

# Aspects of Flat-Space Holography and Higher Spins

DESY Theory Workshop 2023

Michel Pannier

Friedrich-Schiller-Universität Jena  
Theoretisch-Physikalisches Institut

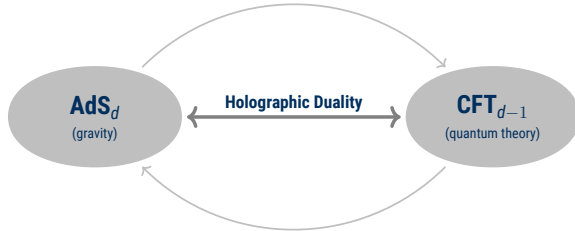
Hamburg, 27 September 2023

# Outline

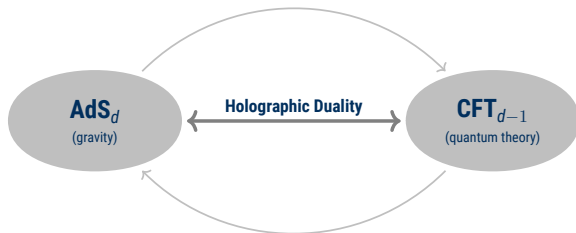
- 1 Motivation: Farewell to AdS
- 2 Peculiarities of Flat-Space Holography
- 3 Higher Spins in Flat Space: Massless and Massive
- 4 Discussion: What's next?

## Motivation: Farewell to AdS

# Expectations on Holography



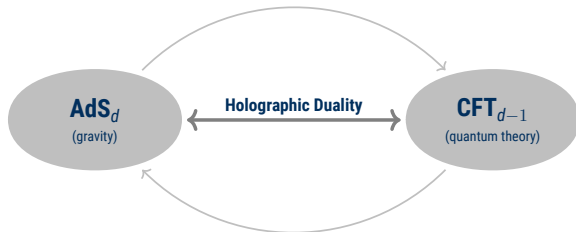
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## Objective I: Quantum Gravity

- emergence of gravity
- unitarity of time evolution
- string-theoretic embedding

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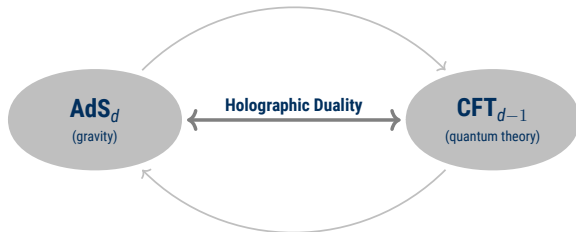
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- promising:  $d = 2 + 1$
- dualities involving higher spin

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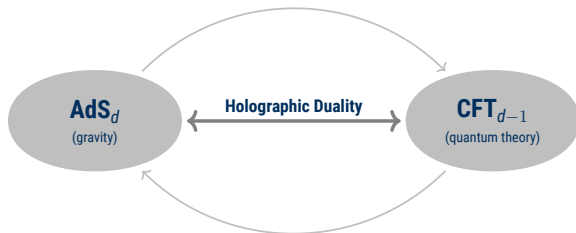
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relevant on cosmological scales
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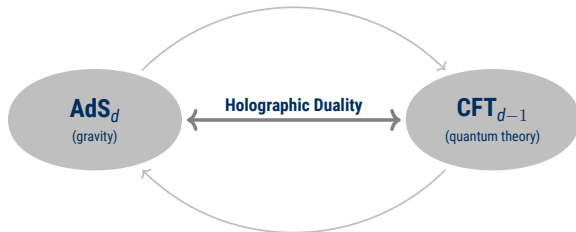
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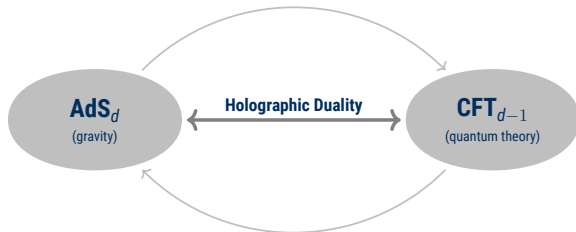
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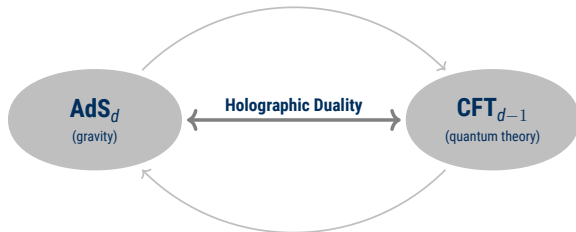
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# Lessons from AdS/CFT

## Towards Quantum Gravity:

Einstein gravity

generalisation ↓ ↑ sub-sector

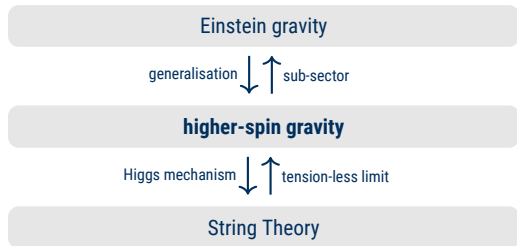
**higher-spin gravity**

Higgs mechanism ↓ ↑ tension-less limit

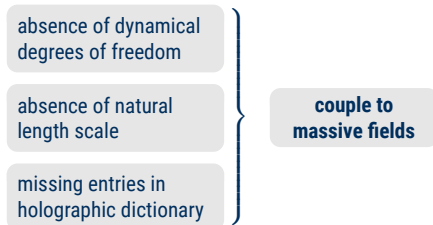
String Theory

# Lessons from AdS/CFT

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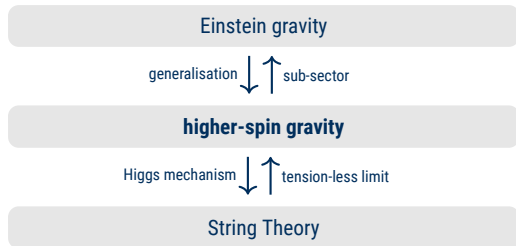


## Towards understanding Holography: (in 3d flat space)



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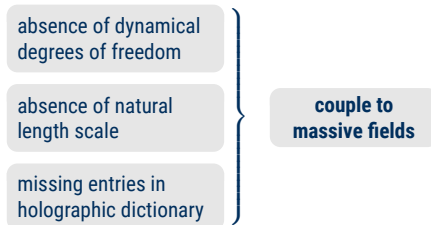
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## Following results: three steps at once

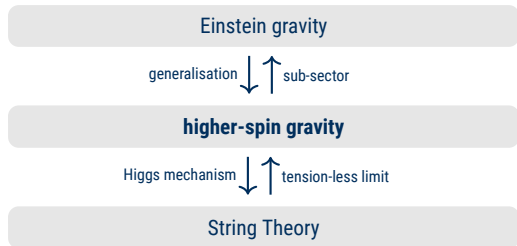
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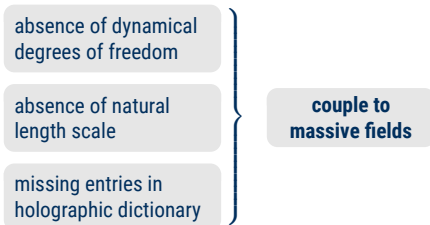


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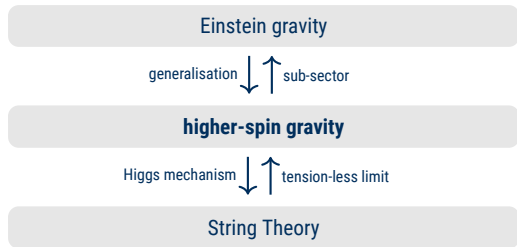
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## Necessary ingredients:

- definition of asymptotics
- higher-spin symmetry algebra
- matter-coupling prescription

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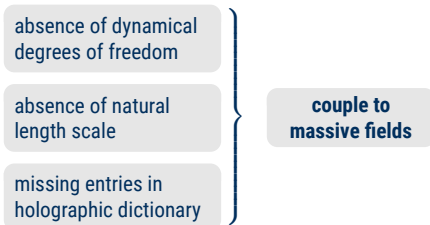
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First-principle considerations,  
no  $\Lambda \rightarrow 0$  limit!

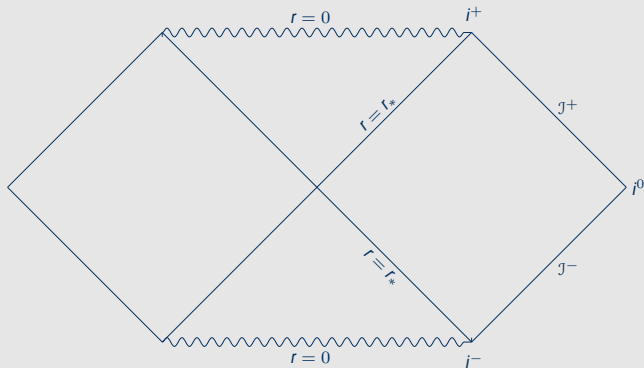


# Peculiarities of Flat-Space Holography

# Carrollian View on Flat Spacetimes

Causal Structure:

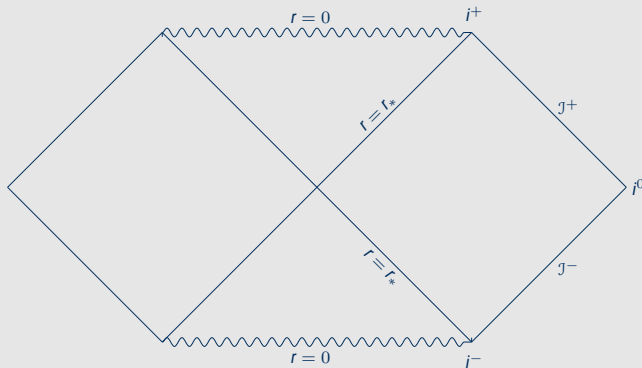
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# Carrollian View on Flat Spacetimes

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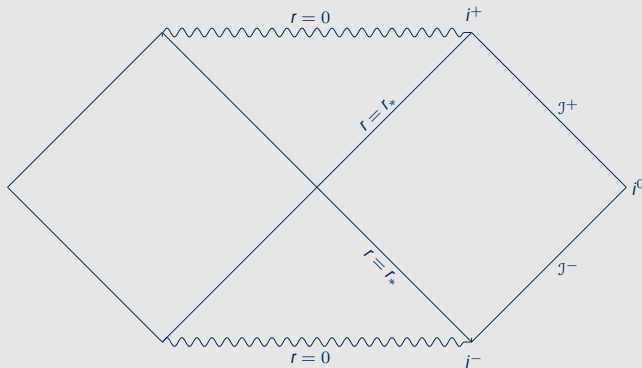
## Special Features:

- lightlike conformal boundary
- different boundary regions
- particular boundary conditions, asymptotic symmetry: BMS  
[Bondi/van der Burg/Metzner, 1962]  
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- dual field theory: Carrollian CFT  
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(alternatively: **Celestial Holography**)  
See also Charlotte's talk!

# Asymptotically Flat Spacetimes

[Cornalba/Costa, hep-th/0203031; Barnich/Troessaert, [1001.1541](#)]

## Most General Asymptotically Flat Metric

$$ds^2 = M(\phi)du^2 - 2dudr + 2N(u, \phi)dud\phi + r^2d\phi^2$$

## Eddington-Finkelstein coordinates $(u, r, \phi)$

- $u$ : outgoing retarded time (light-like)
- $r$ : radial coordinate ( $r \rightarrow \infty$  is asymptotic direction)
- $\phi$ : angular coordinate on  $S^1$

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## Chern-Simons Formulation

$$\omega = \left( J_1 - \frac{M(\phi)}{4} J_{-1} \right) d\phi,$$

$$e = \left( P_1 - \frac{M(\phi)}{4} P_{-1} \right) du + \frac{1}{2} P_{-1} dr + \left( rP_0 - \frac{N(u, \phi)}{2} P_{-1} \right) d\phi$$

[Afshar/Bagchi/Fareghbal/Grumiller/Rosseel, 1307.4768]

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## Isometries: Poincaré algebra

$$\text{iso}(2, 1) \simeq \mathfrak{isl}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R}) \in \mathbb{R}^3$$

$$[J_m, J_n] = (m - n)J_{m+n}$$

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# Higher Spins in Flat Space: Massless and Massive

# The Higher-Spin Symmetry Algebra

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$$\mathcal{M}^2 = (P_0)^2 - P_1 P_{-1},$$

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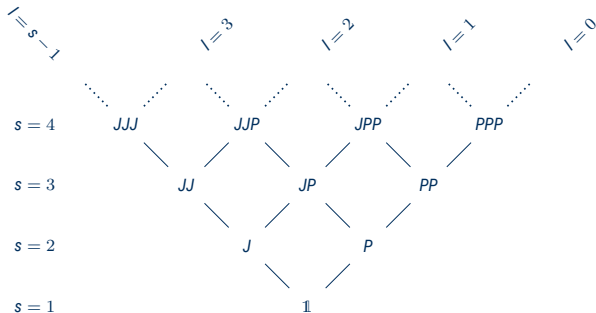
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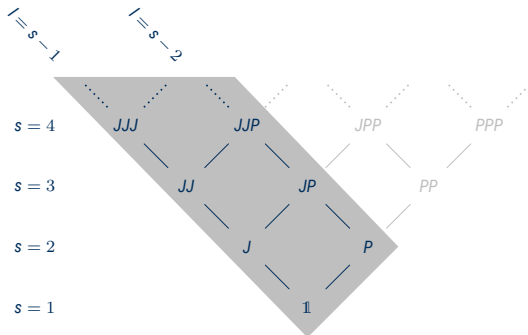
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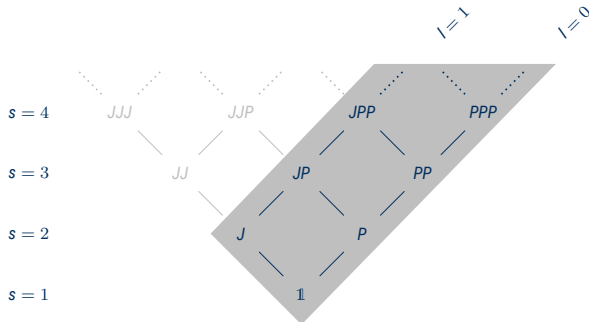
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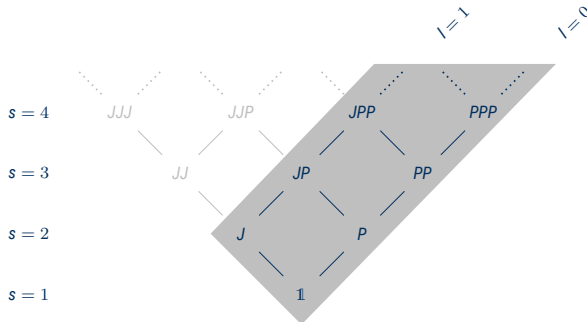
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[Campoleoni/Pekar, [2110.07794](#)]

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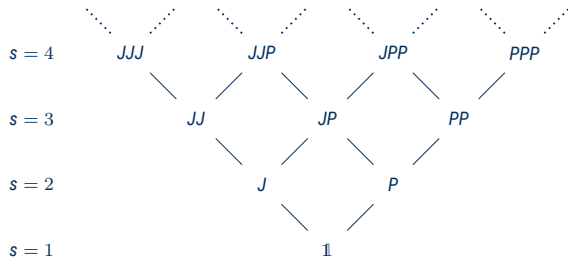
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[Ammon/MP, 2211.12530]

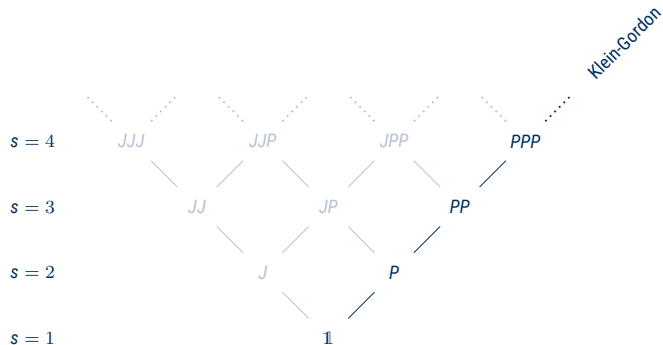
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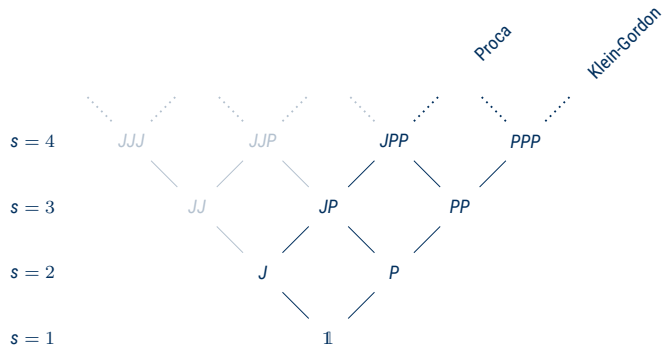
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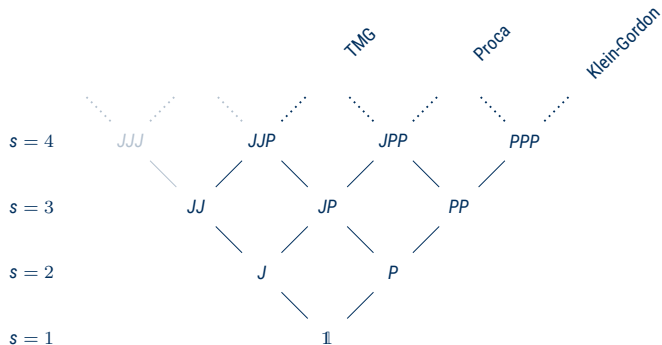
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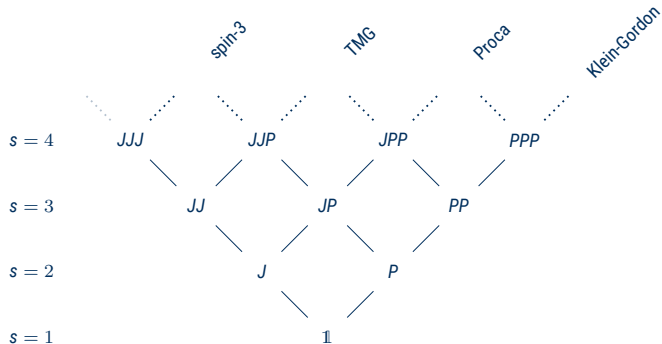
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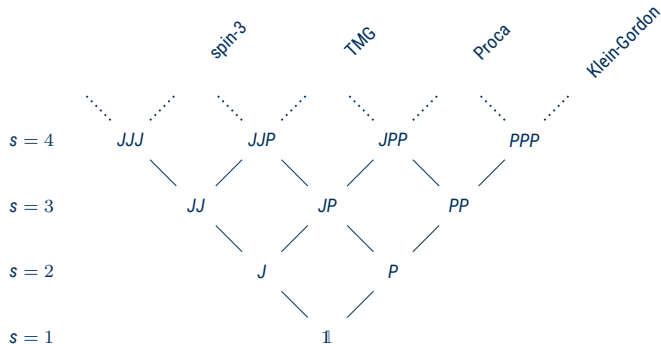
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## Fierz-Pauli Equations

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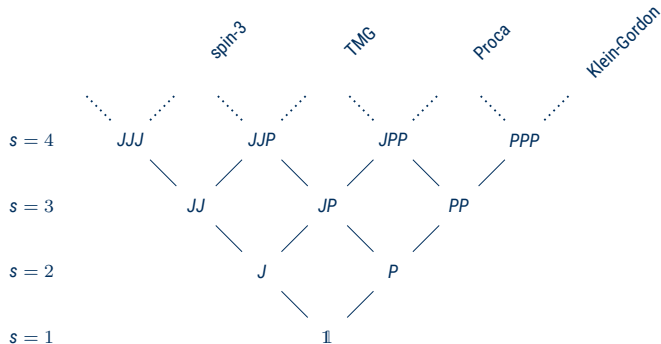
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The master equation provides an unfolding of the complete Fierz-Pauli system!

## Discussion: What's next?

# Further Steps & Outlook

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[Ammon/Gray/Moran/MP/Wölfl, [2012.09173](#)] [MP, 2023]

# Thanks for your attention!

# Boundary Conditions for Flat-Space Higher-Spin

$$\omega = \left( J_1 - \frac{1}{4} \sum_{s=2}^{\infty} Z^{(s)}(\phi) J_{-1} (P_{-1})^{s-2} \right) d\phi,$$
$$e = \left( P_1 - \frac{1}{4} \sum_{s=2}^{\infty} Z^{(s)}(\phi) (P_{-1})^{s-1} \right) du + \frac{1}{2} P_{-1} dr + \left( r P_0 - \frac{1}{2} \sum_{s=2}^{\infty} W^{(s)}(u, \phi) (P_{-1})^{s-1} \right) d\phi$$

$$M(\phi) \equiv Z^{(2)}(\phi),$$

$$N(u, \phi) \equiv W^{(2)}(u, \phi)$$

$$\partial_\phi Z^{(s)}(\phi) = 2 \partial_u W^{(s)}(u, \phi)$$