Julius-Maximilians-
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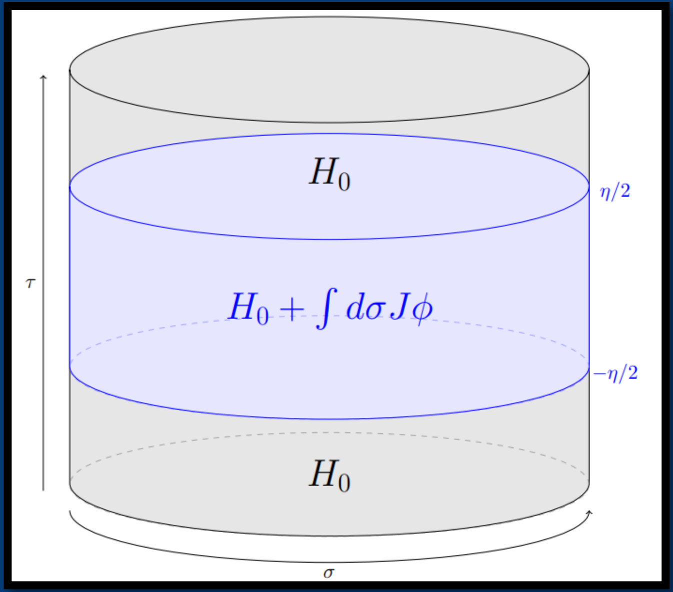
Complexity of a Quantum Circuit with Primary Fields in the Circuit Generator



Based on: Master thesis research with Anna-Lena Weigel and Johanna Erdmenger

DESY Theory Workshop – 28.09.2023 – Tim Schuhmann

Two motivations

Motivation: Complexity

An interesting quantity from quantum information science

Circuit Complexity of a unitary transformation U is the size (i.e. number of gates) of the smallest circuit that implements U

[Aaronson, QIS lecture notes (2021)]

Initial state $|\Psi_i\rangle$ —— Circuit U ——→ Target state $|\Psi_f\rangle$

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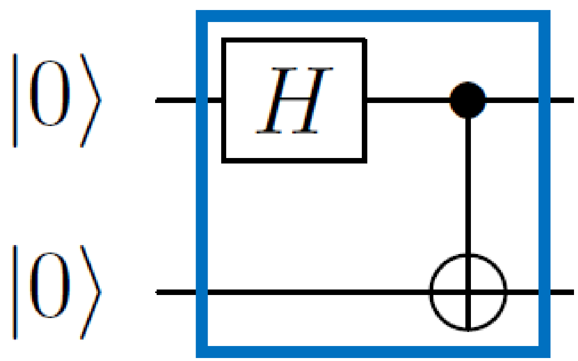
Initial state $|\Psi_i\rangle$ —— Circuit U ——→ Target state $|\Psi_f\rangle$

$|00\rangle$



$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Motivation: Complexity in Holography

A new entry in the holographic dictionary that reaches deep into the bulk

Circuit Complexity of a unitary transformation U is the size (i.e. number of gates) of the smallest circuit that implements U

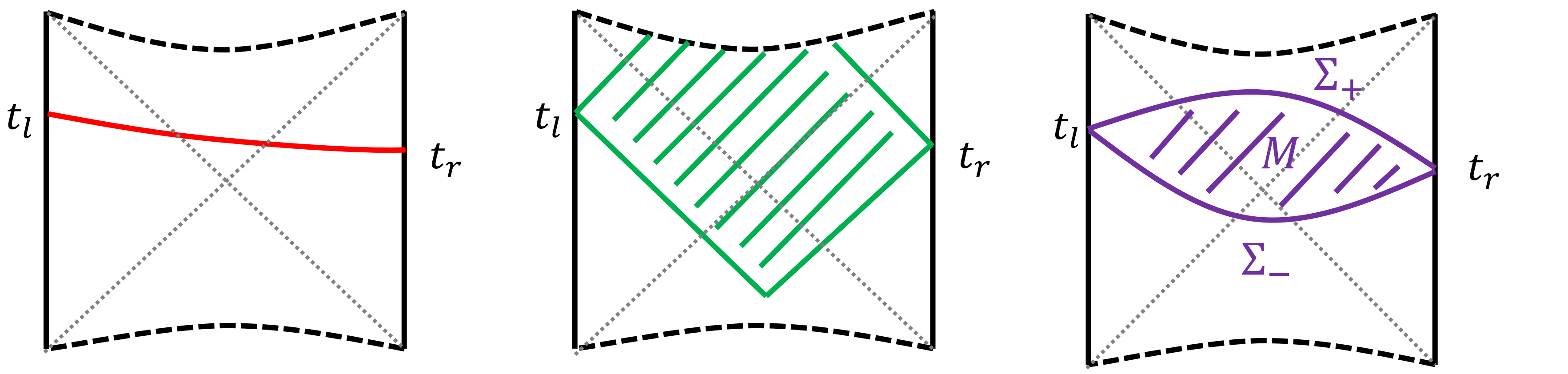
Bulk dual?

[Aaronson, QIS lecture notes (2021)]

CV: Complexity equals volume

CA: Complexity equals action

CAny: Complexity equals anything



Conjectured Complexity duals probe BH interior regions

[Susskind, Stanford (2014)], [Susskind et. al. (2016)], [Belin, Myers et al. (2022)], [Belin, Myers et al. (2023)]

A circuit for conformal transformations

Field theory calculations for an explicit circuit

Nielsen Complexity of **conformal transformations f in CFT_2**

Initial state $|\Psi_i\rangle = |h\rangle$ —— Circuit U ——→ Target state $|\Psi_f\rangle = U_f|h\rangle$

[Nielsen (2005)], [Nielsen et. al. (2006)], [Dowling, Nielsen (2008)], [Magan (2018)], [Caputa, Magan (2019)],

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$U(\tau)$ generated by Schrödinger-like time evolution with generating Hamiltonian

$$H(\tau) = \int_0^{2\pi} d\sigma\, f(\tau,\sigma) * T(\sigma)$$

← generator

velocity

Continuous circuit states: $|\Psi(\tau)\rangle = U(\tau)|h\rangle$

$$U(\tau) = \vec{P}\exp\left(-i\int_{\tau_i}^{\tau} H(\tau')d\tau'\right)$$

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Continuous circuit states: $|\Psi(\tau)\rangle = \vec{P}\exp\left(-i\int_{\tau_i}^{\tau} H(\tau')d\tau'\right)|h\rangle$

Cost function: $F_{FS} = \langle\Psi(\tau)|H(\tau)^2|\Psi(\tau)\rangle - |\langle\Psi(\tau)|H(\tau)|\Psi(\tau)\rangle|^2$

Complexity: known to third order perturbatively, does not agree with CV

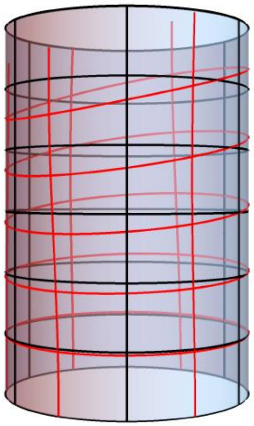
$$C(\tau) = \min_{H(\tau)} \int_{\tau_i}^{\tau_f} d\tau\, F_{FS}(H(\tau), |\Psi(\tau)\rangle)$$

[Nielsen (2005)], [Nielsen et. al. (2006)], [Dowling, Nielsen (2008)], [Magan (2018)], [Caputa, Magan (2019)], [Flory, Heller (2020)], [Flory, Heller (2020)], [Erdmenger, Flory, Gerbershagen, Heller, Weigel (2022)]

A circuit for conformal transformations

Well controlled quantum circuits yield a rich ground to find holographic duals

Boundary/Field theory side		**Bulk/Gravity side:**
Circuit on boundary	$\Leftrightarrow$	Bulk geometry



Conformal transformations in CFT_2:
[Erdmenger, Flory, Gerbershagen, Heller, Weigel (2022),
[Erdmenger, Gerbershagen, Heller, Weigel (2022)]

Motivation: Circuit limitations

Nielsen Complexity is like travelling

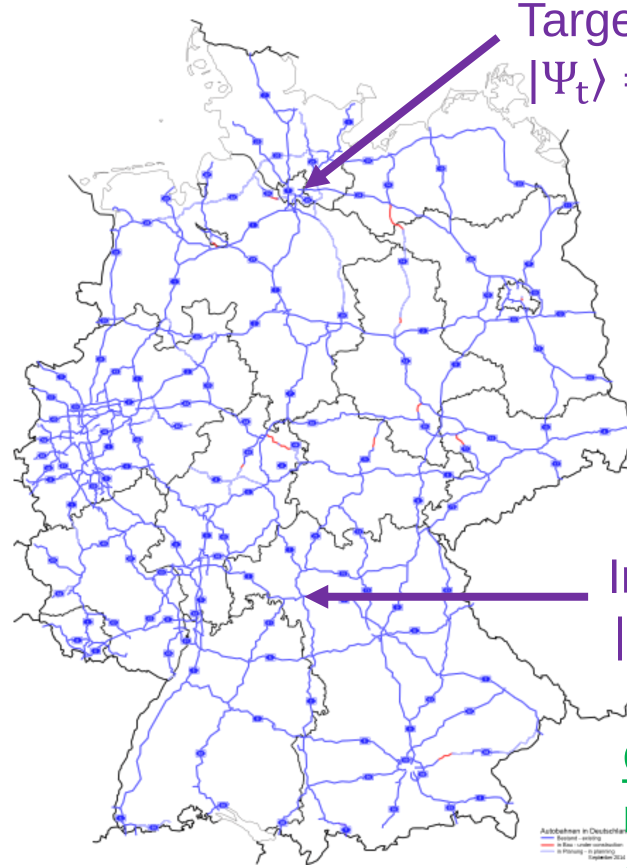
Circuit U takes initial state $|\Psi_i\rangle$ to target state $|\Psi_t\rangle$ through evolution with generating Hamiltonian $H(\tau)$

Choose cost function $F(H(\tau), |\Psi(\tau)\rangle)$

Optimal circuit (Complexity C):

$$C = \min_{H(\tau)} \int_{\tau_i}^{\tau_f} d\tau\, F(H(\tau), |\Psi(\tau)\rangle)$$

Target state: $|\Psi_t\rangle = |Hamburg\rangle$



H: car

F: €/km (fuel,battery)

Initial state: $|\Psi_i\rangle = |W\ddot{U}\rangle$

Optimal route: minimal cost over route

Motivation: Circuit limitations

Continuing the analogy

<u>Suppose:</u>

Travel from Würzburg to **Austin,TX** to a conference

by car !?

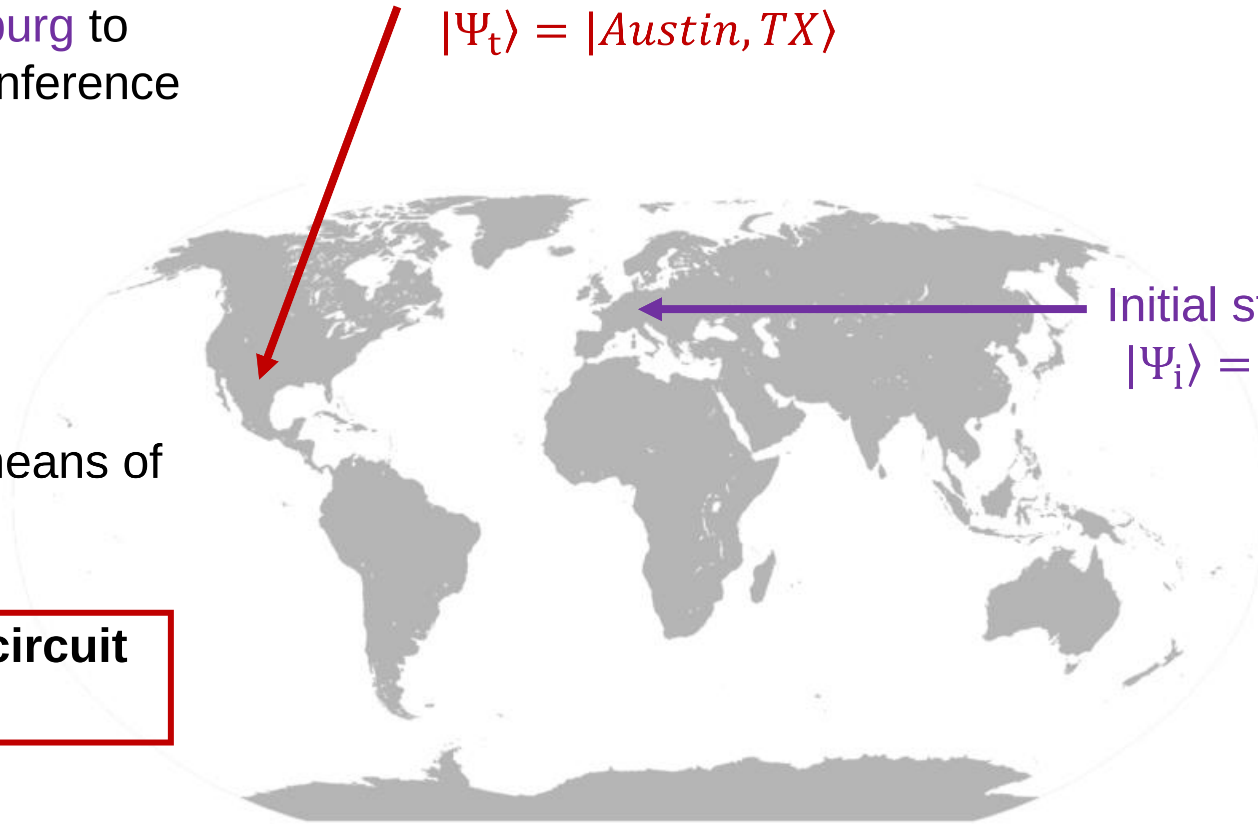
→ Not well posed!

Need to change means of transportation

⇔

Need to change circuit Hamiltonian $H(\tau)$

Target state:
$|\Psi_t\rangle = |Austin, TX\rangle$



Initial state:
$|\Psi_i\rangle = |W\ddot{U}\rangle$

Motivation: Expanding existing circuits

In a nutshell: The second motivation for this research project

Previously described conformal circuits are constrained:

All target states lie in the same conformal family as the reference state.

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All target states lie in the same conformal family as the reference state.

Goal: Access new bulk geometries (e.g. transition from vacuum to BH) via inclusion of a primary operator $\phi(t,\sigma)$ in $H(t)$

$$H_{new}(t) = H(t) + \int_0^{2\pi} \frac{d\sigma}{2\pi} J(t,\sigma) * \phi(t,\sigma)$$

Indicated in

[Black Hole Collapse in the 1/c Expansion (Anous, Hartman, Rovai, Sonner, 2016)]

The main part

A circuit with primaries: Setup

Divide and conquer: Starting with a simple setup

Goal: Access new bulk geometries via inclusion of a primary operator $\phi(t,\sigma)$

$$H_{new}(t) = H_0 + J \int_0^{2\pi} \frac{d\sigma}{2\pi} * \phi(t,\sigma)$$

This is a hard task in general. Simplifying assumptions:

- $H(t) = H_0 = l_0 + \bar{l}_0 - \frac{c+\bar{c}}{24}$ trivial time evolution
- $J(t,\sigma) = J \ll 1 \quad \Rightarrow$ constant small source that can be treated perturbatively
- $h_\phi = \bar{h}_\phi = 1 \quad \Rightarrow$ marginal primary
- CFT_2 on cylinder is nonthermal: $T = 0$
- $\phi^\dagger(t,\sigma) = \phi(t,\sigma) \quad \Rightarrow$ hermitean primaries to ensure unitarity

A circuit with primaries: Setup

Divide and conquer: Starting with a simple setup

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$$H_{new}(t) = H_0 + \underbrace{J \int_0^{2\pi} \frac{d\sigma}{2\pi} * \phi(t,\sigma)}_{H_{int}(t)}$$

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Trivial time evolution + small perturbation: Employ Interaction picture

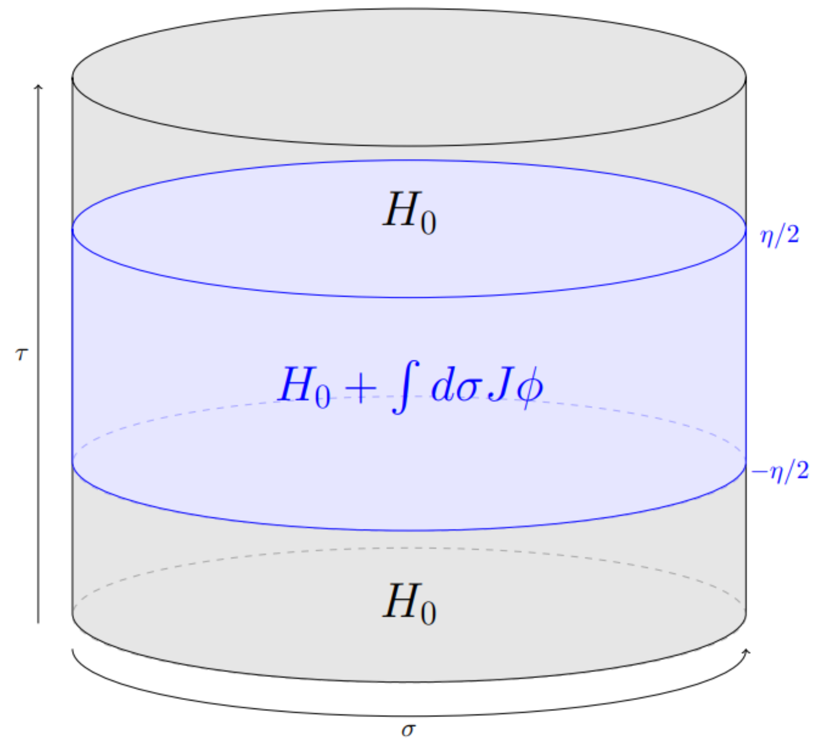
A circuit with primaries: Protocol

Finally: The circuit protocol

Region III ($t > \eta/2$)
Trivial time evolution

Region II ($-\eta/2 < t < \eta/2$)
Primary switched on, full time evol.

Region I ($t < -\eta/2$)
Trivial time evolution, start from $|0\rangle$



A circuit with primaries: States

The states at all times are known

States: via $|\Psi(t)\rangle = U(t)|\Psi_i\rangle$ with $U(t) = T\,\exp(-i\int_{t_i}^{t} H_{int}(t')\,dt')$

Region III ($t > \eta/2$):

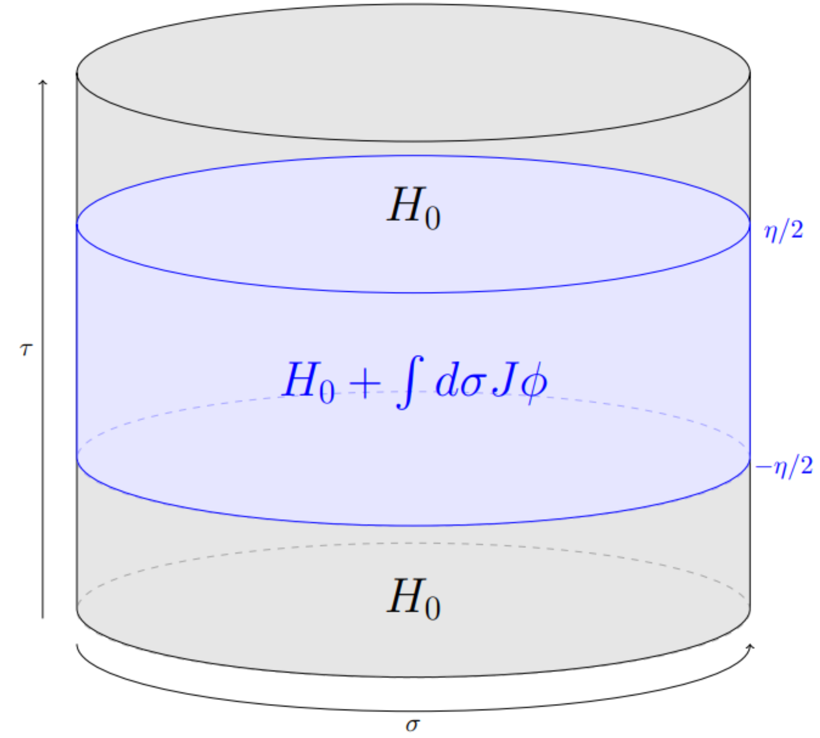
$$|\Psi\rangle_{III} = T\,\exp(-i\int_{-\eta/2}^{\eta/2} H_{int}(t')\,dt')\,|0\rangle$$

Region II ($-\eta/2 < t < \eta/2$):

$$|\Psi(t)\rangle_{II} = T\,\exp(-i\int_{-\eta/2}^{t} H_{int}(t')\,dt')\,|0\rangle$$

Region I ($t < -\eta/2$):

$$|\Psi\rangle_{I} = |\Psi_i\rangle = |0\rangle$$



Energy after the quench

What does this circuit do?

Important indicator → **Calculate energy expectation value after the quench**

$$E(t) = \langle \Psi(t) | H_{new}(t) | \Psi(t) \rangle$$

In Region III: $H_{new}(\tau) = H_0 = l_0 + \bar{l}_0 - (c + \bar{c})/24$

$|\Psi\rangle_{III} = T \exp(-i \int_{-\eta/2}^{\eta/2} H_{int}(t')\, dt')\, |0\rangle$

Method: Expand states to order J^2 with Dyson series

Calculations in Euclidean framework, then Wick rotate back

Employ $(l_0 + \bar{l}_0)|0\rangle = 0$ and commutator $[(l_0 + \bar{l}_0), \phi]$

Regularization: Introduce cutoff $\epsilon \ll 1$ such that $|\tau_1 - \tau_2| > \epsilon$

Renormalization: Take the cutoff independent piece as result

Similar to [Bak, Trivella (2017)]

Energy after the quench

What does this circuit do: Energy scaling suggests BH collapse as bulk dual

Important indicator → **Calculate energy expectation value after the quench**

$$E(t) = \langle \Psi(t) | H_{new}(t) | \Psi(t) \rangle$$

We find in the limit $\eta \ll J \ll 1$ in Region III:

$$\boxed{E_{III,Ren.} \propto \frac{J^2}{\eta^2}}$$

→ Reproduces expected universal scaling of energy after a fast CFT quench
[Buchel, Myers, Niekerk (2013)], [Berenstein, Miller (2014)], [Das, Galante, Myers (2017)]

→ **Energy scaling suggests Vaidya black hole collapse as bulk dual**
[Black Hole Collapse in the 1/c Expansion (Anous, Hartman, Rovai, Sonner, 2016)]

Analysis of the circuit complexity

Cost and circuit complexity are derived analytically

Evaluation of **cost**:

$$F_{FS}(t) = \langle \Psi(t) | H(t)_{new}{}^2 | \Psi(t) \rangle - |\langle \Psi(t) | H(t)_{new} | \Psi(t) \rangle|^2$$

Method, Regularization, Renormalization analogous + to obtain analytic result restrict to euclidean time reversal symmetric primaries $\phi(\tau, \sigma) = \phi(-\tau, \sigma)$

Evaluation of **complexity**:

$$C(t) = \min_{H(t)} \int_{-\infty}^{t} dt' \, F_{FS}(t')$$

By construction: No free dynamical parameters in $H(t)$ → Minimization trivial

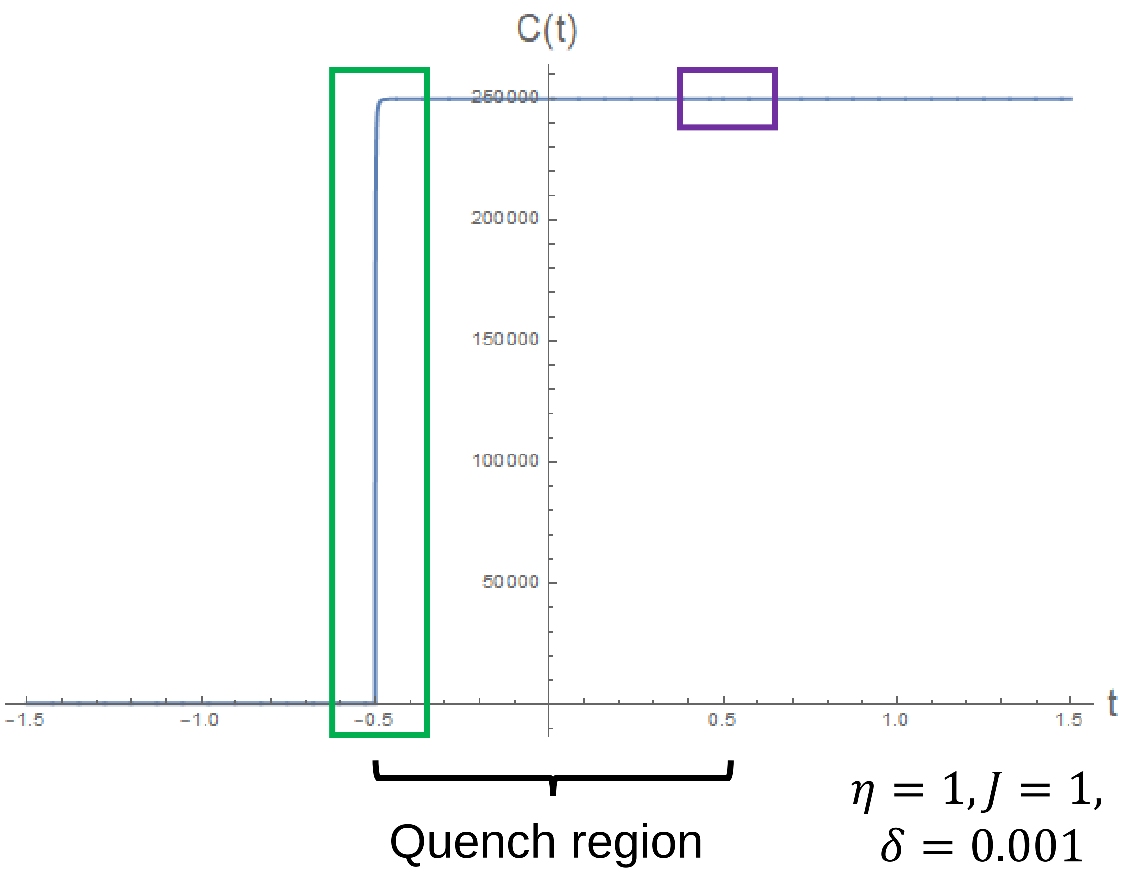
Circuit: Only one specific transformation BUT fully analytically solvable

Integral divergent at $t = -\eta/2$ → Cut out interval of length $\delta \ll 1$ after $t = -\eta/2$

Analysis of the circuit complexity

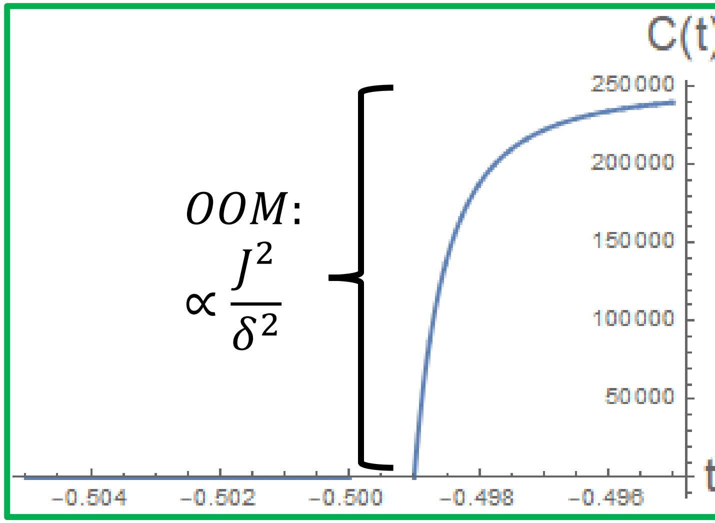
Cost and circuit complexity are derived analytically

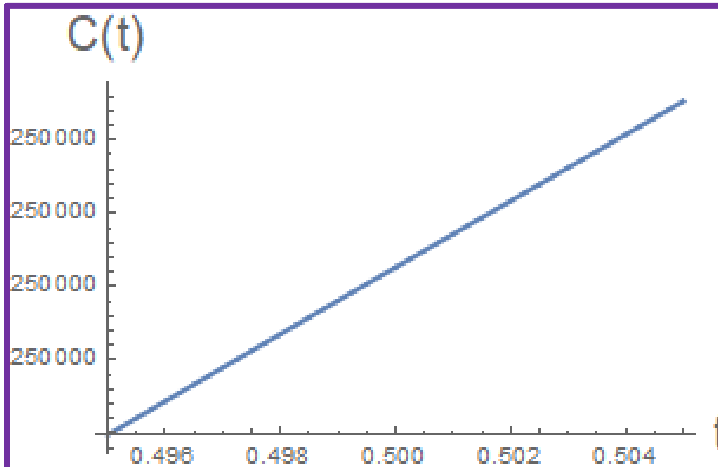
Complexity analysis: General



$\eta = 1, J = 1,$
$\delta = 0.001$

Fast inital growth

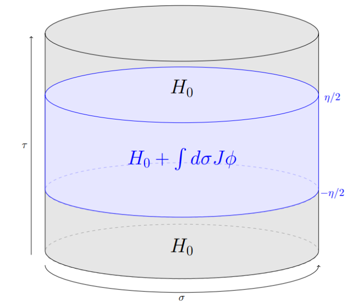




Linear growth after quench

Summary and Outlook

There are more things to do



Summary

- Defined circuit model of trivial time evolution perturbed by marginal primary
- Energy scaling after the quench suggests black hole collapse as bulk dual
- Complexity is derived analytically, grows fast initially and linearly after quench

Outlook

- Combine this framework with conformal transformations analyzed previously
- Allow for more general source functions j such that $J(t,\sigma) = J * j(t,\sigma)$ w/ $J \ll 1$

Backup slides

Holographic duals to conformal circuits

There are explicit gravity duals to simple quantum circuits

Main idea: Identify physical boundary time t with circuit parameter τ

Boundary metric to circuit connection via $H(t) = \int_0^{2\pi} d\sigma \sqrt{\det(g^{(0)})}\, T_t^t$

The full bulk metric can be obtained using the Fefferman-Graham expansion

$$ds^2 = \frac{dr^2}{r^2} + \left(\frac{1}{r^2} g_{ij}^{(0)} + g_{ij}^{(2)} + r^2 g_{ij}^{(4)}\right) dx^i dx^j$$

from boundary data ($\langle T_{ij} \rangle$ determined in background produced by conf. trafos)

$$g_{ij}^{(2)} = -\frac{1}{2} R^{(0)} g_{ij}^{(0)} - \frac{6}{c} \langle T_{ij} \rangle \qquad g_{ij}^{(4)} = \frac{1}{4} \left(g^{(2)} \left(g^{(0)}\right)^{-1} g^{(2)} \right)_{ij}$$

Analysis of the circuit complexity

Cost and circuit complexity are derived analytically

Evaluation of **complexity**:

$$C(t) = \min_{H(t)} \int_{-\infty}^{t} dt' \, F_{FS}(t')$$

Result divergent at $t = -\eta/2$ → Cut out interval of length $\delta \ll 1$ after $t = -\eta/2$

We find

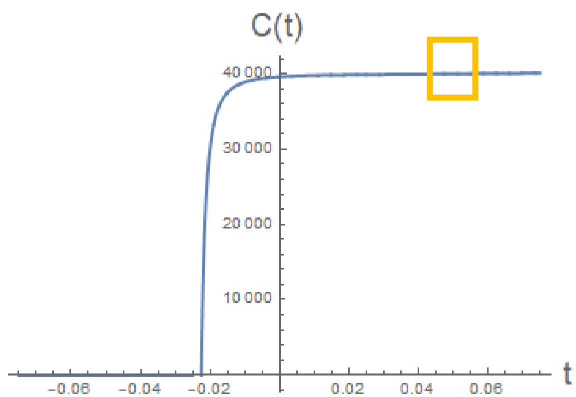
$$C(t) = \begin{cases} 0 \text{ for } t \leq -\frac{\eta}{2} \\ \frac{J^2}{4}\left(\csc^2(\delta) - \csc^2\left(\frac{\eta}{2} + t\right)\right) \text{ for } -\frac{\eta}{2} + \delta < t < \frac{\eta}{2} \\ \frac{J^2}{4}(\csc^2(\delta) - \csc^2(\eta)) + \frac{J^2}{2}(\cot(\eta)\csc^2(\eta))(t - \eta/2) \text{ for } t \geq \frac{\eta}{2} \end{cases}$$

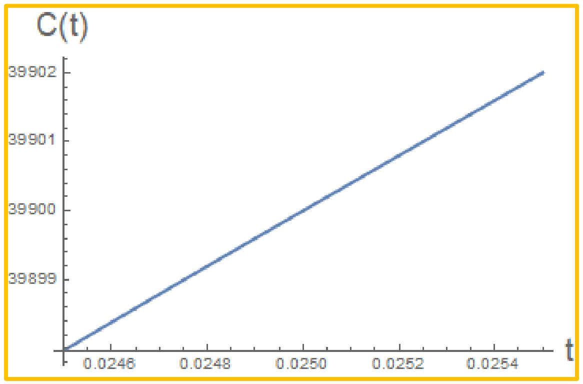
linear growth

Analysis of the circuit complexity

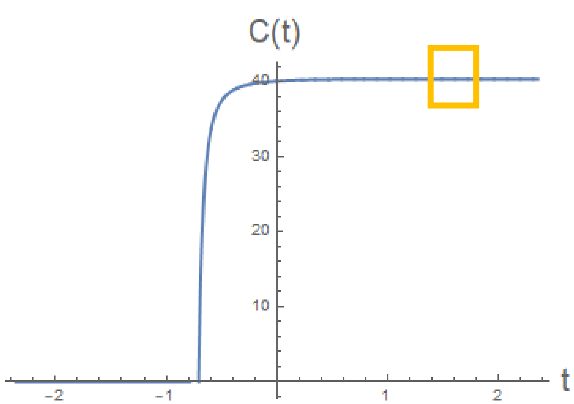
Cost and circuit complexity are derived analytically

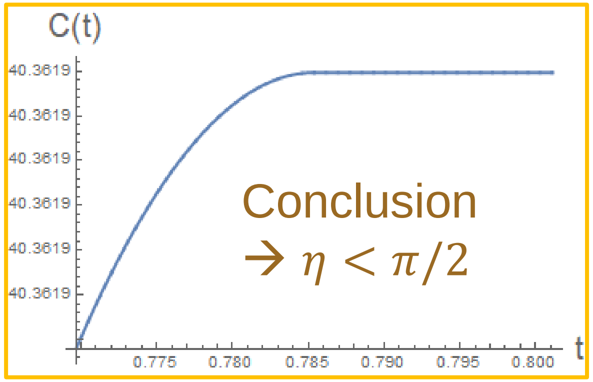
Complexity analysis: Dependence on η





$\eta = 0.05, J = 1,$
$\delta = 0.05\eta$





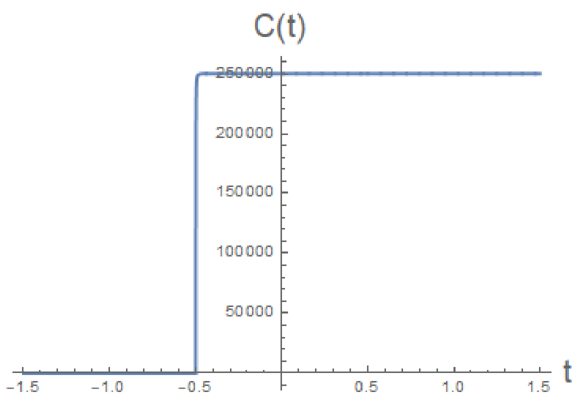
$\eta = \pi/2, J = 1,$
$\delta = 0.05\eta$

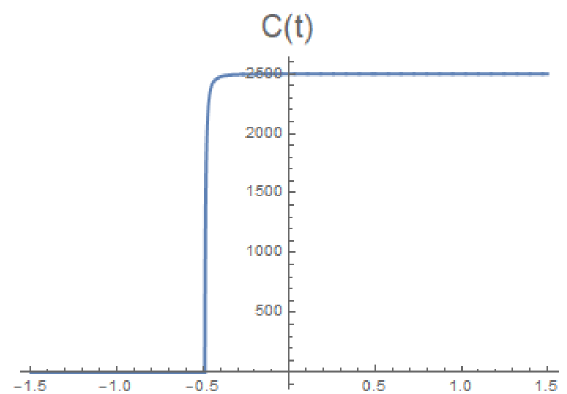
Analysis of the circuit complexity

Cost and circuit complexity are derived analytically

Complexity analysis: Dependence on δ

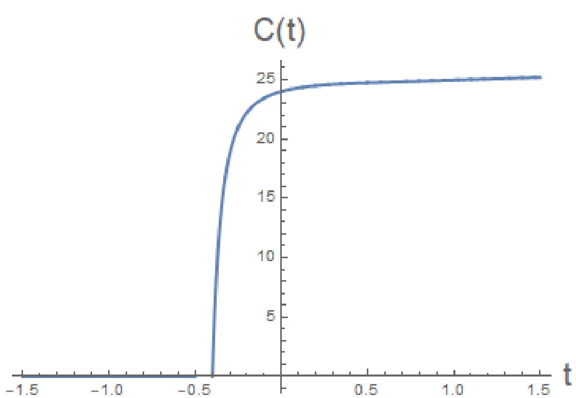
$\eta = 1, J = 1,$
$\delta = 0.001$

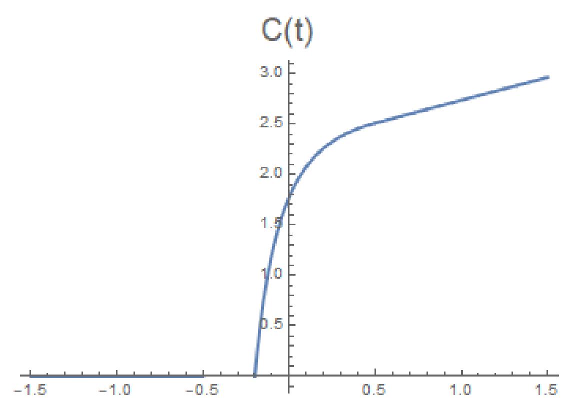




$\eta = 1, J = 1,$
$\delta = 0.01$

$\eta = 1, J = 1,$
$\delta = 0.1$





$\eta = 1, J = 1,$
$\delta = 0.3$