Analytic bootstrap for defect CFTs

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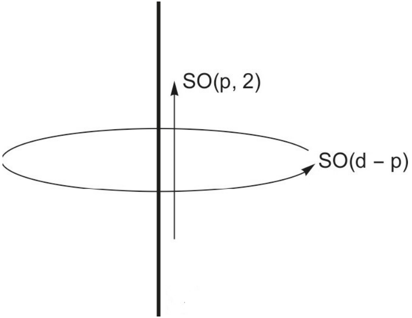
based on 2205.09775, 2212.02524 and upcoming work with L.Bianchi, E. de Sabbata and A. Gimenez Grau

Conformal defects

Conformal defects = extended operators that preserve conformal symmetry.



They have many interesting realizations in condensed matter and high energy physics such as

- Boundary and interfaces.
- Impurities in materials at the critical point.
- Wilson and t'Hooft lines in (super)conformal gauge theories.

Correlators in defect CFTs

A p-dimensional defect breaks the bulk conformal group as

$$SO(d+1,1) \to SO(p+1,1) \oplus SO(d-p)$$

- There are two CFTs: a $p-$dimensional defect CFT $(\hat{\Delta}, s)$ and a bulk CFT (Δ, ℓ).
- Bulk operators acquire non-trivial 1pt functions and couplings to the defect [Billo', Goncalves, Lauria, Meineri '16].

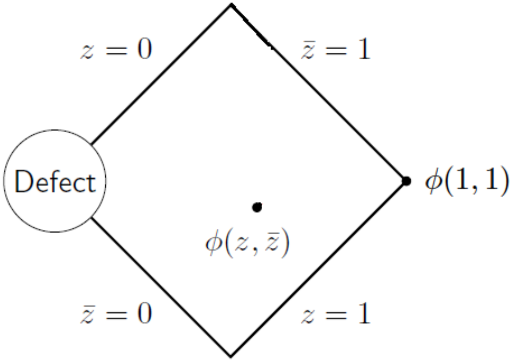
$$\langle \phi(x) \rangle = \frac{a_\phi}{|x_\perp|^{\Delta_\phi}}$$

$$\langle \phi(x) \hat{O}(y) \rangle = \frac{b_{\phi \hat{O}}}{|x_\perp|^{\Delta_\phi - \hat{\Delta}_{\hat{O}}} (|x_\perp|^2 + y^2)^{\hat{\Delta}}}$$

2pt functions in presence of a defect

We are interested in the 2pt function of bulk operators

$$\langle \phi_i(x_1)\,\phi_j(x_2)\rangle = \frac{\delta_{ij}}{\left|x_1^{\perp} x_2^{\perp}\right|^{\Delta_\phi}} F(z,\bar{z})$$



The defect bootstrap

The 2pt function can be expanded using the bulk-bulk OPE

$$\phi \times \phi \sim \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} \mathcal{O}$$

or the bulk-defect OPE

$$\phi \sim \sum_{\widehat{\mathcal{O}}} b_{\phi\widehat{\mathcal{O}}} \widehat{\mathcal{O}}$$

which imply

$$F(z,\bar{z}) = \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} a_{\mathcal{O}} f_{\Delta,\ell}(z,\bar{z}) = \sum_{\widehat{\mathcal{O}}} b^2_{\phi\widehat{\mathcal{O}}} \hat{f}_{\hat{\Delta},s}(z,\bar{z})$$

Consistency between the two expansions constrains the spectrum of the bulk and defect CFTs and fixes $F(z,\bar{z})$.

The analytic bootstrap

The interplay between bulk and defect data is captured by the inversion formula [Lemos, Liendo, Meineri, Sarkar '17]

$$c(\hat{\Delta}, s) = -\sum_{\widehat{\mathcal{O}}} \frac{b^2_{\phi\widehat{\mathcal{O}}}}{\hat{\Delta} - \hat{\Delta}_{\widehat{\mathcal{O}}}} = \int_0^1 d^2z \; I_{\hat{\Delta},s}(z,\bar{z}) \; \mathrm{Disc} F(z,\bar{z})$$

and the dispersion relation [Bianchi, DB '22], [Barrat, Gimenez-Grau, Liendo '22]

$$F(r,w) = \int_0^r \frac{dw'}{2\pi i} \left(\frac{1}{w' - w} + \frac{1}{w' - \frac{1}{w}} - \frac{1}{w'} \right) \mathrm{Disc} F(r,w')$$

$$z = rw \quad \bar{z} = \frac{r}{w}$$

where the discontinuity is

$$\mathrm{Disc} F(z,\bar{z}) = F(z, \bar{z} + i\epsilon) - F(z, \bar{z} - i\epsilon)$$

Subtlety: these formulas may miss low spin $s < s^*$ contributions.

The discontinuity in perturbation theory

If the theory has a small coupling expansion

$$\Delta_{\mathcal{O}} = 2\Delta_\phi + \gamma^{(1)}_{\mathcal{O}} g + O\left(g^2\right) \quad \lambda_{\phi\phi\mathcal{O}} a_{\mathcal{O}} = \lambda_{\phi\phi\mathcal{O}} a^{(0)}_{\mathcal{O}} + \lambda_{\phi\phi\mathcal{O}} a^{(1)}_{\mathcal{O}} g + O\left(g^2\right)$$

the discontinuity reads

$$\text{Disc}\, F(z,\bar{z})|_{O(g)} \sim g \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} a^{(0)}_{\mathcal{O}} \gamma^{(1)}_{\mathcal{O}} f_{\Delta,\ell}(z,\bar{z})$$

- It can be expressed in terms of data of the bulk operators.
- At any given order it depends only on lower order coefficients and on anomalous dimensions (independent of the defect).
- If we know the theory without the defect (e.g. $O(N)$ model), we can compute the discontinuity perturbatively.

Example 1: The localized magnetic field in the O(N) model

Consider the $O(N)$ model

$$S = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi_i)^2 + \frac{1}{2} m^2 (\phi_i)^2 + \frac{\lambda}{4!} (\phi_i \phi_i)^2 \right]$$

and insert the following line operator in the path integral

$$D = e^{-h_0 \int d\tau \, \phi_1(\tau)}$$

- The deformation triggers a RG flow which admits an infrared fixed point in $d = 4 - \varepsilon$ [Cuomo, Komargodski, Mezei '21].
- Physically this is an impurity created by applying an external magnetic field in a critical magnet, but only at a few lattice sites.
- One can compute $\langle \phi_i(x) \phi_j(y) \rangle$ and extract the defect CFT data in ε-expansion from the inversion formula / dispersion relation [Bianchi, DB and de Sabbata '22] [Gimenez-Grau '22].

$\langle\phi\phi\rangle$ in presence of a localized magnetic field for $N = 1$

- We have the bulk OPE

$$\phi \times \phi \sim 1 + \phi^2 + \mathcal{O}_{\ell>0}$$

with

$$\Delta_{\phi^2} = 2\Delta_\phi + \varepsilon\gamma_{\phi^2} + O\left(\varepsilon^2\right) \quad \Delta_\ell = 2\Delta_\phi + \ell + O\left(\varepsilon^2\right)$$

- The discontinuity depends only on one (known) bulk operator!
- Using dispersion / inversion formulae

$$F(z,\bar{z}) = \left(\frac{\sqrt{z\bar{z}}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi} + \varepsilon\frac{3}{8}\underbrace{H(z,\bar{z})}_{\text{special function}} + \text{low spin terms}$$

$$\hat{\Delta} = 1 + s + \varepsilon\frac{1-s}{2s+1} \quad \hat{b}_s^2 = 1 + \varepsilon\frac{-2(s-1)H_s - 3H_{s+\frac{1}{2}}}{2(2s+1)}$$

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- The discontinuity depends only on one (known) bulk operator!
- Using dispersion / inversion formulae + diagrams

$$F(z,\bar{z}) = \left(\frac{\sqrt{z\bar{z}}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi} + \varepsilon\frac{3}{8}H(z,\bar{z}) + \underbrace{a_\phi^{2(0)} + \varepsilon a_\phi^{2(1)}}_{\text{defect identity operator}}$$

$$\hat{\Delta} = 1 + s + \varepsilon\frac{1-s}{2s+1} \quad \hat{b}_s^2 = 1 + \varepsilon\frac{-2(s-1)H_s - 3H_{s+\frac{1}{2}}}{2(2s+1)}$$

General results for $O(N)$

- One can generalize the previous result for the general $O(N)$ case and compute

$$\langle \phi_i(x)\phi_j(y)\rangle = \frac{\hat{F}_S(z,\bar{z})\delta_{i1}\delta_{j1} + \hat{F}_V(z,\bar{z})\left(\delta_{ij} - \delta_{i1}\delta_{j1}\right)}{|x_\perp|^{\Delta_\phi}|y_\perp|^{\Delta_\phi}}$$

and extract the CFT data

$$\hat{\Delta}_S = 1 + s + \varepsilon\frac{1-s}{2s+1} \qquad \hat{b}^2_{S,s} = 1 + \varepsilon\frac{-2(s-1)H_s - 3H_{s+\frac{1}{2}}}{2(2s+1)}$$

$$\hat{\Delta}_V = 1 + s - \varepsilon\frac{s}{2s+1} \quad \hat{b}^2_{V,s} = 1 - \varepsilon\frac{(2s+1)\left(2sH_s + H_{s-\frac{1}{2}}\right) + 2}{2(2s+1)^2}$$

- The results can be checked with explicit diagrammatic calculations.
- One can also extract the bulk 1pt functions $a_{\mathcal{O}_\ell}$ from the full result.

Example 2: The spin impurity in the $O(3)$ model

Now consider the critical $O(3)$ model and insert another defect

$$D_j = Tr_{2j+1}\left(Pe^{\gamma_0 \int d\tau \phi^a T^a}\right)$$

- The theory flows to a fixed point in $d = 4 - \varepsilon$.
- Physically it represent an external atom of spin j in a quantum anti-ferromagnet at the critical point [Sachdev, Buragohain and Vojta '99].
- This setup can be generalized to $SU(N)$ and is relevant for the study of supersymmetric Wilson lines in $\mathcal{N} = 4$ Super Yang-Mills [Beccaria, Giombi and Tseytlin '22].

$\langle \phi^a \phi^a \rangle$ in presence of a spin impurity

The bulk theory is the same as before, the only difference are the tree-level 1pt functions ($a_{\phi^2} \sim \varepsilon$), therefore

$$F(z,\bar{z}) = \left(\frac{\sqrt{z\bar{z}}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi} + \frac{5}{66}\pi^2 j(j+1)\varepsilon^2 H(z,\bar{z}) + \text{low spin terms}$$

and

$$\hat{\Delta} = \Delta_\phi + s + \frac{5\pi^2 j(j+1)\varepsilon^2}{33(2s+1)} \quad \hat{b}_s^2 = \frac{(\Delta_\phi)_s}{s!} - \frac{5\pi^2 j(j+1)\varepsilon^2}{33} \left(\frac{H_s - H_{s-\frac{1}{2}}}{2s+1} - \frac{2}{2s+1} \right)$$

However in this case we have a non trivial low spin ambiguity!

$\langle \phi^a \phi^a \rangle$ in presence of a spin impurity

From a diagrammatic computation we fix the ambiguities

$$F(z,\bar{z}) = \left(\frac{\sqrt{z\bar{z}}}{(1-z)(1-\bar{z})} \right)^{\Delta_\phi} + \frac{5}{66}\pi^2 j(j+1)\varepsilon^2 H(z,\bar{z}) + \\ + \frac{1}{6}\pi^2 j(j+1)\varepsilon + \frac{1}{12}\pi^2 j(j+1)\varepsilon^2 \log\left(\frac{4z\bar{z}}{(1+z\bar{z})^2} \right) + \\ + \frac{5}{66}\pi^2 j(j+1)\varepsilon^2 (1 + \log 2)$$

The extra term consists of contributions from two spin zero operators

$$\hat{\Delta}_- = \frac{\varepsilon}{2} \qquad \hat{b}_-^2 = \frac{\pi^2 j(j+1)}{6} \left(\varepsilon \left(\frac{16}{11} + \frac{5}{11}\log 2 \right) + \varepsilon^2 \log 2 \right)$$

$$\hat{\Delta}_+ = \Delta_\phi + \frac{5\pi^2 j(j+1)\varepsilon^2}{33} \qquad \hat{b}_+^2 = 1 + \frac{\pi^2 j(j+1)\varepsilon^2}{6} \left(\frac{31}{11} - \frac{20}{11}\log 2 \right)$$

Conclusions and Outlook

- We have shown that the analytic bootstrap provides a systematic method to solve conformal defects perturbatively (up to low spin ambiguities).
- We have used the inversion formula / dispersion relation to study 2pt functions of bulk fields in the $O(N)$ model in presence of line defects.
- The bootstrap results are universal, they apply to essentially any line defect in that theory.
- One needs to supplement the bootstrap with other methods to fix the ambiguities.

Future directions:

- Higher dimensional defects (surfaces, boundaries).
- Higher orders in ε or large N.
- Defects in fermionic theories.

Thank you for your attention!