Integrability in non-relativistic string theory

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New perspectives in conformal field theory and gravity

Based on:

arXiv:2102.00008, with A. Fontanella and A. Torrielli,
arXiv:2109.13240,2208.02295,2305.19316, with A. Fontanella,
arXiv:2211.04479, with A. Fontanella and O. Ohlsson Sax.
Also relevant, arXiv:2203.07386, from A. Fontanella and S. van Tongeren (but not me).

Why?

- The success of integrability in $\mathcal{N}=4$ SYM / $\mathrm{AdS}_5\times\mathrm{S}^5$ motivated people to look into settings that are "similar enough" that we can still apply it ($T\bar{T}$, Yang-Baxter deformation, fishnet, defects...).
- NR $\mathrm{AdS}_5\times\mathrm{S}^5$ has vanishing β-function. [Gomis, J. Oh, Z. Yan, 2019][Gallegos, Gursoy, Zinnato, 2019]
- Recently has been shown to admit a Lax connection for the NR limit flat and $\mathrm{AdS}_5\times\mathrm{S}^5$ backgrounds $\Longrightarrow$ Can we use the toolbox we developed for relativistic $\mathrm{AdS}_5\times\mathrm{S}^5$?. [Fontanella, van Tongeren, 2022]
- Studying NR $\mathrm{AdS}_5\times\mathrm{S}^5$ may open the door to non-AdS holography, as the background metric is not asymptotically AdS.

Outline

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What is “NR $AdS_5 \times S^5$”?

We can construct the non-relativistic $AdS_5 \times S^5$ string action by a rescaling and limit of the coordinates equivalent to the İnönü-Wigner contraction

$$\mathfrak{so}(2,4) \oplus \mathfrak{so}(6) \longrightarrow \text{string Newton-Hooke}_5 \oplus \text{Eucl}_5$$

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Let us consider the bosonic relativistic string action in $AdS_5\times S^5$ in terms of a Maurer-Cartan 1-form

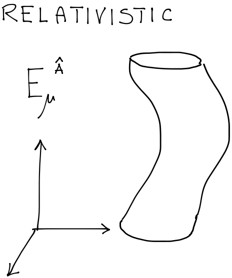
$$S=\int d^2\sigma\,\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}=\int d^2\sigma\,\gamma^{\alpha\beta}\langle\mathcal{P}A,\mathcal{P}A\rangle\ ,$$

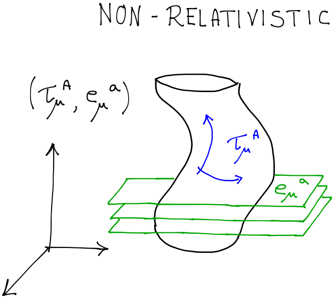
where $\mathcal{P}:\mathfrak{g}\to\mathfrak{g}\setminus\mathfrak{h}$ (projector), $\mathfrak{g}=\{P_{\hat{A}},J_{\hat{A}\hat{B}}\}=\mathfrak{so}(2,4)$, $\mathfrak{h}=\{J_{\hat{A}\hat{B}}\}=\mathfrak{so}(1,4)$, $\mathfrak{g}\setminus\mathfrak{h}=\{P_{\hat{A}}\}$, $\langle\cdot,\cdot\rangle$ usual inner product

$$A_\mu=g^{-1}\partial_\mu g=e_\mu{}^{\hat{A}}P_{\hat{A}}+\omega_\mu{}^{\hat{A}\hat{B}}J_{\hat{A}\hat{B}}\qquad\qquad \langle P_{\hat{A}},P_{\hat{B}}\rangle=\eta_{\hat{A}\hat{B}}\ .$$

The Maurer-Cartan 1-form allows us to translate the İnönü-Wigner contraction into a coordinate rescaling. If we split the range of our index $\hat{A}$, the contraction

$$P_a \to cP_a \ , \qquad J_{Aa} \to cJ_{Aa} \ ,$$





Foliation "2+8" $\hat{A} = (A, a)$

$A = 0, 1 \qquad a = 2, \ldots, 9$

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implies the following expansion of the vielbeins of the metric

$$\text{longitudinal} \qquad E_\mu{}^A = c\tau_\mu{}^A + \frac{1}{c} m_\mu{}^A + \mathcal{O}(c^{-3}) \qquad A = 0, 1$$

$$\text{transverse} \qquad E_\mu{}^a = e_\mu{}^a + \mathcal{O}(c^{-2}) \qquad a = 2, ..., 9$$

$$g_{\mu\nu} = c^2 \tau_\mu{}^A \tau_\nu{}^B \eta_{AB} + [2\tau_{(\mu}{}^A m_{\nu)}{}^B \eta_{AB} + e_\mu{}^a e_\nu{}^b \delta_{ab}] + \mathcal{O}(c^{-2})$$

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Some may be concerned by a divergent metric, but fear not. If we include a closed B-field of the form $B_{\mu\nu} = c^2 \varepsilon_{AB} \tau_\mu{}^A \tau_\nu{}^B$, we can eliminate it

$$\left(\gamma^{\alpha\beta} g_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu}\right)\Big|_{\mathcal{O}(c^2)} \partial_\alpha X^\mu \partial_\beta X^\nu = c^2 \gamma^{00} \mathcal{F}^A \mathcal{F}^B \tilde{\eta}_{AB} \equiv \lambda_A \mathcal{F}^A + \frac{1}{c^2} \lambda_A \lambda^A$$

$$\boxed{S^{\mathsf{NR}} = \int \mathsf{d}^2\sigma \left(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu H_{\mu\nu} + \lambda_A \mathcal{F}^A \right)}$$

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However, it is more useful to write it as

$$\boxed{S^{\mathrm{NR}} = \int \mathrm{d}^2\sigma \left(\gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu H_{\mu\nu} + \varepsilon^{\alpha\beta} (\lambda_+ \theta_\alpha{}^+ \tau_\mu{}^+ + \lambda_- \theta_\alpha{}^- \tau_\mu{}^-) \partial_\beta X^\mu \right)}$$

- world-sheet Zweibein: $\theta_\alpha{}^\pm \equiv \theta_\alpha{}^0 \pm \theta_\alpha{}^1$ $\quad [\, h_{\alpha\beta} = -\theta_{(\alpha}{}^+ \theta_{\beta)}{}^-]$
- $H_{\mu\nu} \equiv 2\tau_{(\mu}{}^A m_{\nu)}{}^B \eta_{AB} + e_\mu{}^a e_\nu{}^b \delta_{ab}$ $\quad \tau_\alpha{}^\pm \equiv \tau_\alpha{}^0 \pm \tau_\alpha{}^1$

string Newton-Cartan $\mathrm{AdS}_5 \times S^5$ data:

$\tau_\mu{}^A \longrightarrow$ AdS_2 (t, z)

$e_\mu{}^a \longrightarrow$ $f(z)\ \mathbb{R}^3 \times \mathbb{R}^5$

$m_\mu{}^A \longrightarrow$ coordinate dependent vielbein

Newton-Cartan $\mathrm{AdS}_5 \times S^5$ data

We have to be careful: $H_{\mu\nu}$, in contrast to $g_{\mu\nu}$, does not transform covariantly under diffeomorphisms.

Compare Cartesian and polar coordinates

$$
\begin{aligned}
ds^2 &= -\left(\frac{4+z_1^2}{4-z_1^2}\right)^2 dt^2 + \left(\frac{4}{4-z_1^2}\right)^2 dz_i dz^i \ , \\
\tau_\mu^A &= \text{diag.}\left(-\frac{4+z_1^2}{4-z_1^2}, \frac{4}{4-z_1^2}, 0, 0, 0\right) \ , \\
m_\mu^A &= \text{diag.}\left(-\frac{2z_m z^m}{4-z_1^2}, \frac{z_m z^m}{4-z_1^2}, 0, 0, 0\right) \ , \\
e_\mu^A &= \text{diag.}\left(0, 0, \frac{4}{4-z_1^2}, \frac{4}{4-z_1^2}, \frac{4}{4-z_1^2}\right) \ ,
\end{aligned}
$$

$$
\begin{aligned}
ds^2 &= -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\Omega_{S^3} \ , \\
\tau_\mu^A &= \text{diag.}\left(-\cosh\rho, 1, 0, 0, 0\right) \ , \\
m_\mu^A &= 0 \ , \\
e_\mu^A &= \text{diag.}\left(0, 0, -\sinh\rho, -\sinh\rho, \beta_2 \sinh\rho\right) \ .
\end{aligned}
$$

Outline

Solving the equations of motion

- Fix conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, $\theta^{\pm} = (-1, \mp 1)$.
- Solve E.o.M. for the Lagrange multipliers.
- Solve E.o.M. for the coordiantes and impose Virasoro constraints.
- Impose closed string boundary conditions to cancel surface term.

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- Impose closed string boundary conditions to cancel surface term.

$$\mathcal{E}^{\lambda_{\pm}} \equiv \varepsilon^{\alpha\beta}\theta_{\alpha}{}^{\pm}\tau_{\mu}{}^{\pm}\partial_{\beta}X^{\mu} = 0$$

Because τ_{μ}^{A} is diagonal and only depends on one of the two longitudinal coordinates, there exist $T = T(t),\ Z = Z(z)$ such that they become

$$\dot{T} + Z' = 0 \qquad\qquad T' + \dot{Z} = 0$$

This T and Z fulfil the usual wave equation, thus

$$T = f_{+}(\sigma_{+}) + f_{-}(\sigma_{-}) \ , \qquad Z = f_{+}(\sigma_{+}) - f_{-}(\sigma_{-}) + \text{const.}$$

But fixing $h_{\alpha\beta} = \eta_{\alpha\beta}$ does not remove all redundancy.

As in the relativistic case, there is a residual $\mathrm{Diff}_+ \oplus \mathrm{Diff}_-$, that allows us to write

$$T = \kappa\tau \qquad\qquad Z = \kappa\sigma$$

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$$T = \kappa\tau \qquad\qquad Z = \kappa\sigma$$

Back to t and z coordinates, this gives us

- Cartesian $\quad t = \kappa\tau \qquad z = -2\tan(\kappa\sigma/2) \qquad \kappa \in \mathbb{Z}$
- Polar $\quad t = \kappa\tau \qquad \rho = \text{gd}^{-1}(\kappa\sigma) \qquad \kappa \in \mathbb{Z}$

where $\text{gd}(x) = \arctan(\sinh(x))$

It seems like NR strings **must have winding**. Interestingly, the same requirement was observed for NR flat space string theory [Gomis, Ooguri, 2000]

Two simplest solutions admitted by NR $\text{AdS}_5 \times \text{S}^5$: [Fontanella, Nieto 2021]

- *Static string* (zero Energy)

$$t = \kappa\tau \qquad \rho = \text{gd}^{-1}(\kappa\sigma) \qquad \text{others} = 0$$

- *BMN-like* (Energy E, linear momentum J)

$$t = \kappa\tau \qquad \rho = \text{gd}^{-1}(\kappa\sigma) \qquad \phi = \nu\tau \qquad \lambda_\pm = \pm\frac{\nu^2}{2\kappa}|\cos(\kappa\sigma)|$$

Dispersion relation $E = \frac{\sqrt{\lambda}\nu^2}{2\kappa} \sim J^2$

Found also a generalisation of the GKP (rotating in transverse AdS)

Outline

A step back: the relativistic case

First, a quick reminder of the relativistic case

$$S = \int \mathrm{d}^2\sigma \, \gamma^{\alpha\beta} \langle \mathcal{P}A, \mathcal{P}A \rangle = \int \mathrm{d}^2\sigma \, \gamma^{\alpha\beta} \langle A^{(1)}, A^{(1)} \rangle \ .$$

$\mathbb{Z}_2$ automorphism $\quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \ , \qquad \mathfrak{g}^{(0)} \equiv \mathfrak{h} \ , \qquad \mathfrak{g}^{(1)} \equiv \mathfrak{p}.$

- E.o.m. $\quad \partial_\alpha(\gamma^{\alpha\beta} A^{(1)}_\beta) + \gamma^{\alpha\beta}[A^{(0)}_\alpha, A^{(1)}_\beta] = 0.$
- Lax pair $\mathscr{L} = A^{(0)} + \ell_1 A^{(1)} - \ell_2 \star A^{(1)}$, with $\ell_1^2 - \ell_2^2 = 1$. [Bena, Polchinski, Roiban, 2003]
- $\mathrm{d}\mathscr{L} + \mathscr{L} \wedge \mathscr{L} = 0$ is equivalent to the E.o.M. and flatness of A.

The Lax connection contains information regarding all the conserved charges (more on that later).

NR $\mathrm{AdS}_5\times\mathrm{S}^5$

Similarly, we want to write string Newton-Cartan $\mathrm{AdS}_5\times\mathrm{S}^5$ as $\frac{\text{Isometry}}{\text{Isotropy}}$.

The nicest way to find which algebras to use is to use Lie Algebra expansion, where we find

$$\text{SNC AdS}_5 \times S^5 = \frac{\text{Lie Algebra Expansion}[\mathfrak{so}(2,4)\oplus\mathfrak{so}(6)]}{\text{String Bargmann}\oplus\{Z_{ab},Z_{a'b'}\}}$$

$$\begin{aligned} J_{AB} &\to \varepsilon_{AB}(M+\epsilon^2 Z) & P_A &\to H_A + \epsilon^2 Z_A \\ J_{Aa} &\to \epsilon G_{Aa} & P_a &\to \epsilon P_a \\ J_{ab} &\to J_{ab} + \epsilon^2 Z_{ab} & P_{a'} &\to \epsilon P_{a'} \\ J_{a'b'} &\to J_{a'b'} + \epsilon^2 Z_{a'b'} & A = 0,1 \quad a &= 2,3,4 \quad a' = 1,\dots,5 \end{aligned}$$

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J_{ab} &\to J_{ab}+\epsilon^2 Z_{ab} & P_{a'} &\to \epsilon P_{a'} \\
J_{a'b'} &\to J_{a'b'}+\epsilon^2 Z_{a'b'} & A=0,1 \quad a&=2,3,4 \quad a'=1,...,5
\end{aligned}$$

Coset numerator $\mathfrak{g}=\{H_A, P_a, P_{a'}, M, G_{Aa}, J_{ab}, J_{a'b'}, Z_A, Z, Z_{ab}, Z_{a'b'}\}$

Coset denominator $\mathfrak{h}=\mathfrak{g}\setminus\{H_A, P_a, P_{a'}\}$

dim($\mathfrak{g}$) - dim($\mathfrak{h}$) $=10$, as needed.

The Maurer-Cartan form captures the SNC data of $\mathrm{AdS}_5 \times S^5$:

$$A_\mu = \tau_\mu{}^A H_A + e_\mu{}^a P_a + e_\mu{}^{a'} P_{a'} + m_\mu{}^A Z_A + \dots$$

but, where do the Lagrange multipliers λ_A enter the construction?

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but, where do the Lagrange multipliers λ_A enter the construction?

We have to include them by hand in the Maurer-Cartan form

$$J \equiv A + \star_2 \Lambda \qquad\qquad \Lambda_\alpha = \lambda_- \theta_\alpha{}^- Z_+ + \lambda_+ \theta_\alpha{}^+ Z_-$$

Then, the NR action takes the from [Fontanella, van Tongeren 2022][Fontanella, Nieto, 2022]

$$S^{\mathrm{NR}} = \int \mathrm{d}^2\sigma \gamma^{\alpha\beta} \langle J_\alpha{}^{(1)}, J_\beta{}^{(1)} \rangle$$

The inner product $\langle \cdot, \cdot \rangle$ was degenerate and not fully adjoint invariant in [Fontanella, van Tongeren 2022], but that was fixed in [Fontanella, Nieto, 2022]

Equations of motion and Lax pair

- Equations of motion

$$\partial_\alpha(\gamma^{\alpha\beta} J^{(1)}_\beta) + \gamma^{\alpha\beta}[J^{(0)}_\alpha, J^{(1)}_\beta] = 0 \qquad \mathcal{E}^{\lambda_\pm} = \varepsilon^{\alpha\beta}\theta_\alpha{}^{\pm} A^{H_\pm}_\beta = 0$$

They contain more e.o.m. than d.o.f., but they are not all independent. Noether identities (due to gauge invariance) relates them.

- Lax connection

$$\mathscr{L}^{\mathrm{NR}} = A^{(0)} + \ell_1 A^{(1)} - \ell_2 \star J^{(1)} \qquad \ell_1^2 - \ell_2^2 = 1 \quad \rightarrow \quad \xi$$

but it needs to be supported by the E.o.M. of $\lambda^\pm$, $\mathcal{E}^{\lambda_\pm} = 0$.

[Fontanella, van Tongeren 2022]

Outline

Spectral curve

One way to compute quantum corrections to the dispersion relation is to use the classical spectral curve. [Kazakov, Marshakov, Minahan, Zarembo, 2004]

The idea is to study the properties of the elliptic curve defined by the eigenvalues of the monodromy matrix

$$\mathcal{M} = \mathrm{P}\exp\left(\int_0^{2\pi} \mathrm{d}^2\sigma \ \mathscr{L}_\sigma^{\mathrm{NR}}(\xi)\right) ,$$

The monodromy matrix is the generating function of conserved charges of the integrable system. Thus, if we compute the eigenvalues of the monodromy matrix on the classical solutions and slightly perturb them, we can get access to the quantum correction to the dispersion relation. [Gromov, Vieira, 2007]

$$\mathcal{M} \longrightarrow p_i(\xi) \longrightarrow \omega_i(n) \longrightarrow E_1 = \frac{1}{2E_0} \sum_{i\in\text{fluc.}} \sum_{n=-\infty}^{\infty} (-1)^F \omega_i(n)$$

However, when we try applying this method, we find some obstructions

- Theorem: On solutions of $\mathcal{E}^{\lambda_\pm} = 0$ all eigenvalues are ξ-independent

[Fontanella, Nieto, Ohlsson Sax 2022]

  Checked on 2 finite reps: spinorial (inherited from the spinorial rep. of $\mathfrak{so}(2,4)$) and adjoint.

- $\mathcal{M}$ evaluated on BMN-like sol. is **non-diagonalisable**

$$\mathcal{M}|_{\text{BMN-like}} = S \begin{pmatrix} {\color{blue}\times} & {\color{red}\times} & {\color{red}\times} & & & \\ & {\color{blue}\times} & {\color{red}\times} & & & \\ & & {\color{blue}\times} & & & \\ & & & {\color{blue}\times} & {\color{red}\times} & \\ & & & & {\color{blue}\times} & \\ & & & & & {\color{blue}\times} \end{pmatrix} S^{-1}$$

${\color{blue}\times}$ no ξ-dep. ${\color{red}\times}$ yes ξ-dep. $\implies$ Is the spectral curve defined by "${\color{red}\times}$"?

The second obstruction is actually easy to understand, as the origin of the non-diagonalisability is very clear: while $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$ is semi-simple, the string Newton-Hooke$_5 \oplus$Eucl$_5$ algebra of the NR action is not.

In fact, the same happens when we construct the Monodromy matrix in flat space, as Poincaré algebra is also not semi-simple. Thus, it is not a peculiarity of our system.

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It seems like we have to generalise our concept of quasi-momenta in the non-diagonalisabe setting

Diagonalisable: $\mathcal{M} = S e^{p_i(\xi) C_i} S^{-1}$ $\quad C_i \in$ Cartan

Non-diagonalisable: $\mathcal{M} = S e^{q_i(\xi) W_i} S^{-1}$ $\quad W_i \in$ MAS

(MAS = maximal abelian subalgebra) [Fontanella, Nieto, Ohlsson Sax 2022]

Classical Integrability

The first obstruction is more delicate. The monodromy matrix is the generating function of conserved quantities, as the vanishing curvature of the Lax implies

$$\partial_\tau \mathrm{Tr}[\mathcal{M}(\xi)] = 0$$

Expanding $\mathrm{Tr}[\mathcal{M}(\xi)]$ around a particular point, e.g. $\xi = 0$

$$\mathrm{Tr}[\mathcal{M}(\xi)] = H_0 + \xi H_1 + \xi^2 H_2 + \ldots \quad \Longrightarrow \quad \partial_\tau H_n = 0 \quad \forall\ n$$

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From previous theorem, $\partial_\xi \mathrm{Tr}\mathcal{M} = 0$. Where is the usual tower of charges?

We believe it to be a consequence of using non-unitary representations. The problem should disappear in unitary ones, but that would imply using infinite-dimensional representations.

Outline

NR strings from relativistic strings

To understand a bit better our results, we want to see better how they emerge from relativistic strings. For that, we have to identify the relativistic string that becomes the BMN-like string in the NR limit.

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Let us start with a well-known classical solution: the folded string, given by

$$t = \kappa\tau \ , \qquad \beta_1 = \omega\tau \ , \qquad \phi_1 = \nu\tau \ ,$$
$$\sinh\rho = \sqrt{\frac{c^2\kappa^2 - \nu^2}{\omega^2 - c^2\kappa^2}} \ \mathrm{sn}\left(\frac{\sqrt{\omega^2 - c^2\kappa^2}}{c}\sigma, \frac{c^2\kappa^2 - \nu^2}{c^2\kappa^2 - \omega^2}\right) \ .$$

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Because we are interested in the large c limit, it is more convenient to use

$$\sinh\rho = \sqrt{\frac{c^2\kappa^2 - \nu^2}{c^2\kappa^2 - \omega^2}} \, \mathrm{sc}\left(\frac{\sqrt{c^2\kappa^2 - \omega^2}}{c}\sigma, \frac{\nu^2 - \omega^2}{c^2\kappa^2 - \omega^2} \right) \ .$$

NR strings from relativistic strings

Let us consider the $\omega = 0$ and $c \to \infty$ limit. This gives us

$$\sinh \rho \approx \operatorname{sc}(\kappa\sigma, 0) = \tan(\kappa\sigma) \Longrightarrow \rho = \operatorname{gd}^{-1}(\kappa\sigma) \ .$$

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Luckily, the B-field that we had to introduce saves the day by subtracting an infinite contribution. The total energy is then

$$E = \kappa c^2 \sqrt{\lambda} \left[1 - \frac{\mathsf{E}\left(\frac{\nu^2}{c^2\kappa^2}\right)}{\mathsf{K}\left(\frac{\nu^2}{c^2\kappa^2}\right)} \right] \approx \frac{\sqrt{\lambda}\nu^2}{2\kappa} + \mathcal{O}(c^{-2}) \ ,$$

which perfectly matches the one we computed.

Outline

Summary of NR strings in $AdS_5 \times S^5$

- $\mathcal{E}^{\lambda_\pm} = 0$ force us to have winding.
- We have proposed a coset formulation of NR $AdS_5 \times S^5$, together with a Lax connection.
- We can construct the Monodromy matrix, but it is non-diagonalisable and its trace is independent of ξ.
- We have shown that the simplest classical NR strings are the limit of a folded string.

Open problems

- Include fermions.
- Understanding the spectral curve: representations “a là Wigner”?
- Quantum corrections to the dispersion relation and S-matrix.
- NR limit of other interesting classical solutions, e.g. giant magnons.
- Identify the “dual” limit on $\mathcal{N} = 4$ SYM on “boundary geometry”

Thanks for your attention!