JT gravity on hyperbolic lattices and discrete holography

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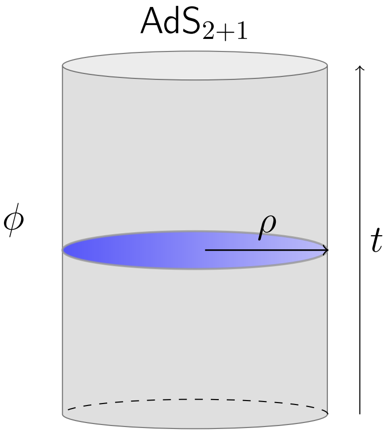
Motivation

- The AdS/CFT-correspondence [9711200] states that a (quantum) gravity theory in $(d+1)$ dimensional asymptotic AdS spacetime, has a dual description by a CFT without gravity, living on the d dimensional boundary of the spacetime
- Question: Can the holographic principle be realised in discrete spacetime?
- First step: Aperiodic spin chains as possible boundary theories (see [2205.05693] and [2212.11292])



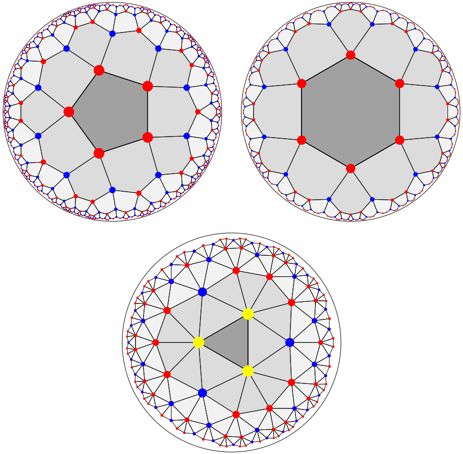
[2205.05693]

JT gravity on a hyperbolic lattice

- We discretize a Cauchy slice of AdS_3, which can be viewed as EAdS_2

$$ds^2 = (2R)^2 \frac{d\rho^2 + \rho^2\, d\phi^2}{(1-\rho^2)^2}$$

  through a $\{p, q\}$ tiling, where q regular p-gons meet at each vertex
- Since the underlying space has negative curvature, p and q satisfy $(p-2)(q-2) > 4$



[2205.05693]

JT gravity on a hyperbolic lattice

- The discrete JT gravity action on a triangular lattice Δ is

$$\mathcal{S}_{JT} = -S_0\chi(\Delta) - \sum_{v\in\Delta^\circ} \phi(v)\left(\varepsilon(v) + \frac{a(v)}{R^2}\right) - \sum_{v\in\partial\Delta} \phi(v)(\psi(v) - \psi_c),$$

with a counterterm ψ_c and generating functional

$$\mathcal{Z} = \int \mathcal{D}l_{ij}^2 \mathcal{D}\phi \, \mathrm{e}^{-\mathcal{S}_{JT}}$$

- Integrating out the dilaton and setting $R = 1$ yields $\varepsilon(v) = -a(v)$
- For a $\{3, q\}$ tiling this fixes the simplicial geometry

$$\left(2 - \frac{q}{3}\right)\pi = \varepsilon = -a = -\frac{q a_\Delta}{3} = -\frac{q s^2}{4\sqrt{3}}$$

JT gravity on a hyperbolic lattice

- The path integral reduces to a sum over boundary fluctuations
- Since we consider the disk, we fix the boundary length to β, with boundary dilaton value ϕ_b, and for a $\{3, q\}$ the counterterm is $\psi_c = \frac{\pi}{6}\left(6 - q + \sqrt{(q-6)(q-2)}\right)$
- We parameterize the boundary by a discrete curve Γ over the edges of the triangles and obtain the disk partition function

$$
\begin{aligned}
Z(\beta) &= \mathrm{e}^{S_0 \chi(\Delta)} \sum_{|\Gamma|=\beta} \exp\left(\phi_b \sum_{v \in \partial\Delta} (\psi(v) - \psi_c)\right) \\
&= \mathrm{e}^{(S_0 + 2\pi\phi_b)\chi(\Delta)} \sum_{|\Gamma|=\beta} \exp\left[-\phi_b \left(\sum_{v \in \Delta^\circ} \varepsilon(v) + \psi_c V_\partial\right)\right]
\end{aligned}
$$

JT gravity on a hyperbolic lattice

- Since $\varepsilon = -a$ the sum in the exponent reduces to $a \cdot V_{\text{int}} = A_\Gamma$, the area enclosed by Γ
- We calculate this area in the Schwarzian limit, where we take β and ϕ_b to be large, i.e. we take a saddle point approximation
- Since $\beta = |\Gamma| = s \cdot V_\partial$ is fixed, we have to maximize the area A_Γ, in order to find the saddle point
- The configuration with maximal area is given by a lattice consisting of n concentric layers, where n is related to β and has to be large
- We introduce a vector $\vec{v}(n) = (v_1(n), v_2(n))^T$, where $v_{1/2}(n)$ is the number of red/blue vertices in the n-th layer
- Thus, $\vec{v}(1) = \vec{0}$, $\vec{v}(2) = (3q - 12, 3)^T$ and in general $\vec{v}(n+1) = M_{\{3,q\}} \cdot \vec{v}(n)$

JT gravity on a hyperbolic lattice

We solve this recursion relation and obtain the total number of internal vertices for $n \gg 1$ as [2306.07203]

$$V_{\text{int}}(n) = 3 + \sum_{i=2}^{n-1} v_1(n) + v_2(n)$$
$$\approx \frac{3}{2}\left(1 + \frac{q-2}{\sqrt{q^2 - 8q + 12}}\right)\frac{\lambda_+^n - \lambda_+}{\lambda_+ - 1},$$

as well as the number of boundary vertices

$$V_\partial(n) = v_1(n) + v_2(n) = \frac{3}{2}\left(1 + \frac{q-2}{\sqrt{q^2 - 8q + 12}}\right)\lambda_+^n$$

JT gravity on a hyperbolic lattice

- We arrive at

$$A(n(\beta)) = \frac{q}{3} a_\Delta \cdot V_{\text{int}} \approx \frac{q}{3} \cdot \frac{a_\Delta}{\lambda_+ - 1} V_\partial(n) = \frac{q}{3s} \cdot \frac{a_\Delta}{\lambda_+ - 1} \beta$$

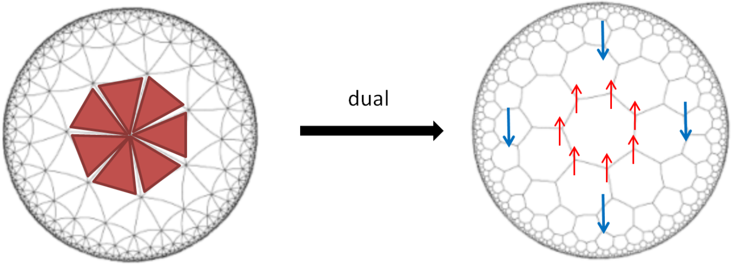
- Thus, the disk partition function is given by

$$Z(\beta) \sim \mathrm{e}^{\phi_b (A(n(\beta)) - \psi_c V_\partial)} \sim \exp\left(-\frac{\phi_b}{s}\left(\psi_c - \frac{q}{3} \cdot \frac{a_\Delta}{\lambda_+ - 1}\right)\beta\right)$$

- We note that this is a zero-loop result, while in the continuous case, the disk partition function is obtained from a one-loop calculation

Discrete JT gravity as an Ising model

- In order to include spacetime fluctuations, we have to sum over all lattice configurations, with fixed boundary length
- This is done by associating a spin up to all triangles in a fixed sublattice, while all other spins point down
- The sum over different triangulations becomes a sum over spin configurations on the dual lattice $\{q, 3\}$



Discrete JT gravity as an Ising model

- The action becomes

$$\frac{\mathcal{S}_{JT}}{B} = -\gamma_\phi \chi(\mathbb{D}) - \frac{1}{2\pi}\sum_{v\in\partial\Delta}(\psi(v)-\psi_c) = -\gamma_\phi - \frac{q}{6} + \frac{\varepsilon}{4\pi}F + \frac{2\psi_c-\varepsilon}{4\pi}V_\partial$$
$$= -\gamma_\phi - \frac{q}{6} - h\sum_{r\in\Delta^\times}\frac{s_r+1}{2} - J\sum_{\langle rr'\rangle\in\Delta^\times}\frac{s_r s_{r'}-1}{2},$$

with spins $s_r = \pm 1$ on the dual lattice $\Delta^\times$, boundary condition $s_\infty = -1$ and

$$\gamma_\phi = \frac{S_0}{2\pi\phi_b},\ B = 2\pi\phi_b,\ h = -\frac{\varepsilon}{4\pi} = \frac{q-6}{12},\ J = \frac{2\psi_c-\varepsilon}{4\pi} = \frac{1}{12}\sqrt{(q-6)(q-2)}$$

- We also have to impose $1 = \chi(\mathbb{D}) = \frac{1}{2}(F + 3V_\partial + 2V_{\text{int}})$

Discrete JT gravity as an Ising model

- The disk partition function of discrete JT gravity is thus given by

$$Z(\beta) = \mathrm{Tr}\ \mathrm{e}^{-\beta M} = \sum_{\{s\}} \mathrm{e}^{-\mathcal{S}} \delta(V_\partial B - \beta)\,,$$

  where the boundary length is given in units of the dilaton
- A related quantity is the resolvent function

$$R(E) = \mathrm{Tr}\ \frac{1}{E-M} = -\int_0^\infty d\beta\, Z(\beta)\, \mathrm{e}^{\beta E} = -\sum_{\{s\}} \mathrm{e}^{-BH}\,,$$

  which is equal to the canonical partition function of an Ising model at temperature $T = 1/B$ and Hamiltonian

$$H = \mathcal{S}/B - EV_\partial = -\gamma_\phi - \frac{q}{6} - h \sum_{r\in\Delta^\times} \frac{s_r + 1}{2} - (J-E) \sum_{\langle rr'\rangle\in\Delta^\times} \frac{s_r s_{r'} - 1}{2}$$

Discrete JT gravity as an Ising model

- The magnetization is given by $m = \frac{1}{V} \sum_{r \in \Delta^\times} \langle s_r \rangle$ and we assume the fluctuation of each spin δs_r around its expectation value is small
- Thus $s_r s_{r'} \approx m \cdot [(s_r + s_{r'} - m)]$
- We find the mean-field Hamiltonian

$$H_{MF} = \text{const.} - h_{\text{eff}} \sum_{r \in \Delta^\times} s_r \,,$$

where we introduce the effective magnetic field

$$h_{\text{eff}} = \frac{1}{2}(h + 3m(J - E))$$

Discrete JT gravity as an Ising model

- The mean-field partition function is given by

$$Z_{MF} = \mathrm{Tr}\ \mathrm{e}^{-BH_{MF}} \sim \left[2\cosh(Bh_{\text{eff}})\right]^V ,$$

and we find the self consistency condition

$$m = \frac{1}{VB} \frac{\partial(\log Z_{MF})}{\partial h_{\text{eff}}} = \tanh(Bh_{\text{eff}}) = \tanh\left[\frac{B}{2}(h + 3m(J - E))\right]$$

- We graphically solve this equation in order to find the critical temperature $T_C = 1/B_C$ of the phase transition

E	$q = 7$	$q = 8$	$q = 8$
0	$B_C = 5,5$	$B_C = 3,9$	$B_C = 3,1$
$-J$	$B_C = 2,3$	$B_C = 1,6$	$B_C = 1,2$
$-2J$	$B_C = 1,4$	$B_C = 1$	$B_C = 0,75$
$-3J$	$B_C = 1$	$B_C = 0,7$	$B_C = 0,55$

Discrete JT gravity as an Ising model

- While qualitatively correct, the mean-field approach can only be used to give an estimation for the true critical temperature, since fluctuations are important in lower dimensions
- We thus simulate the model for temperatures close to the expected phase transition
- We create a Markov chain of configurations with Metropolis probability

$$W_f(\phi \to \phi') = C \cdot \begin{cases} \exp(-B\,\delta H) & \text{if } \delta H > 0\,, \\ 1 & \text{else}\,, \end{cases}$$

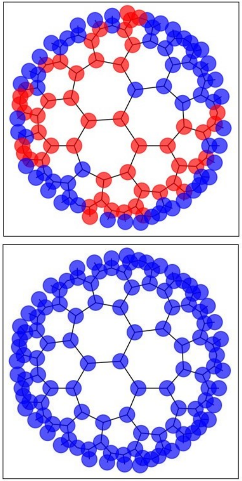
where $\delta H = H(\phi') - H(\phi)$

Discrete JT gravity as an Ising model

- For each update step we impose the condition that the topology is unchanged, i.e.

$$1 = \frac{1}{2}(F + 3V_\partial + 2V_{\text{int}})$$

- We find that the critical temperatures are indeed lower than the ones predicted through mean-field theory



Conclusion and outlook

Conclusion

- We choose a discrete analog of JT gravity as a bulk theory for discrete holography, and calculated its partition function in the Schwarzian limit
- We map the gravity theory to an Ising model on the dual lattice, and study its phase transition analytically and numerically

Outlook

- Numerically evaluate the partition function of the Ising model, which gives the resolvent of discrete JT gravity
- Find a matrix model formulation of the gravity theory, similar to [1903.11115]