TBA and GSE of the twisted $AdS_3 \times S^3 \times T^4$ superstring

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AdS_3 problem

- String model solvability and Gauge/Gravity completeness
- AdS_3/CFT_2 integrability and exact techniques
- $AdS_3 \times S^3 \times \mathcal{M}^4$ complete solvability problem
  - S-matrix (dressing phases)
  - **G**round **S**tate **E**nergy
  - Finite-size effects
  - Twisted sector on the worldsheet (Non-BPS vacua)
  - The full spectrum

Outline

- **T**hermodynamic **B**ethe **A**nsatz and *Mirror* formulation
- $AdS_3 \times S^3 \times T^4$ RR-flux: Twisted vacua and Y-system
- Generalised Lüscher formalism
- Mixed flux GSE: TBA and $AdS_3 \times S^3 \times S^3 \times S^1$ lightcone

The scope of AdS_n

The Gauge/Gravity duality: mapping of string theory on AdS space and conformal field theory – **string energy spectrum/spectrum of scaling dimensions** of the CFT.

Integrability perspective: global symmetries, *e.g.* $AdS_5 \times S^5$ superspace isometries/ $\mathcal{N} = 4$ superconformal symmetry – observable computation. Further progress includes completeness

- $AdS_4 \times \mathbb{CP}^3$ (BA, QSC)

- $\boldsymbol{AdS_3} \times \boldsymbol{S}^3 \times \boldsymbol{\mathcal{M}}^4 = \begin{cases} \boldsymbol{\mathcal{M}}^4 = \boldsymbol{T}^4 \\ \boldsymbol{\mathcal{M}}^4 = \boldsymbol{S}^3 \times \boldsymbol{S}^1 \end{cases}$ (TBA, QSC proposals)

- $AdS_2 \times S^2 \times T^6$ (Generalised ABA conjecture)

From group-theoretic point

$AdS_n \times S^n = \hat{G}/H$ supercosets, with superisometry $\hat{G}$ include:

- $AdS_5 \times S^5 \longrightarrow \dfrac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$
- $AdS_3 \times S^3 \longrightarrow \dfrac{PSU(1,1|2) \times PSU(1,1|2)}{SO(1,2) \times SO(3)}$
- $AdS_2 \times S^2 \longrightarrow \dfrac{PSU(1,1|2)}{SO(1,1) \times SO(2)}$

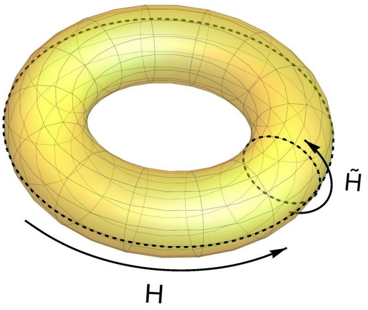
In general the AdS_3 background can be equipped with 3-form fluxes (RR and NSNS)

Mirror formulation

A theory on a torus can
be described by the partition function
$\mathcal{Z}$ on the circle at finite temperature.

Evolution of
such theory can be given through either
of the cycles, where $p \to -i\tilde{H}$, $H \to i\tilde{p}$

Space and time
are interchanged in the Mirror Model
$\sigma \to \tilde{\tau} = -i\sigma$ and $\tau \to \tilde{\sigma} = i\tau$, which
constitutes the double Wick-rotation
of the initial superstring model.



Evolution torus

Mirror formulation

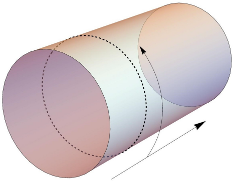
It can be shown that mirror partition function $\tilde{\mathcal{Z}}$ agrees with the initial one

$$\tilde{\mathcal{Z}} = \sum_k \left\langle \tilde{\psi}_k \middle| e^{-L\tilde{H}} \middle| \tilde{\psi}_k \right\rangle = \int \mathcal{D}\tilde{p}\,\mathcal{D}x\, e^{\int_0^R \mathrm{d}\tau \int_0^L \mathrm{d}\sigma (ipx' - \tilde{H})} \qquad \widetilde{\mathcal{Z}}(L,R) = \mathcal{Z}(L,R)$$

where in the original model, the size is L and R is the inverse temperature β [Zamolodchikov '90]. In the *mirror model* it results in size R and temperature $1/L$ (swap).

Considering $R \to +\infty$ forms *decompactifying limit*:

- Zero T, Finite volume
- Finite T, Infinite volume



Hence in the infinite volume limit finds relation on GSE (original) and bulk free energy (mirror)

$$R \to +\infty : \qquad E(L) = L\tilde{f}(L)$$

where $\tilde{f}$ at $1/L$ can be obtained from the Mirror TBA.

$AdS_3 \times S^3 \times T^4$ Mirror TBA

$$S(p_1, p_2) = \Sigma \cdot \hat{S}(p_1, p_2) \otimes \hat{S}(p_1, p_2)$$

The mirror derivation follows

$$S\text{-matrix} \rightarrow \text{Bethe-Yang} \rightarrow \text{Densities} \xrightarrow{\text{TDL}} \text{Mirror TBA}$$

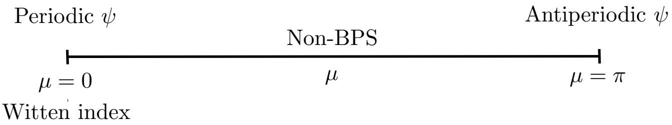
Fundamentally the AdS_3 MTBA depends on **mirror momentum $\tilde{p}$** (rapidity) and a set of ***Y*-functions**:

- $N_x^{(Q)}$ particles with $\sum_{Q=1}^{\infty} N_x^{(Q)} = N_x$ described by $Y_Q/\bar{Y}_Q$-functions
- $N_0^{(\dot{\alpha})}$ massless excitations $Y_0^{(\dot{\alpha})}$, with $-1 < x_k^{(\dot{\alpha})} < 1$ $x_k \in \mathbb{R}$
- $N_y^{(\alpha)}$ auxiliary particles (roots) by $Y_{\pm}^{(\alpha)}$, with $|y_\alpha| = 1$.

that arise in the string hypothesis.

Non-BPS vacuum

As indicated above, for non-BPS vacua one must introduce twist dependence



GSE twist interpolation between supervacuum and Non-BPS sector (even and odd winding sectors)

$AdS_3 \times S^3 \times T^4$ RR Mirror TBA

From the $AdS_3 \times S^3 \times T^4$ mirror TBA the ground state energy receives contributions from both chiral **massive** sectors as well as **massless** excitations

$$E(\mu, h, L) = -\sum_{Q=1}^{+\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{d\widetilde{p}^Q}{du} \log\left[(1+Y_Q)\left(1+\overline{Y}_Q\right)\right]$$
$$-\sum_{\dot{\alpha}=1}^{N_0} \int_{|u|>2} \frac{du}{2\pi} \frac{d\widetilde{p}^0}{du} \log\left[1+Y_0^{(\dot{\alpha})}\right]$$

which implies energy dependence on twist $\boldsymbol{\mu}$, string tension $\boldsymbol{h} = \frac{\sqrt{\lambda}}{2\pi}$ and lightcone momentum $\boldsymbol{L}$ (gauge fixed).

In the temporal gauge $L = J$, where J associated to the $U(1)$ isometry of S^3 and gets quantised.

Massive

The analytic structure of the TBA system and its solvability can be gained from massive (massless) sector. Hence the equation for left particles

$$
\begin{aligned}
-\log Y_Q = L\widetilde{\mathcal{E}}_Q - \log\left(1+Y_{Q'}\right)\star K^{Q'Q}_{\mathfrak{sl}(2)} - \log\left(1+\overline{Y}_{Q'}\right)\star \widetilde{K}^{Q'Q}_{\mathfrak{su}(2)} \\
- \sum_{\dot\alpha}\log\left(1+Y_0^{(\dot\alpha)}\right)\check{\star} K^{0Q} \\
- \sum_{\alpha=1,2}\log\left(1-\frac{e^{i\mu_\alpha}}{Y_+^{(\alpha)}}\right)\hat{\star} K^{yQ}_+ - \sum_{\alpha=1,2}\log\left(1-\frac{e^{i\mu_\alpha}}{Y_-^{(\alpha)}}\right)\hat{\star} K^{yQ}_-
\end{aligned}
$$

$$
\mu_\alpha = (-1)^\alpha \mu, \qquad \alpha = \{1,2\}
$$

allows to evaluate the leading contributing terms when perturbed in μ. K^{ab} are kernels in the appropriate mirror particle sector.

Right and massless equations admit similar analysis.

Auxiliary

On the other hand, the coupled system on $\boldsymbol{y}_{\pm}$ particles should be taken separately since all the terms appear to contribute at the same level

$$\log Y_{+}^{(\alpha)} = -\log\left(1+Y_Q\right)\star K_{+}^{Qy} + \log\left(1+\overline{Y}_Q\right)\star K_{-}^{Qy} - \sum_{\dot{\alpha}} \log\left(1+Y_0^{(\dot{\alpha})}\right)\check{\star} K^{0y}$$

Analytic structure and closure of TBA indicates dependence of Y-function and identifies the contribution order $\mathcal{O}[\mu]$, when perturbed in μ.

Small twist

From *kernel convolutions* with *constant densities* it becomes possible to obtain Ansätze for Y-functions, after which the TBA can be solved and results in

$$Y_{\{Q,\overline{Q}\}} \approx \mu^2 \left[\frac{x_Q^+}{x_Q^-}\right]^L \qquad Y_0^{(\dot{\alpha})} \approx \mu^2 \left[\frac{x_0^+}{x_0^-}\right]^L \qquad \widetilde{\mathcal{E}}_Q = \log \frac{x_Q^-}{x_Q^+} \ .$$

The massive mirror energy $\widetilde{\mathcal{E}}_Q$ derives

$$\widetilde{\mathcal{E}}_Q = \log \frac{x\left(u - i\frac{Q}{h}\right)}{x\left(u + i\frac{Q}{h}\right)} = 2\,\mathrm{arcsinh}\left(\frac{\sqrt{\left(\widetilde{p}^Q\right)^2 + Q^2}}{2h}\right),$$

At the level of Y-ansätze, it can be proven that all functions start to contribute at $\mathcal{O}[\mu^2]$.

One can now obtain the GSE at small μ and arbitrary L

$$E(\mu,h,L) \approx -\mu^2 \left[\sum_{Q=1}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\widetilde{p}^Q 2 e^{-L\tilde{\mathcal{E}}_Q} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\widetilde{p}^Q 2 e^{-L\tilde{\mathcal{E}}_0} \right] \\ = -\frac{\mu^2}{\pi} \left[\mathcal{I} + \frac{8hL}{4L^2-1} \right] \tag{1}$$

where the massive term $\mathcal{I}$ can be transformed into

$$\mathcal{I} = L \sum_{k=0}^{\infty} (-1)^k 4^{k+L} \frac{\Gamma\left(k+L-\frac{1}{2}\right)\Gamma\left(k+L+\frac{1}{2}\right)}{\Gamma(k+1)\Gamma(k+2L+1)} h^{2k+2L} \zeta(2k+2L-1)$$

One can find that the massless integral can be computed analytically for $L > \frac{1}{2}$, whereas the massive part $\mathcal{I}$ becomes the single convergent sum for

$$\mathcal{I}_{\text{conv}}: \qquad L > 1\,, \;\; |h| \leq \frac{1}{2}$$

Large circumference

For the case of large L and arbitrary μ the TBA solves, when $Y \sim e^{-L\tilde{\mathcal{E}}_x}$ and provides

$$E(\mu, h, L) \approx -\frac{4}{\pi}\sin^2\left(\frac{\mu}{2}\right)\left[\mathcal{I} + \frac{8hL}{4L^2-1}\right] .$$

Hence this solution nontrivially results in replacing twists factor

$$\mu^2 \quad \to \quad 4\sin^2\left(\frac{\mu}{2}\right) ,$$

for massless and massive integrals. Clearly, it recovers $\mu \ll 1$ case.

Lüscher formalism

The finite-size effects are related to wrapping interactions that come from the dynamics of the virtual particles (on finite volume).

[Arutyunov, Bajnok, Frolov, Janik, Łukowski, Zamaklar ...]

From the perspective of finite-size corrections to GSE, it appears possible to consider Lüscher formalism by introducing massless deformation and twisting

$$E(\mu, h, L) = -2 \sum_{Q=1}^{+\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{d\widetilde{p}^{Q}}{du} e^{-L\tilde{\mathcal{E}}_Q} \mathfrak{F}_Q$$
$$- n_0 \int_{|u|>2} \frac{du}{2\pi} \frac{d\widetilde{p}^{0}}{du} e^{-L\tilde{\mathcal{E}}_0} \mathfrak{F}_0 + \mathcal{O}\left[e^{-2L\tilde{\mathcal{E}}_x}\right]$$

The Tr_x is over appropriate $x = \{Q/\overline{Q}/0\}$ 2-dim representations $X = X_1 \otimes X_2$ (1 boson and fermion $\in X_i$). So the computation provides

$$E(\mu, h, L) = -\frac{4}{\pi} \sin^2\left(\frac{\mu}{2}\right) \int_{-\infty}^{+\infty} \sum_{Q=1}^{+\infty} d\widetilde{p}^Q \, e^{-L\widetilde{\mathcal{E}}_Q}$$

$$+ -\frac{4}{\pi} n_0 \sin^2\left(\frac{\mu}{2}\right) \frac{8hL}{4L^2 - 1} + \mathcal{O}\left[e^{-2L\tilde{\mathcal{E}}_Q}\right]$$

$$\mathfrak{F}_x = \mathrm{Tr}_x e^{i(\pi+\mu)F} = \left(1 - e^{-i\mu}\right)\left(1 - e^{i\mu}\right) = 4 \sin^2\left(\frac{\mu}{2}\right)$$

where $F \to F_{x_1} - F_{x_1}$ plays a role of the fermion number operator. After recasting the massive integral, we find 1 to 1 agreement

$$\boxed{E(\mu, h, L) = -\frac{4}{\pi} \sin^2\left(\frac{\mu}{2}\right) \left[\mathcal{I} + \frac{8hL}{4L^2 - 1}\right] + \mathcal{O}\left[e^{-2L\tilde{\mathcal{E}}_Q}\right] .}$$

Lightcone $AdS_3 \times S^3 \times T^4$ sigma model

$AdS_3 \times S^3 \times T^4$ Near-BMN (mixed)

$$S = \int \mathrm{d}\tau \int_0^{\mathcal{J}} \mathrm{d}\sigma \, \mathcal{L}(x, \dot{x}) \qquad \mathcal{L} = \mathcal{L}^{(2)} + \frac{1}{h}\mathcal{L}^{(4)} + \frac{1}{h^2}\mathcal{L}^{(6)} + \dots$$

where the orders with odd field configuration are absent due to perturbative properties of $AdS_n \times S^n$. [Sundin et. al.]

$$\mathcal{L}^{(2)} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{m}}$$

where the terms describe massless and massive sectors

$$\mathcal{L}_0 = |\partial_i u_1|^2 + |\partial_i u_2|^2 + i\bar{\chi}_L^{\dot{r}} \partial_- \chi_L^{\dot{r}} + i\bar{\chi}_R^{\dot{r}} \partial_+ \chi_R^{\dot{r}}$$

$$\begin{aligned} \mathcal{L}_{\mathrm{m}} = {} & |\partial_i \tilde{z}|^2 - \hat{q}^2 |\tilde{z}|^2 + |\partial_i \tilde{y}|^2 - \hat{q}^2 |\tilde{y}|^2 \\ & + i\bar{\chi}_L^r \partial_- \chi_L^r + i\bar{\chi}_R^r \partial_+ \chi_R^r - \hat{q}\bar{\chi}_L^r \chi_R^r - \hat{q}\bar{\chi}_R^r \chi_L^r \qquad r = 1, 2 \end{aligned}$$

$AdS_3 \times S^3 \times T^4$ Near-BMN GSE

The leading computation for the large $\mathcal{J} = L/h$ derives

$$E^M = \frac{2}{\gamma} \sum_{k\backslash\{0\}} \left[(1 - e^{2i\pi k\tilde{\mu}})\, \Omega(k)^2 \left(\frac{4\pi |k| K_1\left(\frac{2\pi\hat{q}|k|}{\gamma}\right)}{\gamma\hat{q}} \right) \right]$$

$$= -\frac{4}{\pi} \sin^2\left(\frac{\mu}{2}\right) \sqrt{\hat{q}\,\frac{2\pi}{\mathcal{J}}}\; e^{-\hat{q}\,\mathcal{J}} + \mathcal{O}\left[e^{-\mathcal{J}} \mathcal{J}^{-\frac{3}{2}}\right] \qquad \Omega(z)^2 = -\frac{\gamma^2\hat{q}^2}{4\pi^2 z^2}$$

$$\boxed{E^{\text{nBMN}}_{T^4} = -\frac{\mu^2}{\pi\mathcal{J}} - \frac{4}{\pi} \sin^2\left(\frac{\mu}{2}\right) \sqrt{\hat{q}\,\frac{2\pi}{\mathcal{J}}}\; e^{-\hat{q}\,\mathcal{J}} + \mathcal{O}\left[e^{-\mathcal{J}} \mathcal{J}^{-\frac{3}{2}}\right]}$$

where for $\hat{q} = 1$ it reproduces the pure RR case.

Mixed flux $AdS_3 \times S^3 \times T^4$

Mixed flux $AdS_3 \times S^3 \times S^3 \times S^1$

From the lightcone perspective, it is possible to construct twisted superstring sigma models equipped with RR-NSNS flux. The fields become charged under $\mathfrak{su}(2)_\bullet$ and $\mathfrak{su}(2)_\circ$.

In the context of the $AdS_3 \times S^3 \times \mathcal{M}^4$ lightcone superstring one can consider boso-fermionic twisting in generic form

$$\phi_k(\tau, \sigma + \mathcal{J}) = e^{-i\mu_k^{\mathrm{B}}} \phi_k(\tau, \sigma) \quad \text{and} \quad \psi_{k,\alpha}(\tau, \sigma + \mathcal{J}) = e^{-i\mu_{k,\alpha}^{\mathrm{F}}} \psi_{k,\alpha}(\tau, \sigma)$$

with $\alpha = 1, 2$ and $\mathcal{J} = L/h$.

From the twisted $AdS_3 \times S^3 \times S^3 \times S^1$, the dispersion relation can be given as

$$E = \sum_{n=-\infty}^{\infty} \sqrt{2\rho q(n + \tilde{\mu}_B) m_{B,i} + \rho^2 (n + \tilde{\mu}_B)^2 + m_{B,i}^2} - \sum_{n=-\infty}^{\infty} \sqrt{2\rho q(n + \tilde{\mu}_F) m_{F,i} + \rho^2 (n + \tilde{\mu}_F)^2 + m_{F,i}^2}$$

$$\hat{q}^2 + q^2 = 1\,, \;\; \rho = 2\pi/\mathcal{J}$$

Mixed flux GSE from lightcone

In general for N complex massless and massive bosons and fermions we derive

$$E^{N_{B,F}} = \sum_{i=1}^{N_0^B} \left(\frac{|\mu_{0,B,i}|}{\mathcal{J}} - \frac{\mu_{0,B,i}^2}{2\pi\mathcal{J}} - \frac{\pi}{3\mathcal{J}} \right) - \sum_{i=1}^{N_0^F} \left(\frac{|\mu_{0,F,i}|}{\mathcal{J}} - \frac{\mu_{0,F,i}^2}{2\pi\mathcal{J}} - \frac{\pi}{3\mathcal{J}} \right)$$
$$- \frac{2\hat{q}}{\pi} \sum_{w=1}^{+\infty} \left[\sum_{l=1}^{N_B} \frac{m_{B,l}}{w} K_1 \left(\mathcal{J} w m_{B,l}\, \hat{q} \right) \cos \left(\mathcal{J} w\, m_{B,l} \sqrt{1-\hat{q}^2} + w\mu_{B,l} \right) \right.$$
$$\left. - \sum_{n=1}^{N_F} \frac{m_{F,n}}{w} K_1 \left(\mathcal{J} w m_{F,n}\, \hat{q} \right) \cos \left(\mathcal{J} w\, m_{F,n} \sqrt{1-\hat{q}^2} + w\mu_{F,n} \right) \right]$$

Mixed flux GSE from lightcone

For the case of mixed flux, when $\mathcal{M}^4 = T^4$ there are **two particles** in each sector, hence one obtains for the large $\mathcal{J}$

$$E^{LC} = \frac{|\mu_1 - \mu_2| + |\mu_1 + \mu_2| - 2\mu_2}{\mathcal{J}} - \frac{\mu_1^2}{\pi \mathcal{J}} - \frac{4}{\pi} \sin^2\left(\frac{\mu}{2}\right) \sqrt{\hat{q}\frac{2\pi}{\mathcal{J}}} \cos\left(\mathcal{J}\sqrt{1-\hat{q}^2}\right) e^{-\mathcal{J}\hat{q}}$$

By universal twisting for massless excitations ($\mu_{1,2} = \mu$), we acquire for the mixed AdS_3 GSE [Frolov, AP, Sfondrini '23]

$$E^{MF}_{T^4} = -\frac{\mu^2}{\pi \mathcal{J}} - \frac{4}{\pi} \sin^2\left(\frac{\mu}{2}\right) \sqrt{\hat{q}\frac{2\pi}{\mathcal{J}}} \cos\left(\mathcal{J}\sqrt{1-\hat{q}^2}\right) e^{-\mathcal{J}\hat{q}} + \mathcal{O}\left[\mathcal{J}^{-\frac{3}{2}} \cos(q\mathcal{J}) e^{-\mathcal{J}\hat{q}}\right]$$

Mixed flux GSE: TBA proposal

For this purpose we derive the mixed flux mirror energy in terms of mirror momentum $\tilde{p}$ and bound state number Q

$$\tilde{\mathcal{E}}_Q = \frac{1}{h}\sqrt{\tilde{p}^2 + \hat{q}^2 Q^2} + iq\frac{Q}{h}$$

By making a proposal for the mixed flux TBA and recalling large L behaviour, we find for the GSE

$$E^{MF}_{T^4} = -\frac{\mu^2}{2\pi}\int_{-\infty}^{+\infty}\sum_{Q=1}^{+\infty} d\widetilde{p}^Q (e^{-L\widetilde{\mathcal{E}}_Q} + e^{-L\overline{\widetilde{\mathcal{E}}}_Q}) - \frac{n_0\mu^2}{2\pi}\int_{-\infty}^{+\infty} d\widetilde{p}^0 e^{-L\widetilde{\mathcal{E}}_0}$$

$$= -\frac{\mu^2}{\pi\mathcal{J}} - \frac{\mu^2}{\pi}\sqrt{\hat{q}\frac{2\pi}{\mathcal{J}}}\cos\left(\mathcal{J}\sqrt{1-\hat{q}^2}\right)e^{-\mathcal{J}\hat{q}} + \dots,$$

that is in full agreement with the $AdS_3 \times S^3 \times T^4$ lightcone computation for large $\mathcal{J}$.

Conclusions

- Proven TBA system closure and solvability for the twisted $AdS_3 \times S^3 \times T^4$ with RR-flux
- Derived GSE in μ, h, L
- Developed generalised Lüscher formalism that consistently accounts for massless modes
- In large $\mathcal{J} = L/h$ the GSE $E_{\text{MTBA}} = E_{\text{LC}}$ (Near-BMN)
- Proposed $AdS_3 \times S^3 \times T^4$ mixed flux GSE from Mirror TBA, which appears in full agreement with the $AdS_3 \times S^3 \times S^3 \times S^1$ lightcone derivation

Further Directions

- Twisted QSC construction and observables [in progress]
- The full excited TBA (contour deformation)
- Mixed flux $AdS_3 \times S^3 \times T^4$ TBA system
- Twisted superstring sigma models
- NLO TBA (all sectors) and Lüscher formalism
- Twisted hybrid formalism for AdS_3 and n-pt functions for (generic q)

Thank You

Bethe-Yang system

Left magnon equation

$$1 = e^{i\tilde{p}_k R} \prod_{\substack{j=1\\ j\neq k}}^{N_1} S^{11}_{\mathfrak{sl}}\left(u_k, u_j\right) \prod_{j=1}^{N_{\bar{1}}} \widetilde{S}^{11}_{\mathfrak{sl}}\left(u_k, u_j\right) \prod_{\dot{\alpha}=1}^{2} \prod_{j=1}^{N_0^{(\alpha)}} S^{10}\left(u_k, u_j^{(\dot{\alpha})}\right) \prod_{\alpha=1}^{2} \prod_{j=1}^{N_y^{(\alpha)}} S^{1y}\left(u_k, y^{(\alpha)}\right)$$

$$S^{11}_{\mathfrak{sl}}\left(u_k, u_j\right) = \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1 - \frac{1}{x_k^- x_j^+}}{1 - \frac{1}{x_k^+ x_j^-}} \left(\sigma_{kj}^{\bullet\bullet}\right)^{-2} \qquad \widetilde{S}^{11}_{\mathfrak{sl}}\left(u_k, u_j\right) = e^{ip_k} \frac{1 - \frac{1}{x_k^+ x_j^+}}{1 - \frac{1}{x_k^- x_j^-}} \frac{1 - \frac{1}{x_k^- x_j^+}}{1 - \frac{1}{x_k^+ x_j^-}} \left(\widetilde{\sigma}_{kj}^{\bullet\bullet}\right)^{-2}$$

$$S^{10}\left(u_k, u_j\right) = e^{-\frac{i}{2}p_k} e^{-ip_j} \frac{1 - x_k^+ x_j}{x_k^- - x_j} \left(\sigma_{kj}^{\bullet\circ}\right)^{-2} \qquad S^{1y}\left(u_k, y_j\right) = e^{-\frac{i}{2}p_k} \frac{1 - \frac{1}{x_k^- y_j}}{1 - \frac{1}{x_k^+ y_j}} = e^{\frac{i}{2}p_k} \frac{x_k^- - \frac{1}{y_j}}{x_k^+ - \frac{1}{y_j}}.$$

which after fusion results in the left Q-bound state equation

$$1 = e^{i\tilde{p}_a R} \prod_{\substack{b=1\\ b\neq a}}^{N_L} S^{Q_a Q_b}_{\mathfrak{sl}}\left(u_a, u_b\right) \prod_{b=1}^{N_R} \widetilde{S}^{Q_a \bar{Q}_b}_{\mathfrak{sl}}\left(u_a, u_b\right)$$

$$\times \prod_{\dot{\alpha}=1}^{2} \prod_{j=1}^{N_0^{(\alpha)}} S^{Q_a 0}\left(u_a, x_j^{(\dot{\alpha})}\right) \prod_{\alpha=1}^{2} \prod_{b=1}^{N_y^{(\alpha)}} S^{Q_a y}\left(u_a, y_b^{(\alpha)}\right)$$

Kernels and Convolution

The kernels depend on the associated S-matrix

$$K_{ij}(u,v) = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}u}\log S_{ij}(u,v)$$

hence are identified by the scattering data.

Depending on the scattered sectors, the associated cut structure is also reflected in an appropriate convolution bounds, *i.e.*

$$\star \leftrightarrow \int_{-\infty}^{+\infty} \mathrm{d}u \qquad \hat{\star} \leftrightarrow \int_{-2}^{+2} \mathrm{d}u \qquad \check{\star} \leftrightarrow \left(\int_{-\infty}^{-2} + \int_{+2}^{+\infty}\right) \mathrm{d}u$$

The involved $\star$-left action for any domain defines

$$\rho_i \star K_{ij}(v) \equiv \sum_i \int \mathrm{d}u\, \rho_i(u) K_{ij}(u,v)$$

The massive mirror energy $\widetilde{\mathcal{E}}_Q$ derives

$$\widetilde{\mathcal{E}}_Q = \log \frac{x\left(u - i\frac{Q}{h}\right)}{x\left(u + i\frac{Q}{h}\right)} = 2\,\mathrm{arcsinh}\left(\frac{\sqrt{\left(\widetilde{p}^{Q}\right)^2 + Q^2}}{2h}\right),$$

where the dependence is spanned by the Q-particle mirror momentum $\widetilde{p}_Q$ (real particles possess rapidity $\tilde{p}(u)$, $u \in \mathbb{R}$, whereas massive bound states are on the u-plane with complex momentum and energy).

In the present framework, the Zhukovsky relation is implemented on long-cut

$$x(u) = \frac{1}{2}\left(u - i\sqrt{4 - u^2}\right)$$

with Zhukovsky variable x and $-2 > u > 2$ for $u \in \mathbb{R}$.

AdS_5 and AdS_3 regimes

Large h, L is fixed

$$\mathcal{I}(h \gg L, L) \approx \frac{\pi}{L^2 - 1} h^2 \quad \mathcal{I}_{AdS_5}(h \gg L, L) \approx \frac{3\pi}{L^4 - 5L^2 + 4} h^4$$

AdS_3, h-regimes

$$\mathcal{I}(h \ll 1, L \gg h) = \sqrt{\frac{\pi}{L}}\, h^{2L} \qquad \mathcal{I}(h \gg 1, L \gg h) = \sqrt{2\pi \frac{h}{L}}\, e^{-\frac{L}{h}}$$

$\mathcal{J} = L/h$, $\mathcal{J}$ is fixed

$$\mathcal{I}(\mathcal{J} \gg 1) \approx \frac{\sqrt{2\pi} e^{-\mathcal{J}}}{\sqrt{\mathcal{J}}} \qquad \mathcal{I}_{AdS_5}(\mathcal{J} \gg 1) \approx \frac{\sqrt{2\pi} e^{-\mathcal{J}}}{\sqrt{\mathcal{J}}}$$

$$\mathcal{I}(\mathcal{J} \ll 1) \approx \frac{\pi}{\mathcal{J}^2} \qquad \mathcal{I}_{AdS_5}(\mathcal{J} \ll 1) \approx \frac{3\pi}{\mathcal{J}^4}$$

Small h, massless modes contribute already at linear order

$$E(\mu, L, h \ll 1) = -\frac{4}{\pi} \sin^2\left(\frac{\mu}{2}\right) \left(n_0 \frac{4hL}{4L^2 - 1} + \sqrt{\pi}\, \frac{\Gamma(L - \frac{1}{2})}{\Gamma(L)}\, \zeta(2L - 1)\, h^{2L} \right),$$

whereas in $AdS_5 \times S^5$ at $\mathcal{O}[\mu^2]$.

AdS_5 and AdS_3 GSE

By considering GSE at distinct $\mathcal{J}$ regimes for $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$

$$E_{AdS_3}(\mathcal{J} \ll 1) = -\frac{\mu^2}{\pi}\left(\frac{n_0 - 1}{\mathcal{J}} - \frac{\pi}{\mathcal{J}^2}\right) \qquad E_{AdS_5}(\mathcal{J} \ll 1) = -\frac{6\mu^2}{\mathcal{J}^4}$$

$$E_{AdS_3}(\mathcal{J} \gg 1) = -\frac{\mu^2}{\pi}\left(\frac{n_0}{\mathcal{J}} - \sqrt{\frac{2\pi}{\mathcal{J}}}\, e^{-\mathcal{J}}\right) \qquad E_{AdS_5}(\mathcal{J} \gg 1) = -2\,\frac{\mu^2}{\pi}\frac{\sqrt{2\pi}}{\sqrt{\mathcal{J}}} e^{-\mathcal{J}}$$

important that at large $\mathcal{J}$ **massless modes** start to dominate in the $AdS_3 \times S^3 \times T^4$ case.