Higher-Point Conformal Blocks From the Oscillator Formalism

New Perspectives in Conformal Field Theory and Gravity

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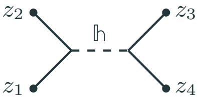
1. Motivation

What are conformal blocks?

- conformal block decomposition of CFT correlation functions

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle = \sum_{\mathbb{h}} C_{12\mathbb{h}} C_{34\mathbb{h}} G^{(4)}_{\mathbb{h}}(z_1, \ldots, z_4)$$

$$= \sum_{\mathbb{h}} C_{1234\mathbb{h}}$$



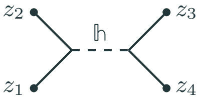
  splits *model-dependent (conformal) data* $\{\mathbb{h}, C_{ijk}\}$ from universal building blocks $G^{(4)}_{\mathbb{h}}$

- "universal" within a fixed setting: blocks depend on e. g.
  - dimension d of base space
  - $d = 2$: global vs. Virasoro blocks
  - topology of base space (sphere, torus, ...)

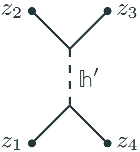
Why should we care about (higher-point) blocks?

- conformal bootstrap program: consistency constrains on conformal data

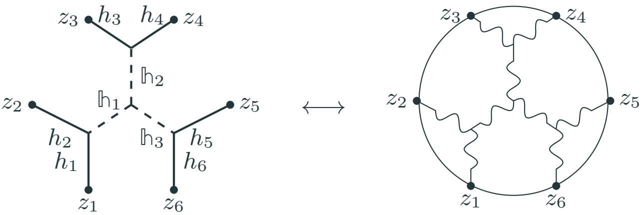
$$\sum_{\mathbb{h}} C^{\mathbb{h}}_{h_1 h_2 h_3 h_4} \cdot$$



$$= \sum_{\mathbb{h}'} C^{\mathbb{h}'}_{h_1 h_3 h_2 h_4} \cdot$$



  - higher-point schemes?
- AdS/CFT-correspondence Maldacena, 1998
  - Global conformal blocks are dual to bulk *geodesic Witten diagrams* Witten, 1998; Hijano et al., 2016



2. Oscillator Representation Method for Global Blocks on the Sphere

How to compute conformal blocks?

- directly solve Ward identities + Casimir equations (Dolan and Osborn, 2001, 2004)
  - difficulty quickly increases with degree of blocks
- recurrence relation method (Zamolodchikov, 1984)
  - works fine order by order
  - challenging for finding closed form expressions

How to compute conformal blocks?

- shadow operator method (Ferrara and Parisi, 1972, Ferrara et al., 1972)
  - insert (shadow)projector into correlator

$$\Psi_{\mathbb{h}}(z_1, \ldots, z_4) = \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\tilde{P}_{\mathbb{h}}\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle$$

  - gives linear combination of conformal block and shadow block
  - conformal block obtained in an extra step
- oscillator representation method (Beşken, Datta, and Kraus, 2020a)
  - express projector as $P_{\mathbb{h}} = \int [\mathrm{d}^2 u] |\bar{u}\rangle\langle u|$ in terms of generalized coherent states

$$G^{(4)}_{\mathbb{h}}(z_1, \ldots, z_4) = \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)P_{\mathbb{h}}\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle$$

  - no shadow part!

Oscillator Representations of Verma Modules

- oscillator representation $L_n \mapsto \ell_n = u^{(1-n)}\partial_u + (1-n)\mathbb{h}u^{-n}$, $n = 0, \pm 1$ on weighted Bergman spaces

$$\mathcal{B}_{\mathbb{h}} := \left\{ f : \mathbb{D} \to \mathbb{C} | f \text{ holomorphic}, \int_{\mathbb{D}} [\mathrm{d}^2 u] \, |f(u)|^2 < \infty \right\}$$

- HW state $|\mathbb{h}\rangle = 1$, descendant states $|\mathbb{h}, n\rangle =$ monomials u^n
  - orthogonality relation

$$(u^m, u^n) = \frac{n!}{(2\mathbb{h})_n} \delta_{m,n} \tag{1}$$

- existence of reproducing kernel gives rise to resolution of one (Hall, 1999)

$$P_{\mathbb{h}} = \int [\mathrm{d}^2 u] |\bar{u}\rangle\langle u|$$

Decomposing the Blocks

- four-point block

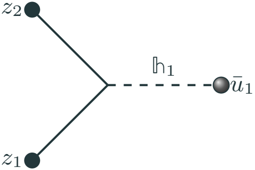
$$
\begin{aligned}
G^{(4)}_{\mathbb{h}} &= \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)P_{\mathbb{h}}\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle \\
&= \int [\mathrm{d}^2 u]\, \underbrace{\langle 0|\, \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\, |\bar{u}\rangle}_{\chi_{\mathbb{h}}(z_1,z_2;\bar{u})} \underbrace{\langle u|\, \mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\, |0\rangle}_{\psi_{\mathbb{h}}(z_3,z_4;u)}
\end{aligned}
$$

- five-point block

$$
\begin{aligned}
G^{(5)}_{\mathbb{h}_1,\mathbb{h}_2} &= \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)P_{\mathbb{h}_1}\mathcal{O}_3(z_3)P_{\mathbb{h}_1}\mathcal{O}_4(z_4)\mathcal{O}_5(z_5)\rangle \\
&= \int [\mathrm{d}^2 u_1] \int [\mathrm{d}^2 u_2]\, \underbrace{\langle 0|\, \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\, |\bar{u}_1\rangle}_{\chi_{\mathbb{h}_1}(z_1,z_2;\bar{u}_1)} \\
&\quad \cdot \underbrace{\langle u_1|\, \mathcal{O}_3(z_3)\, |\bar{u}_2\rangle}_{\Omega_{\mathbb{h}_1,\mathbb{h}_2}(z_3;u_1,\bar{u}_2)} \underbrace{\langle u_2|\, \mathcal{O}_4(z_4)\mathcal{O}_5(z_5)\, |0\rangle}_{\psi_{\mathbb{h}_2}(z_4,z_5;u_2)}
\end{aligned}
$$

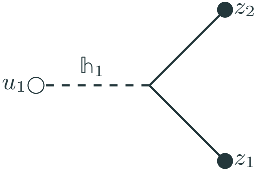
- n-point (comb) block: same shape with $(n-3)$ Ω's

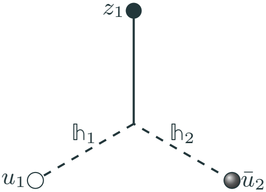
- wavefunctions allow for a diagrammatic interpretation



$\sim \ \chi_{\mathbb{h}}(z_1, z_2; \bar{u})$

$\psi_{\mathbb{h}}(z_1, z_2; u) \ \sim$



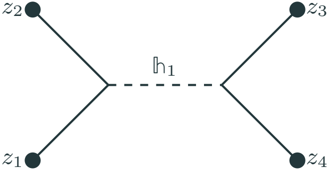


$\sim \ \Omega_{\mathbb{h}_1, \mathbb{h}_2}(z; u_1, \bar{u}_2)$

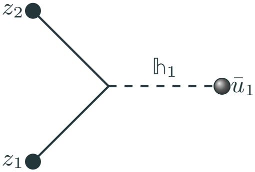
Gluing the Blocks: Four-Point Blocks

- computation of the four-point block

$G^{(4)}_{\mathbb{h}_1} =$



$= \int [\mathrm{d}^2 u_1]$



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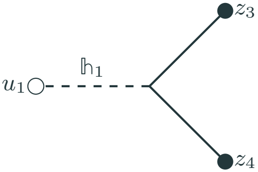
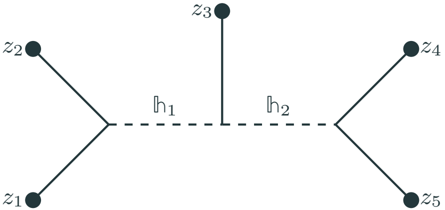


$$= \mathcal{L}^{(4)}\ \mathfrak{z}^{\mathbb{h}_1}\ {}_2F_1(\mathbb{h}_1 + h_2 - h_1, \mathbb{h}_1 - h_3 + h_4; 2\mathbb{h}_1; \mathfrak{z})$$

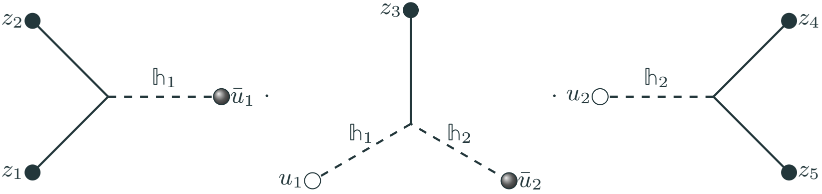
Gluing the Blocks: Five-Point Blocks

- computation of the five-point block

$G^{(5)}_{\mathbb{h}_1,\mathbb{h}_2} =$

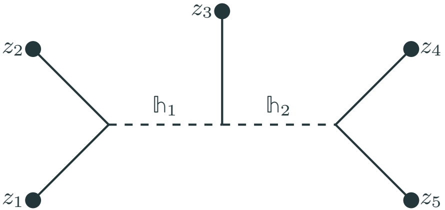


$= \int [\mathrm{d}^2 u_1] \int [\mathrm{d}^2 u_2]$

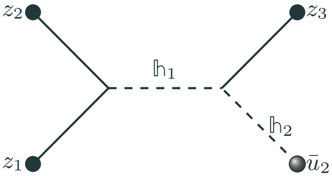


- computation of five-point block (alternative):

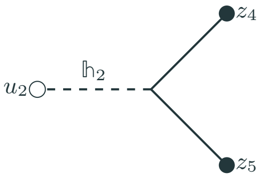
$G^{(5)}_{\mathbb{h}_1,\mathbb{h}_2} =$



$= \int [\mathrm{d}^2 u_2]$



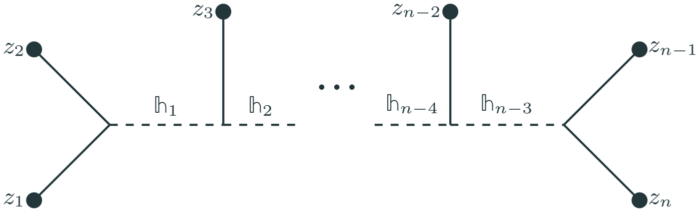
$\cdot$



- computation of the n-point comb block:

$$G^{h_1,\ldots,h_n}_{\mathbb{h}_1,\ldots,\mathbb{h}_{n-3}}(z_1,\ldots,z_n)$$

$=$

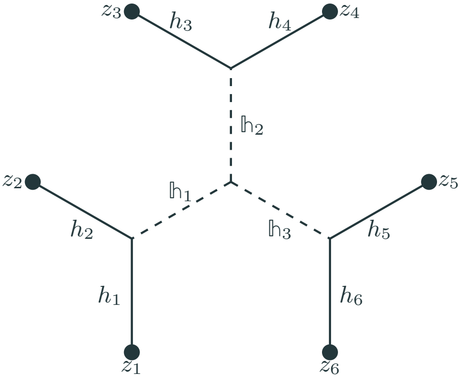


$$= \mathcal{L}^{(n)} \prod_{i=1}^{n-3} \mathfrak{z}_i^{\mathbb{h}_i} \; F_K(\ldots;\mathfrak{z}_1,\ldots,\mathfrak{z}_{n-3})$$

$\rightarrow$ matches (Rosenhaus, 2019)

- What about different channels? E. g. six-point star channel

$S^{h_1,\ldots,h_6}_{\mathbb{h}_1,\mathbb{h}_2,\mathbb{h}_3} =$

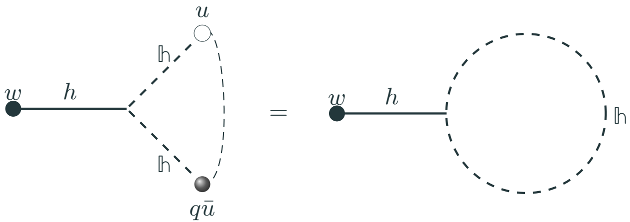


  - construction works for $\mathbb{h}_2 \geq \mathbb{h}_1 + \mathbb{h}_3$

3. Generalizations of the Method

- torus conformal blocks (Hollweck, 2022)

$$G_h^{(1)}(q,w) \quad = $$



- Virasoro conformal blocks (Beşken, Datta, and Kraus, 2020b)
  - infinite series of weighted Bargman-Segal spaces
  - solving for $\psi_{\mathbb{h}}$ and $\chi_{\mathbb{h}}$ much harder
  - semi-classical limit!
- $\mathfrak{bms}$-blocks in Carrollian CFTs (Ammon et al., 2021)
- conformal blocks in higher dimensions (e. g. $d = 4$)

Oscillator Formalism: From $d = 2$ to $d = 4$

(based on Calixto and Perez-Romero, 2010, 2011, 2014)

- oscillator variable becomes matrix valued

$$u \in \mathbb{D} \quad \rightarrow \quad U \in \mathbb{D}_4 = \{U \in \mathbb{C}^{2\times 2} : 1 - U^\dagger U > 0\}$$

- orthonormal eigenbasis generalized Wigner-D-matrices

$$\varphi_n(u) = u^n \quad \rightarrow \quad \varphi^{j,m}_{q_a,q_b}(U) \sim \det(U)^m \mathcal{D}^j_{q_a,q_b}(U)$$

- projector from coherent states

$$P_\Delta = \int [\mathrm{d}U] |U^\dagger\rangle\langle U|$$

unitary oscillator representation ✓

- next step: compute wavefunctions

$$\Psi_\Delta(x^\mu, y^\mu; U), \quad X_\Delta(x^\mu, y^\mu; U^\dagger), \quad \Omega_\Delta(x^\mu; U_1, U_2^\dagger)$$

4. Conclusion and Outlook

- take-home message: The oscillator formalism provides an efficient method for the computation of conformal blocks.
  - intuitive diagrammatic formulation
  - applicable in different regimes of CFT
- future directions
  - continue investigation of different channels
  - derive closed form solutions for higher-point blocks in $d = 4$
  - implement spin in $d = 4$

Thank you for your attention!

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6. Backup Slides

More on Casimir Equations

- second-order Casimir of $\mathfrak{sl}(2,\mathbb{R})$

$$C_2 = -L_0^2 + \frac{1}{2}\{L_{-1}, L_1\}$$

- eigenvalue equation with projector

$$C_2 P_{\mathbb{h}} = P_{\mathbb{h}} C_2 = \mathbb{h}(1-\mathbb{h}) P_{\mathbb{h}}\,. \tag{2}$$

- multi-point Casimir operator

$$\mathcal{C}_2^{(i_1,\dots,i_m)} = -\left(\mathcal{L}_0^{(i_1)} + \cdots + \mathcal{L}_0^{(i_m)}\right)^2 + \frac{1}{2}\left\{\mathcal{L}_{-1}^{(i_1)} + \cdots + \mathcal{L}_{-1}^{(i_m)}, \mathcal{L}_1^{(i_1)} + \cdots + \mathcal{L}_1^{(i_m)}\right\}$$

Six-Point Casimir Equations

- comb channel blocks

$$\left(\mathcal{C}_2^{(3,4,5,6)} + \mathbb{h}_1(\mathbb{h}_1 - 1)\right) G^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0\,,$$
$$\left(\mathcal{C}_2^{(4,5,6)} + \mathbb{h}_2(\mathbb{h}_2 - 1)\right) G^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0\,,$$
$$\left(\mathcal{C}_2^{(5,6)} + \mathbb{h}_3(\mathbb{h}_3 - 1)\right) G^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0$$

- star channel blocks

$$\left(\mathcal{C}_2^{(1,2)} + \mathbb{h}_1(\mathbb{h}_1 - 1)\right) S^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0\,,$$
$$\left(\mathcal{C}_2^{(3,4)} + \mathbb{h}_2(\mathbb{h}_2 - 1)\right) S^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0\,,$$
$$\left(\mathcal{C}_2^{(5,6)} + \mathbb{h}_3(\mathbb{h}_3 - 1)\right) S^{(6)}_{\mathbb{h}_1\mathbb{h}_2\mathbb{h}_3}(z_1, \ldots, z_6) = 0$$

Casimir Equations for Comb Blocks

- Casimir equations for general n-point comb block

$$
\begin{aligned}
\left(\mathcal{C}_2^{(3,4,5,\ldots,n)} + \mathbb{h}_1(\mathbb{h}_1 - 1)\right) G^{(n)}_{\mathbb{h}_1\ldots\mathbb{h}_{n-3}}(z_1,\ldots,z_n) &= 0\,,\\
\left(\mathcal{C}_2^{(4,5,\ldots,n)} + \mathbb{h}_2(\mathbb{h}_2 - 1)\right) G^{(n)}_{\mathbb{h}_1\ldots\mathbb{h}_{n-3}}(z_1,\ldots,z_n) &= 0\,,\\
&\vdots\\
\left(\mathcal{C}_2^{(n-1,n)} + \mathbb{h}_{n-3}(\mathbb{h}_{n-3} - 1)\right) G^{(n)}_{\mathbb{h}_1\ldots\mathbb{h}_{n-3}}(z_1,\ldots,z_6) &= 0\,.
\end{aligned}
$$

Remarks on Hypergeometric Functions

- (ordinary) hypergeometric function as power series

$$ {}_2F_1(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} $$

  $\rightarrow$ solution to the differential equation

$$ \left[z(1-z)\partial_z^2 + [c-(a+b+1)z]\partial_z - ab \right] F(a,b,c;z) = 0 $$

- generalized hypergeometric function

$$ {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!} $$

Comb Function

- definition as power series

$$F_K \left[\begin{matrix} a_1, b_1, \ldots, b_{n-4}, a_2 \\ c_1, \ldots, c_{n-3} \end{matrix} ; z_1, \ldots z_{n-3} \right]$$

$$= \sum_{k_1, \ldots, k_{n-3}}^{\infty} \frac{(a_1)_{k_1} (b_1)_{k_1+k_2} (b_2)_{k_2+k_3} \ldots (b_{n-4})_{k_{n-4}+k_{n-3}} (a_2)_{k_{n-3}}}{(c_1)_{k_1} \ldots (c_{n-3})_{k_{n-3}}} \cdot \frac{z_1^{k_1}}{k_1!} \ldots \frac{z_{n-3}^{k_{n-3}}}{k_{n-3}!}$$

- satisfies splitting equations

  Rosenhaus, 2019

Lauricella Hypergeometric Functions

- definition as power series

$$F_D^{(3)}(a, b_1, b_2, b_3, c; z_1, z_2, z_3) = \sum_{j_1, j_2, j_3 = 0}^{\infty} \frac{(a)_{j_1+j_2+j_3} (b_1)_{j_1} (b_2)_{j_2} (b_3)_{j_3}}{(c)_{j_1+j_2+j_3}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \frac{z_3^{j_3}}{j_3!}$$

- integral representation

$$F_D^{(3)}(a, b_1, b_2, b_3, c; z_1, z_2, z_3) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \cdot \int_0^1 \mathrm{d}t \, t^{a-1} (1-t)^{c-a-1} (1 - z_1 t)^{-b_1} (1 - z_2 t)^{-b_2} (1 - z_3 t)^{-b_3}$$