0} \tag{30}$$

where u, v are tangent vectors at the point f on $\text{Diff}_+ S^1 / U(1)$



Quantum information metric

$$\mathcal{G}_f(u,v) = -\frac{\partial^2}{\partial\lambda_1\partial\lambda_2} S(\sigma_{f+\lambda_1 u}\|\sigma_{f+\lambda_2 v})\Big|_{\lambda_1=\lambda_2=0} \tag{31}$$

- This metric is known as the Bogoliubov–Kubo–Mori (BKM) metric (also as the quantum Fisher information metric)

[Kubo '57, Mori '65, Petz–Toth '93]

- The result at the identity $f = \text{id}$ is

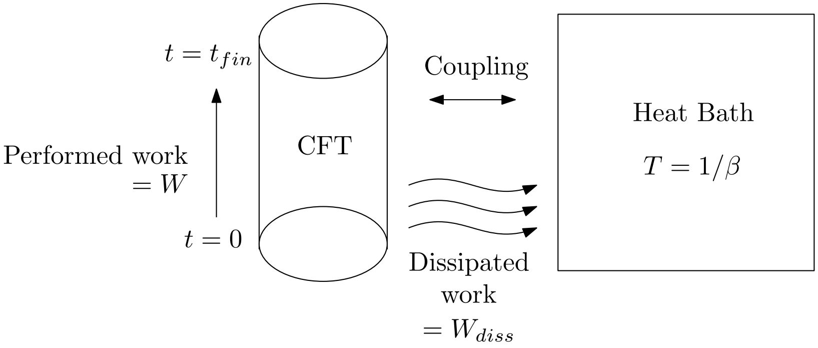
$$\mathcal{G}_{f=\text{id}}(u,v) = \frac{c\beta}{24\pi}\int_0^{2\pi} dx \left[u''(x)\,v''(x) + \frac{48\pi\langle T\rangle_\beta}{c}\,u'(x)\,v'(x)\right]. \tag{32}$$

[De Boer–Godet–JK–Keski-Vakkuri '23]

One application (among others)

Work dissipation in driven open CFT

- Couple the CFT to an external heat bath at finite temperature ⇒ open CFT



Work dissipation in driven open CFT

- The action of a point particle on $\frac{\text{Diff}_+ S^1}{U(1)}$ in the BKM metric computes dissipated work W_{diss}

  [Scandi–Perarnau-Llobet '18, Miller–Scandi–Anders–Perarnau-Llobet '19]

- Geodesics of the BKM metric are driving protocols that minimize dissipated work W_{diss}
- Geodesics can be found explicitly in the high-temperature $\beta \to 0$ limit

  [De Boer–Godet–JK–Keski-Vakkuri '23]

Thank you