Physics and Modularity of Calabi-Yau Manifolds

Based on published and upcoming work with P. Candelas, X. de la Ossa, H. Jockers, S. Kotlewski, and J. McGovern

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Motivation: Number Theory and Physics

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- Quantisation conditions: $H^n(X, \mathbb{Z})$ natural in both physics and number theory.
- Local-global principles: by studying spaces defined over finite fields, we learn about spaces over $\mathbb{R}$ or $\mathbb{C}$.

Modularity and Physics

In this talk, I will focus on one particular aspect of relation of number theory and physics, *modularity of Calabi-Yau manifolds* **[Weil, Serre, Wiles, Taylor, Moore, Candelas, de la Ossa, Elmi, van Straten, Rodriguez-Villegas, Klemm, Hulek, Verrill, ...]**.

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- Find rational CFTs by studying properties of f_X.

Overview of Modularity

Elliptic curves

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There is a modular form of weight 2:

$$f(q) = q - 2q^3 + q^5 - 3q^7 + q^9 - 2q^{11} - 2q^{13} - 2q^{15} - 6q^{17} + \ldots$$

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However, in very special cases this factorises into

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Cohomologically this indicates that there is 2-dim sublattice $\Gamma \subset H^3(X, \mathbb{Z})$ of Hodge type $(1, 2) + (2, 1)$.

Example: Flux Vacua and Modularity

We study supersymmetric flux vacua in type IIB supergravity theory compactified on Calabi-Yau threefolds X. This gives a 4-dimensional $\mathcal{N} = 1$ theory

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In terms of cohomology these imply that

$$H, F \in \left(H^{(2,1)}(X, \mathbb{C}) \oplus H^{(1,2)}(X, \mathbb{C}) \right) \cap H^3(X, \mathbb{Z}) \ ,$$

which is a strong condition on the Calabi-Yau manifold X.

Solutions via Modularity

The flux vectors F and H generate a two-dimensional sublattice of $H^3(X, \mathbb{Z})$ of Hodge type $(1,2) + (2,1)$.

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Earlier, I claimed that in such situations the polynomial $R(X, T)$ factorises into

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We can thus find Calabi-Yau manifolds X for which the flux vacuum equations can be solved by studying these polynomials!

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In this way, we can essentially find supersymmetric flux vacua by using number theory only!

Future directions

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- Periods, special functions, amplitudes, L-function values.