

Higher-Point Conformal Blocks From the Oscillator Formalism

New Perspectives in Conformal Field Theory and Gravity

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1. Motivation

What are conformal blocks?

- conformal block decomposition of CFT correlation functions

$$\begin{aligned}\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4) \rangle &= \sum_{\mathfrak{h}} C_{12\mathfrak{h}} C_{34\mathfrak{h}} G_{\mathfrak{h}}^{(4)}(z_1, \dots, z_4) \\ &= \sum_{\mathfrak{h}} C_{1234\mathfrak{h}} \begin{array}{c} z_2 \bullet \quad \quad \bullet z_3 \\ \quad \diagdown \quad \quad \diagup \\ \quad \quad \mathfrak{h} \quad \quad \text{---} \quad \quad \\ \quad \diagup \quad \quad \diagdown \\ z_1 \bullet \quad \quad \bullet z_4 \end{array}\end{aligned}$$

splits *model-dependent (conformal) data* $\{\mathfrak{h}, C_{ijk}\}$ from **universal building blocks** $G_{\mathfrak{h}}^{(4)}$

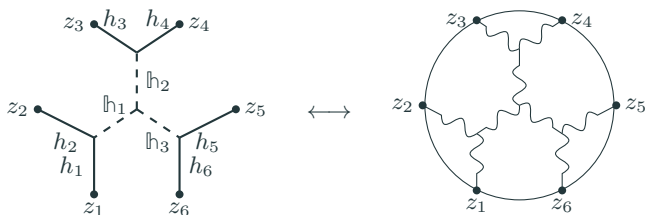
- “universal” within a fixed setting: blocks depend on e. g.
 - ▶ dimension d of base space
 - ▶ $d = 2$: global vs. Virasoro blocks
 - ▶ topology of base space (sphere, torus, ...)

Why should we care about (higher-point) blocks?

- conformal bootstrap program: consistency constraints on conformal data

$$\sum_{\mathbb{h}} C_{h_1 h_2 h_3 h_4}^{\mathbb{h}} \cdot \begin{array}{c} z_2 \quad z_3 \\ \diagdown \quad \diagup \\ \text{---} \mathbb{h} \text{---} \\ \diagup \quad \diagdown \\ z_1 \quad z_4 \end{array} = \sum_{\mathbb{h}'} C_{h_1 h_3 h_2 h_4}^{\mathbb{h}'} \cdot \begin{array}{c} z_2 \quad z_3 \\ \diagdown \quad \diagup \\ \text{---} \mathbb{h}' \text{---} \\ \diagup \quad \diagdown \\ z_1 \quad z_4 \end{array}$$

- higher-point schemes?
- AdS/CFT-correspondence [Maldacena, 1998](#)
 - Global conformal blocks are dual to bulk *geodesic Witten diagrams* [Witten, 1998](#); [Hijano et al., 2016](#)



2. Oscillator Representation Method for Global Blocks on the Sphere

How to compute conformal blocks?

- directly solve Ward identities + Casimir equations ([Dolan and Osborn, 2001, 2004](#))
 - ▶ difficulty quickly increases with degree of blocks
- recurrence relation method ([Zamolodchikov, 1984](#))
 - ▶ works fine order by order
 - ▶ challenging for finding closed form expressions

How to compute conformal blocks?

- shadow operator method (Ferrara and Parisi, 1972, Ferrara et al., 1972)
 - ▶ insert (shadow)projector into correlator

$$\Psi_{\mathbb{h}}(z_1, \dots, z_4) = \langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \tilde{P}_{\mathbb{h}} \mathcal{O}_3(z_3) \mathcal{O}_4(z_4) \rangle$$

- ▶ gives linear combination of conformal block and shadow block
 - ▶ conformal block obtained in an extra step
- oscillator representation method (Beşken, Datta, and Kraus, 2020a)
 - ▶ express projector as $P_{\mathbb{h}} = \int [d^2u] |\bar{u}\rangle \langle u|$ in terms of generalized coherent states

$$G_{\mathbb{h}}^{(4)}(z_1, \dots, z_4) = \langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) P_{\mathbb{h}} \mathcal{O}_3(z_3) \mathcal{O}_4(z_4) \rangle$$

- ▶ no shadow part!

Oscillator Representations of Verma Modules

- oscillator representation $L_n \mapsto \ell_n = u^{(1-n)} \partial_u + (1-n)\hbar u^{-n}$, $n = 0, \pm 1$ on **weighted Bergman spaces**

$$\mathcal{B}_{\hbar} := \left\{ f : \mathbb{D} \rightarrow \mathbb{C} \mid f \text{ holomorphic, } \int_{\mathbb{D}} [d^2 u] |f(u)|^2 < \infty \right\}$$

- HW state $|\hbar\rangle = 1$, descendant states $|\hbar, n\rangle =$ monomials u^n
 - orthogonality relation

$$(u^m, u^n) = \frac{n!}{(2\hbar)_n} \delta_{m,n} \quad (1)$$

- existence of **reproducing kernel** gives rise to resolution of one (Hall, 1999)

$$P_{\hbar} = \int [d^2 u] |\bar{u}\rangle \langle u|$$

Decomposing the Blocks

- four-point block

$$\begin{aligned}
 G_{\mathfrak{h}}^{(4)} &= \langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) P_{\mathfrak{h}} \mathcal{O}_3(z_3) \mathcal{O}_4(z_4) \rangle \\
 &= \int [d^2 u] \underbrace{\langle 0 | \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) | \bar{u} \rangle}_{\chi_{\mathfrak{h}}(z_1, z_2; \bar{u})} \underbrace{\langle u | \mathcal{O}_3(z_3) \mathcal{O}_4(z_4) | 0 \rangle}_{\psi_{\mathfrak{h}}(z_3, z_4; u)}
 \end{aligned}$$

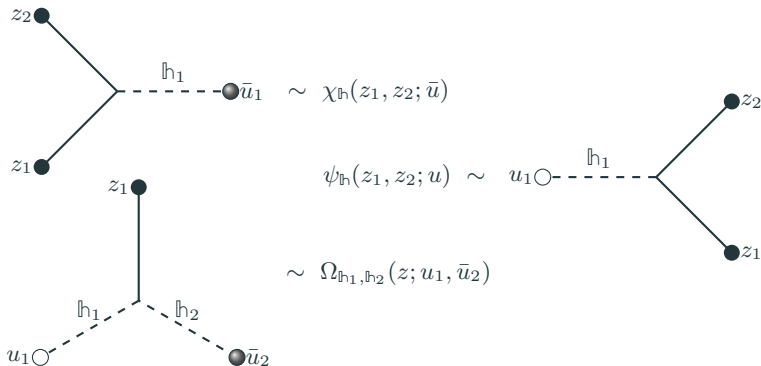
- five-point block

$$\begin{aligned}
 G_{\mathfrak{h}_1, \mathfrak{h}_2}^{(5)} &= \langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) P_{\mathfrak{h}_1} \mathcal{O}_3(z_3) P_{\mathfrak{h}_1} \mathcal{O}_4(z_4) \mathcal{O}_5(z_5) \rangle \\
 &= \int [d^2 u_1] \int [d^2 u_2] \underbrace{\langle 0 | \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) | \bar{u}_1 \rangle}_{\chi_{\mathfrak{h}_1}(z_1, z_2; \bar{u}_1)} \\
 &\quad \cdot \underbrace{\langle u_1 | \mathcal{O}_3(z_3) | \bar{u}_2 \rangle}_{\Omega_{\mathfrak{h}_1, \mathfrak{h}_2}(z_3; u_1, \bar{u}_2)} \underbrace{\langle u_2 | \mathcal{O}_4(z_4) \mathcal{O}_5(z_5) | 0 \rangle}_{\psi_{\mathfrak{h}_2}(z_4, z_5; u_2)}
 \end{aligned}$$

- n -point (comb) block: same shape with $(n - 3)$ Ω 's

Oscillator Diagrams

- wavefunctions allow for a diagrammatic interpretation



Gluing the Blocks: Four-Point Blocks

- computation of the **four-point block**

$$\begin{aligned}
 G_{\mathfrak{h}_1}^{(4)} &= \text{Diagram 1} \\
 &= \int [d^2 u_1] \text{Diagram 2} \cdot \text{Diagram 3} \\
 &= \mathcal{L}^{(4)} \mathfrak{z}^{\mathfrak{h}_1} {}_2F_1(\mathfrak{h}_1 + h_2 - h_1, \mathfrak{h}_1 - h_3 + h_4; 2\mathfrak{h}_1; \mathfrak{z})
 \end{aligned}$$

Diagram 1: A four-point block with external points z_1, z_2, z_3, z_4 and an internal dashed line labeled $\mathfrak{h}_1$. The points z_1 and z_2 are on the left, z_3 and z_4 are on the right, and the dashed line connects the two vertices.

Diagram 2: A four-point block with external points z_1, z_2, z_3, z_4 and an internal dashed line labeled $\mathfrak{h}_1$. The points z_1 and z_2 are on the left, z_3 and z_4 are on the right, and the dashed line connects the two vertices to a central point labeled $\bar{u}_1$.

Diagram 3: A four-point block with external points z_1, z_2, z_3, z_4 and an internal dashed line labeled $\mathfrak{h}_1$. The points z_1 and z_2 are on the left, z_3 and z_4 are on the right, and the dashed line connects the two vertices to a central point labeled u_1 .

Gluing the Blocks: Five-Point Blocks

- computation of the **five-point block**

$$\begin{aligned}
 G_{\mathfrak{h}_1, \mathfrak{h}_2}^{(5)} = & \text{Diagram 1} = \int [d^2 u_1] \int [d^2 u_2] \\
 & \cdot \text{Diagram 2} \cdot \text{Diagram 3}
 \end{aligned}$$

Diagram 1: A five-point block with external points z_1, z_2, z_3, z_4, z_5 . Solid lines connect z_1, z_2 to a vertex, z_4, z_5 to another vertex, and z_3 to a central vertex. Dashed lines connect the two side vertices to the central vertex, labeled $\mathfrak{h}_1$ and $\mathfrak{h}_2$.

Diagram 2: A three-point block with external points z_1, z_2 and a shaded internal point $\bar{u}_1$. Solid lines connect z_1, z_2 to a vertex, and a dashed line connects this vertex to $\bar{u}_1$, labeled $\mathfrak{h}_1$.

Diagram 3: A three-point block with external points u_1 (open circle), $\bar{u}_2$ (shaded circle), and z_3 . Dashed lines connect $u_1, \bar{u}_2$ to a central vertex, and a solid line connects this vertex to z_3 . The dashed lines are labeled $\mathfrak{h}_1$ and $\mathfrak{h}_2$.

Diagram 4: A three-point block with external points u_2 (open circle) and z_4, z_5 . A dashed line connects u_2 to a vertex, and solid lines connect this vertex to z_4, z_5 . The dashed line is labeled $\mathfrak{h}_2$.

Gluing the Blocks: Five-Point Blocks

- computation of **five-point block** (alternative):

$$\begin{aligned}
 G_{h_1, h_2}^{(5)} &= \\
 &= \int [d^2 u_2] \cdot
 \end{aligned}$$

The diagram illustrates the computation of the five-point block $G_{h_1, h_2}^{(5)}$. It shows two equivalent representations. The top representation is a tree-level diagram with a central dashed line. The left vertex of the dashed line connects to two external points z_1 and z_2 . The right vertex connects to two external points z_4 and z_5 . A third external point z_3 is connected to the central dashed line. The bottom representation shows the same diagram but with an additional external point $\bar{u}_2$ connected to the central dashed line, and an integration measure $\int [d^2 u_2]$ is shown.

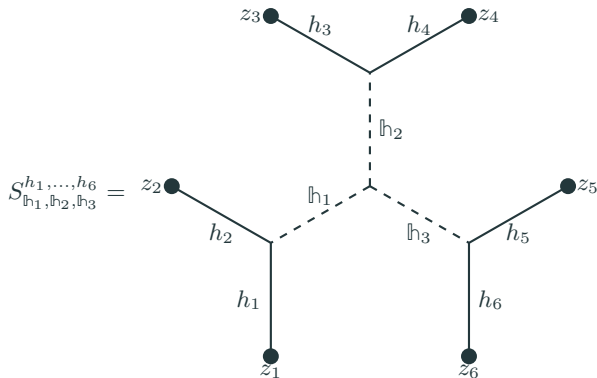
Gluing the Blocks: n -Point Blocks

- computation of the n -point comb block:

$$\begin{aligned}
 & G_{\mathfrak{h}_1, \dots, \mathfrak{h}_{n-3}}^{h_1, \dots, h_n}(z_1, \dots, z_n) \\
 = & \quad \begin{array}{c} \text{Diagram of an } n\text{-point comb block} \\ \text{with vertices } z_1, \dots, z_n \text{ and internal labels } \mathfrak{h}_1, \dots, \mathfrak{h}_{n-3} \end{array} \\
 = & \mathcal{L}^{(n)} \prod_{i=1}^{n-3} \mathfrak{z}_i^{\mathfrak{h}_i} F_K(\dots; \mathfrak{z}_1, \dots, \mathfrak{z}_{n-3})
 \end{aligned}$$

→ matches (Rosenhaus, 2019)

- What about different channels? E. g. six-point star channel



- construction works for $h_2 \geq h_1 + h_3$

3. Generalizations of the Method

Directions of Generalization

- **torus** conformal blocks (Hollweck, 2022)

$$G_h^{(1)}(q, w) = \text{diagram} = \text{diagram}$$

The diagram shows the equality of two Feynman diagrams. On the left, a solid line with a black dot labeled w and a label h above it connects to a vertex. From this vertex, two dashed lines branch out: one to a white circle labeled u with a label $\mathbb{h}$ above it, and another to a black dot labeled $q\bar{u}$ with a label $\mathbb{h}$ above it. On the right, a solid line with a black dot labeled w and a label h above it connects to a vertex, which then connects to a dashed circle with a label $\mathbb{h}$ on its right side.

- **Virasoro** conformal blocks (Beşken, Datta, and Kraus, 2020b)
 - ▶ infinite series of weighted Bargman-Segal spaces
 - ▶ solving for $\psi_{\mathbb{h}}$ and $\chi_{\mathbb{h}}$ much harder
 - ▶ semi-classical limit!
- **bms-blocks** in Carrollian CFTs (Ammon et al., 2021)
- conformal blocks in **higher dimensions** (e.g. $d = 4$)

Oscillator Formalism: From $d = 2$ to $d = 4$

(based on [Calixto and Perez-Romero, 2010, 2011, 2014](#))

- oscillator variable becomes **matrix valued**

$$u \in \mathbb{D} \quad \rightarrow \quad U \in \mathbb{D}_4 = \{U \in \mathbb{C}^{2 \times 2} : 1 - U^\dagger U > 0\}$$

- orthonormal eigenbasis generalized Wigner-D-matrices

$$\varphi_n(u) = u^n \quad \rightarrow \quad \varphi_{q_a, q_b}^{j, m}(U) \sim \det(U)^m \mathcal{D}_{q_a, q_b}^j(U)$$

- projector** from coherent states

$$P_\Delta = \int [dU] |U^\dagger\rangle \langle U|$$

unitary oscillator representation ✓

- next step: compute wavefunctions

$$\Psi_\Delta(x^\mu, y^\mu; U), \quad X_\Delta(x^\mu, y^\mu; U^\dagger), \quad \Omega_\Delta(x^\mu; U_1, U_2^\dagger)$$

4. Conclusion and Outlook

- take-home message: The oscillator formalism provides an efficient method for the computation of conformal blocks.
 - ▶ intuitive diagrammatic formulation
 - ▶ applicable in different regimes of CFT
- future directions
 - ▶ continue investigation of different channels
 - ▶ derive closed form solutions for higher-point blocks in $d = 4$
 - ▶ implement spin in $d = 4$

Thank you for your attention!

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6. Backup Slides

- second-order Casimir of $\mathfrak{sl}(2, \mathbb{R})$

$$C_2 = -L_0^2 + \frac{1}{2}\{L_{-1}, L_1\}$$

- eigenvalue equation with projector

$$C_2 P_{\mathfrak{h}} = P_{\mathfrak{h}} C_2 = \mathfrak{h}(1 - \mathfrak{h}) P_{\mathfrak{h}}. \quad (2)$$

- multi-point Casimir operator

$$\begin{aligned} \mathcal{C}_2^{(i_1, \dots, i_m)} = & - \left(\mathcal{L}_0^{(i_1)} + \dots + \mathcal{L}_0^{(i_m)} \right)^2 \\ & + \frac{1}{2} \left\{ \mathcal{L}_{-1}^{(i_1)} + \dots + \mathcal{L}_{-1}^{(i_m)}, \mathcal{L}_1^{(i_1)} + \dots + \mathcal{L}_1^{(i_m)} \right\} \end{aligned}$$

Six-Point Casimir Equations

- comb channel blocks

$$\begin{aligned}\left(\mathcal{C}_2^{(3,4,5,6)} + \mathfrak{h}_1(\mathfrak{h}_1 - 1)\right) G_{\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3}^{(6)}(z_1, \dots, z_6) &= 0, \\ \left(\mathcal{C}_2^{(4,5,6)} + \mathfrak{h}_2(\mathfrak{h}_2 - 1)\right) G_{\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3}^{(6)}(z_1, \dots, z_6) &= 0, \\ \left(\mathcal{C}_2^{(5,6)} + \mathfrak{h}_3(\mathfrak{h}_3 - 1)\right) G_{\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3}^{(6)}(z_1, \dots, z_6) &= 0\end{aligned}$$

- star channel blocks

$$\begin{aligned}\left(\mathcal{C}_2^{(1,2)} + \mathfrak{h}_1(\mathfrak{h}_1 - 1)\right) S_{\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3}^{(6)}(z_1, \dots, z_6) &= 0, \\ \left(\mathcal{C}_2^{(3,4)} + \mathfrak{h}_2(\mathfrak{h}_2 - 1)\right) S_{\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3}^{(6)}(z_1, \dots, z_6) &= 0, \\ \left(\mathcal{C}_2^{(5,6)} + \mathfrak{h}_3(\mathfrak{h}_3 - 1)\right) S_{\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_3}^{(6)}(z_1, \dots, z_6) &= 0\end{aligned}$$

- Casimir equations for general n -point comb block

$$\begin{aligned} \left(\mathcal{C}_2^{(3,4,5,\dots,n)} + \mathbb{h}_1(\mathbb{h}_1 - 1) \right) G_{\mathbb{h}_1 \dots \mathbb{h}_{n-3}}^{(n)}(z_1, \dots, z_n) &= 0, \\ \left(\mathcal{C}_2^{(4,5,\dots,n)} + \mathbb{h}_2(\mathbb{h}_2 - 1) \right) G_{\mathbb{h}_1 \dots \mathbb{h}_{n-3}}^{(n)}(z_1, \dots, z_n) &= 0, \\ &\vdots \\ \left(\mathcal{C}_2^{(n-1,n)} + \mathbb{h}_{n-3}(\mathbb{h}_{n-3} - 1) \right) G_{\mathbb{h}_1 \dots \mathbb{h}_{n-3}}^{(n)}(z_1, \dots, z_n) &= 0. \end{aligned}$$

Remarks on Hypergeometric Functions

- (ordinary) hypergeometric function as power series

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

→ solution to the differential equation

$$\left[z(1-z)\partial_z^2 + [c - (a+b+1)z]\partial_z - ab \right] F(a, b, c; z) = 0$$

- generalized hypergeometric function

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}$$

- definition as power series

$$\begin{aligned}
 F_K & \left[\begin{matrix} a_1, b_1, \dots, b_{n-4}, a_2 \\ c_1, \dots, c_{n-3} \end{matrix}; z_1, \dots, z_{n-3} \right] \\
 &= \sum_{k_1, \dots, k_{n-3}}^{\infty} \frac{(a_1)_{k_1} (b_1)_{k_1+k_2} (b_2)_{k_2+k_3} \dots (b_{n-4})_{k_{n-4}+k_{n-3}} (a_2)_{k_{n-3}}}{(c_1)_{k_1} \dots (c_{n-3})_{k_{n-3}}} \\
 & \quad \cdot \frac{z_1^{k_1}}{k_1!} \dots \frac{z_{n-3}^{k_{n-3}}}{k_{n-3}!}
 \end{aligned}$$

- satisfies splitting equations

Rosenhaus, 2019

Lauricella Hypergeometric Functions

- definition as power series

$$\begin{aligned} F_D^{(3)}(a, b_1, b_2, b_3, c; z_1, z_2, z_3) \\ = \sum_{j_1, j_2, j_3=0}^{\infty} \frac{(a)_{j_1+j_2+j_3} (b_1)_{j_1} (b_2)_{j_2} (b_3)_{j_3}}{(c)_{j_1+j_2+j_3}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \frac{z_3^{j_3}}{j_3!} \end{aligned}$$

- integral representation

$$\begin{aligned} F_D^{(3)}(a, b_1, b_2, b_3, c; z_1, z_2, z_3) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \\ &\cdot \int_0^1 dt t^{a-1} (1-t)^{c-a-1} (1-z_1 t)^{-b_1} (1-z_2 t)^{-b_2} (1-z_3 t)^{-b_3} \end{aligned}$$