Aspects of Four Point One Loop Superstring Amplitudes

in Celestial Holography

Based on arXiv:2310.xxxx

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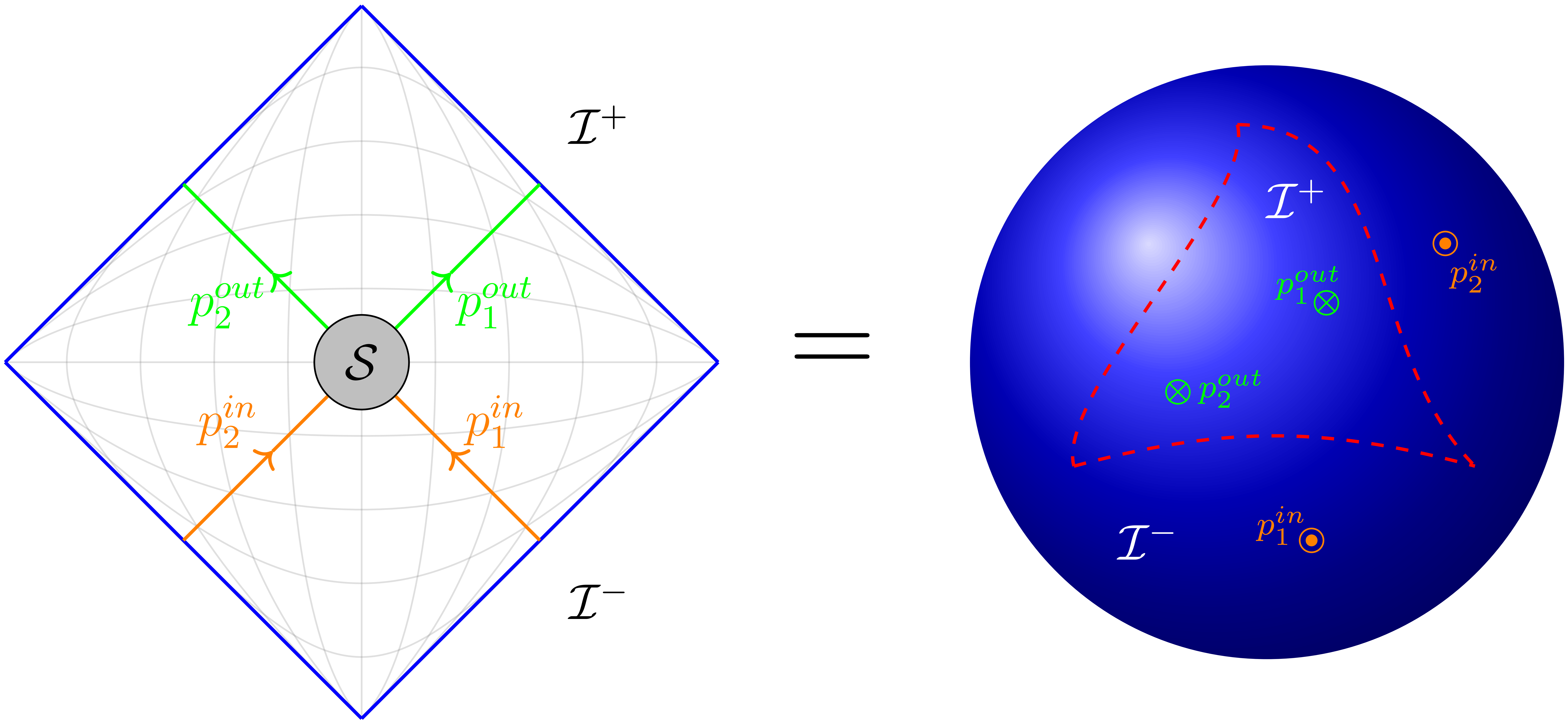
Preliminaries

In one picture



Preliminaries

When in doubt, Mellin transform

Mellin transform with respect to ω_i the energy of the i-th particle

$$\tilde{\mathscr{A}}\left(\{\Delta_i, z_i, \bar{z}_i\}\right) = \prod_{i=1}^{n} \int_0^\infty d\omega_i\, \omega_i^{\Delta_i - 1} \mathscr{A}\left(\{\omega_i, z_i, \bar{z}_i\}\right)$$

Conformal scaling dimensions Δ_i on principal continuous series

$$\Delta_i = 1 + i\lambda_i, \quad \lambda_i \in \mathbb{R}$$

[1705.01027, Pasterski, Shao]

Preliminaries

Some notation

$$p^\mu = \omega q^\mu = \frac{\omega}{2}\left(1 + z\bar{z}, z + \bar{z}, -i\left(z + \bar{z}\right), 1 - z\bar{z}\right)$$

Spinor-Helicity Formalism

$$\langle ij \rangle = \sqrt{\omega_i \omega_j} z_{ij}\,, \quad [ij] = -\sqrt{\omega_i \omega_j} \bar{z}_{ij}\,, \qquad \left(z_{ij} \equiv z_i - z_j, \quad \bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j\right)$$

Mandelstam variables

$$s_{ij} \equiv 2 p_i p_j = \langle ij \rangle [ji] = \omega_i \omega_j z_{ij} \bar{z}_{ij}$$

[1806.05688, Stieberger, Taylor]

Preliminaries

Momentum Conservation

$$\mathscr{A} = A\,\delta^{(4)}\left(p\right)$$

$$\delta^{(4)}\left(p\right) \equiv \delta^{(4)}\left(\sum_{j=1}^{4} \varepsilon_j \omega_j q_j\right) = \frac{4}{\omega_4 (z_{14}\bar{z}_{14})(z_{23}\bar{z}_{23})} \delta(r-\bar{r}) \prod_{i=1}^{3} \delta\left(\omega_i - \chi_i\,\omega_4\right)$$

$$\chi_1 = \frac{z_{24}\bar{z}_{34}}{z_{12}\bar{z}_{13}}, \qquad \chi_2 = -\frac{z_{14}\bar{z}_{34}}{z_{12}\bar{z}_{23}}, \qquad \chi_3 = -\frac{z_{24}\bar{z}_{14}}{z_{23}\bar{z}_{13}}$$

[1806.05688, Stieberger, Taylor]

Preliminaries

Four-Point structures

$$s_{12} = s_{34}, \quad s_{13} = s_{24}, \quad s_{14} = s_{23}, \qquad s_{12} + s_{13} + s_{14} = 0$$

$$A_{YM}^{tree}(1^-,2^-,3^+,4^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

Conformal cross ratio & scattering angle

$$r = \frac{z_{12} z_{34}}{z_{23} z_{41}}, \qquad \frac{s_{23}}{s_{12}} = \frac{1}{r} = \sin^2\left(\frac{\theta}{2}\right) \Rightarrow \Theta(r-1)$$

Kinematical Factor

$$K\left(h_i, \bar{h}_i\right) = \prod_{i<j}^{4} z_{ij}^{h/3 - h_i - h_j} \bar{z}_{ij}^{\bar{h}/3 - \bar{h}_i - \bar{h}_j}$$

[1806.05688, Stieberger, Taylor]

Preliminaries

Gluons as Conformal Operators

$$\mathscr{A}\left(1^-, 2^-, 3^+, 4^+\right)$$

$$\Delta_i = h_i + \bar{h}_i, \qquad J_i = h_i - \bar{h}_i,$$

$$\Delta \equiv \sum_i \Delta_i = 4 - 2\beta, \qquad \beta = -\frac{i}{2}\sum_{i=1}^{4} \lambda_i$$

$$h_1 = \frac{i}{2}\lambda_1, \qquad \bar{h}_1 = 1 + \frac{i}{2}\lambda_1$$

$$h_2 = \frac{i}{2}\lambda_2, \qquad \bar{h}_2 = 1 + \frac{i}{2}\lambda_2$$

$$h_3 = 1 + \frac{i}{2}\lambda_3, \qquad \bar{h}_3 = \frac{i}{2}\lambda_3$$

$$h_4 = 1 + \frac{i}{2}\lambda_4, \qquad \bar{h}_4 = \frac{i}{2}\lambda_4 .$$

[1706.03917, Pasterski, Shao, Strominger]

Tree-Level Open Superstring Amplitude

in Celestial Holography

$$\tilde{\mathscr{A}}^{tree}(1^-,2^-,3^+,4^+) = 4(\alpha')^{\beta} K\left(h_i, \bar{h}_i\right) \delta(r-\bar{r})\theta(r-1)\, r^{\frac{5-\beta}{3}}(r-1)^{\frac{2-\beta}{3}}\, I(r,\beta)$$

$$I(r,\beta) = \pi\delta(i\beta) + \frac{1}{2}\frac{\pi}{\sin\pi\beta}(-r)^{\beta-1}\sum_{k=0}^{\infty}(-r)^{-k}\zeta\left(\beta-k,\{1\}^k\right)$$

$$S_{n,k}(z=1) = \zeta\left(n+1,\{1\}^{k-1}\right) = \sum_{n_1>n_2>\ldots>n_k}\frac{1}{n_1^{n+1}\, n_2\,\cdots\, n_k} \quad \text{MZV of depth k}$$

[1806.05688, Stieberger, Taylor]

One-Loop Open Superstring Amplitude

The star of the show

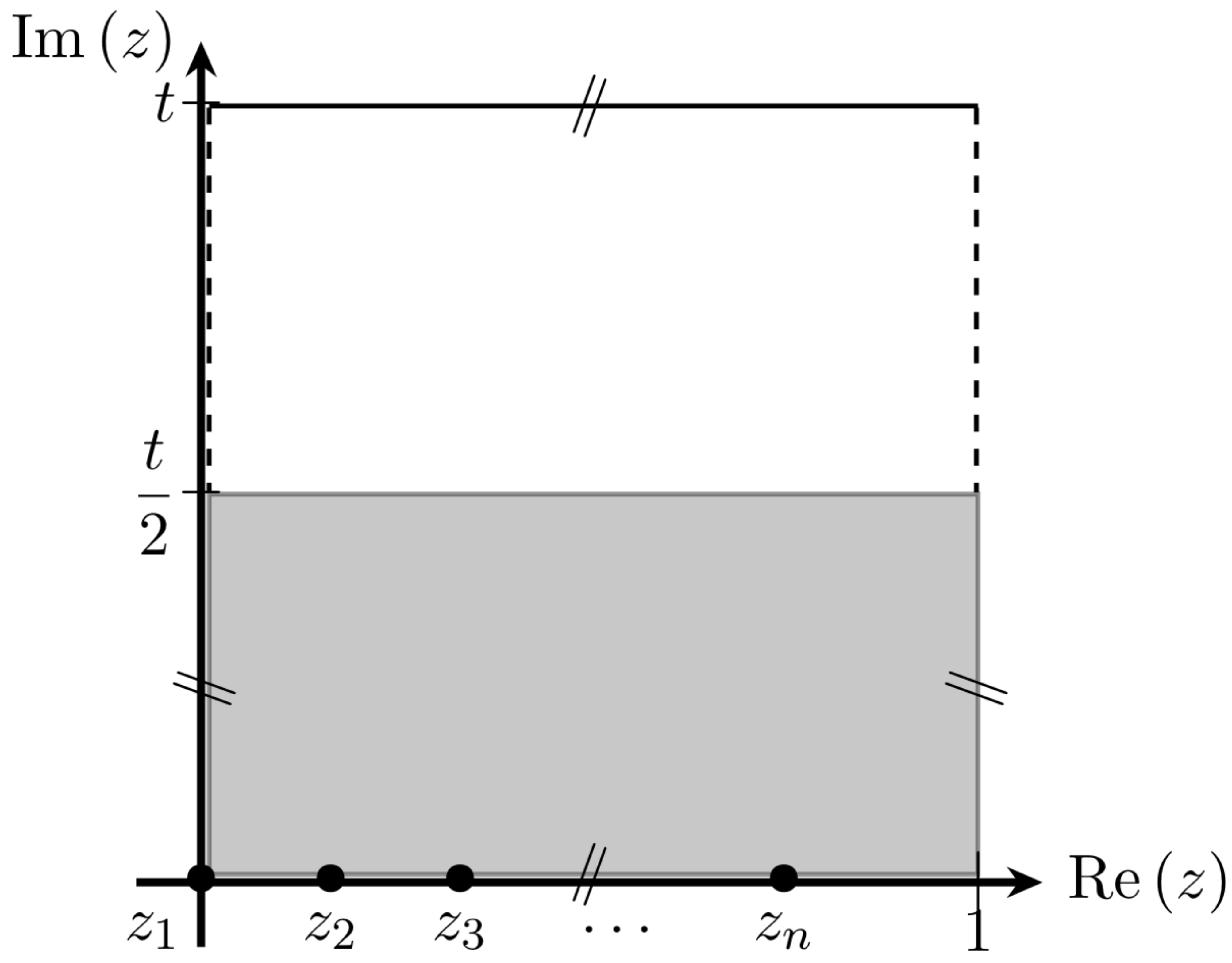
$$\mathscr{A}^{Loop} = s_{12}s_{23}\mathscr{A}^{tree}_{YM}(1,2,3,4)\int_0^1 \frac{\mathrm{d}q}{q}\Big\{\mathrm{Tr}(t^1t^2t^3t^4)\left[N_G\, I_{1234}(q) - 32\, I_{1234}(-q)\right]$$

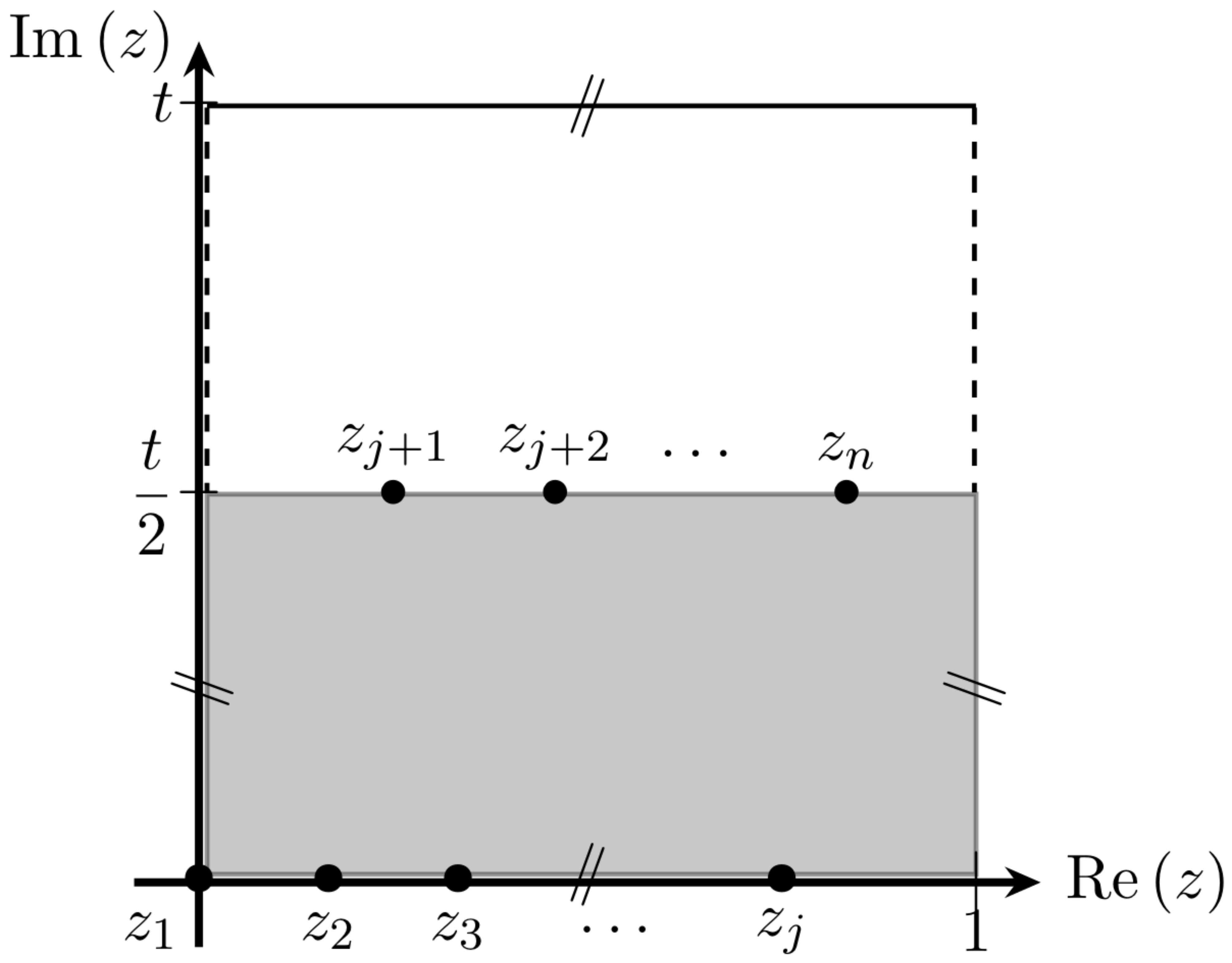
$$+\mathrm{Tr}(t^1t^2)\,\mathrm{Tr}(t^3t^4) I_{12|34}(q) + \mathrm{cyc}(2,3,4)\Big\}\,,$$

$$q = e^{i2\pi\tau} = e^{-2\pi t}$$

[1412.5535, 1704.03449, Broedel, Mafra, Matthes, Schlotterer]

Obligatory Worldsheet Picture





[1412.5535, 1704.03449, Broedel, Mafra, Matthes, Schlotterer]

Integrals over Worldsheet coordinates

(Iterated) Integrals & Genus-one Greens function

$$I_{1234}(q) \equiv \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \, \delta\left(x_1\right) \prod_{j<k}^{4} \exp\left[\alpha' s_{jk} P_{jk}\right]$$

$$I_{12|34}(q) = \int_0^1 dx_4 \int_0^1 dx_3 \int_0^1 dx_2 \int_0^1 dx_1 \, \delta\left(x_1\right) \exp\left[\alpha'\left(s_{12}P_{12} + s_{34}P_{34} + \sum_{\substack{i=1,2 \\ j=3,4}} s_{ij} P(x_{ij} - \frac{\tau}{2})\right)\right]$$

$$P\left(\mathsf{z}_i - \mathsf{z}_j\right) = \frac{1}{2}\left|\log \frac{\theta_1\left(\mathsf{z}_i - \mathsf{z}_j, \tau\right)}{\theta_1'\left(0,\tau\right)}\right|^2 - \frac{\pi}{\mathfrak{I}\left(\tau\right)}\left[\mathfrak{I}\left(\mathsf{z}_i - \mathsf{z}_j\right)\right]^2$$

$$P_{ij} \equiv P\left(x_i - x_j\right)$$

$$P'_{ij} = P\left(x'_{ij}\right) \equiv P\left(x_{ij} - \frac{\tau}{2}\right)$$

[1412.5535, 1704.03449, Broedel, Mafra, Matthes, Schlotterer]

Celestial Integrands

The annulus

$$\tilde{\mathcal{I}}_{An} = \frac{8\pi^3 g_{10}^2}{\alpha'} \prod_{i=1}^{4} \int_0^\infty d\omega_i \, \omega_i^{\Delta_i - 1} \, s_{12} s_{23} \, \mathscr{A}_{YM}^{tree} \left(1^-, 2^-, 3^+, 4^+\right) \prod_{j<k} \exp\left[-\alpha' s_{jk} P_{jk}\right]$$

$$= 16\pi^3 g_{10}^2 \, (\alpha')^{\beta-3} \, \Gamma\left(2-\beta\right) K\left(h_i, \bar{h}_i\right) \delta(r-\bar{r}) \theta\left(r-1\right) r^{\frac{2}{3}(1+\beta)} (r-1)^{\frac{2-\beta}{3}} \left(\mathscr{P}_1 + \frac{\mathscr{P}_2}{r}\right)^{\beta-2}$$

$$\mathscr{P}_1 = P_{13} + P_{24} - P_{12} - P_{34}, \qquad \mathscr{P}_2 = P_{13} + P_{24} - P_{14} - P_{23} \, .$$

Similar expression found in [2307.03551, Donnay, Giribet, González, Puhm, Rojas]

Celestial Integrands

The Möbius strip

$$\mathcal{I}_{Moeb}(q) = -\frac{N_G}{32}\mathcal{I}_{An}(-q)$$

Celestial Integrands

The non-planar part

$$\tilde{\mathscr{I}}_{non-planar}(q) = \tilde{\mathscr{I}}_{An}(q)\Big|_{(\mathscr{P}_1,\mathscr{P}_2)\to(\mathscr{P}'_1,\mathscr{P}'_2)}$$

$$\mathscr{P}'_1 = P'_{13} + P'_{24} - P_{12} - P_{34}, \qquad \mathscr{P}'_2 = P'_{14} + P'_{23} - P_{12} - P_{34}$$

The Problem from here

Remaining Worldsheet Integrals

$$\tilde{\mathscr{A}}^{1-Loop}_{String,\,An} = \int_0^1 \frac{\mathrm{d}q}{q} \underbrace{\int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \, \delta\left(x_1\right) \, \tilde{\mathscr{I}}_{An}}_{Hard}$$

Exchange order

$$\tilde{\mathscr{A}}^{1-Loop}_{String\,An} = \int_0^1 \frac{\mathrm{d}q}{q} \prod_{i=1}^{n} \int_0^{\infty} d\omega_i \, \omega_i^{\Delta_i - 1} \underbrace{\int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \, \delta\left(x_1\right) \mathscr{I}_{An}}_{Known}$$

Iterated Integral Expansion

Warning: Explicit

$$
\begin{aligned}
I_{1234}\left(q\right) = \omega(0,0,0) \; &- (\alpha')\; 2\,\omega(0,1,0,0)\,(s_{12}+s_{23}) \\
&+ (\alpha')^2 \left[2\,\omega(0,1,1,0,0)\left(s_{12}^2+s_{23}^2\right) - 2\,\omega(0,1,0,1,0)\,s_{12}s_{23} \right] \\
&+ (\alpha')^3 \left[\beta_5\,(s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3) + \beta_{2,3}\,s_{12}s_{23}(s_{12}+s_{23})\right] + \mathcal{O}(\alpha'^4)
\end{aligned}
$$

$$
\beta_5 = \frac{4}{3}\left[\omega(0,0,1,0,0,2)+\omega(0,1,1,0,1,0)-\omega(2,0,1,0,0,0)-\zeta_2\,\omega(0,1,0,0)\right]
$$

$$
\begin{aligned}
\beta_{2,3} = &-\frac{1}{3}\omega(0,0,1,0,2,0)+\frac{3}{2}\omega(0,1,0,0,0,2)+\frac{1}{2}\omega(0,1,1,1,0,0) \\
&+2\omega(2,0,1,0,0,0)+\frac{4}{3}\omega(0,0,1,0,0,2)+\frac{10}{3}\zeta_2\omega(0,1,0,0)\;.
\end{aligned}
$$

[1412.5535, 1704.03449, Broedel, Mafra, Matthes, Schlotterer]

Some explicit eMZV's

Just numbers after all

$$\omega(0,0,0) = \frac{1}{6}$$

$$\omega(0,1,0,0) = \frac{\zeta_3}{8\zeta_2} + \frac{3}{2\pi^2}\sum_{m,n=1}^{\infty}\frac{1}{m^3}q^{mn}$$

$$\omega(0,1,1,0,0) = \frac{\zeta_2}{15} - \frac{1}{2\pi^2}\sum_{m,n=1}^{\infty}\frac{n}{m^4}q^{mn} + \frac{1}{3}\sum_{m,n=1}^{\infty}\frac{n}{m^2}q^{mn}$$

$$\omega(0,1,0,1,0) = -\frac{\zeta_2}{60} + \frac{2}{\pi^2}\sum_{m,n=1}^{\infty}\frac{n}{m^4}q^{mn} - \frac{1}{3}\sum_{m,n=1}^{\infty}\frac{n}{m^2}q^{mn}$$

[1412.5535, 1704.03449, Broedel, Mafra, Matthes, Schlotterer]

Celestial Iterated Integral Expression

And resulting Amplitude structure

$$I^{loop}_{planar}(n,m) = \frac{8\pi^3 g_{10}^2}{\alpha'} \prod_{i=1}^{4} \int_0^\infty d\omega_i^{\Delta_i - 1} s_{12} s_{23} \mathscr{A}^{tree}_{YM}(1^-,2^-,3^+,4^+)(\alpha' s_{12})^n (\alpha' s_{23})^m$$

$$= 32\pi^3 g_{10}^2 (\alpha')^{\beta-3} \, (-1)^{\frac{4}{3}(1+\beta)+i(\lambda_2+\lambda_3)} K\left(h_i, \bar{h}_i\right) \delta(r-\bar{r})\theta\,(r-1)\, r^{\frac{2}{3}(1+\beta)-m}(r-1)^{\frac{2-\beta}{3}} \delta(i(2+(n+m)-\beta))$$

$$\tilde{\mathscr{A}}^{Loop}_{String,\,An}(1^-,2^-,3^+,4^+) \propto \int_0^1 \frac{dq}{q} \Bigg(\omega(0,0,0)\delta(i\Delta) + 2\,\omega(0,1,0,0)\frac{1+r}{r}\delta(i(\Delta+2))$$
$$+2\left[\,\omega(0,1,1,0,0)\,\left(1+r^2\right) - \,\omega(0,1,0,1,0)\,r\,\right]\frac{1}{r^2}\delta(i(\Delta+4))$$
$$+\left[\beta_5\,\left(1+r^2\right)\,+\,\beta_{2,3}\,r\right]\frac{(1+r)}{r^3}\delta(i(\Delta+6)) + \cdots \Bigg)$$

Non-planar contribution

$$I_{12|34}(q) = q^{s_{12}/4}\Bigg\{ 1 + (\alpha')^2\, s_{12}^2\Big(\frac{7\zeta_2}{6} + 2\,\omega(0,0,2)\Big) + (\alpha')^3\Big[2s_{23}\,\big(s_{12}+s_{23}\big)\Big(\frac{\zeta_2}{3} + \omega(0,0,2)\Big)$$
$$-4\,\zeta_2\,\omega(0,1,0,0)\,s_{12}^3 - s_{12}s_{23}\,\big(s_{12}+s_{23}\big)\Big(\frac{5}{3}\,\omega(0,3,0,0) + 4\,\zeta_2\,\omega(0,1,0,0) - \frac{1}{2}\zeta_3\Big)\Big] + \mathcal{O}(\alpha'^4)\Bigg\}\ ,$$

[1704.03449, Broedel, Mafra, Matthes, Schlotterer]

$$I_{NP}^{loop}(n,m) = \frac{8\pi^3 g_{10}^2}{\alpha'}\prod_{i=1}^{4}\int_0^\infty d\omega_i\,\omega_i^{\Delta_i-1}\,s_{12}s_{23}\,\mathcal{A}_{YM}^{tree}\left(1^-,2^-,3^+,4^+\right)\,(\alpha' s_{12})^n\,(\alpha' s_{23})^m\,\exp\left[-\frac{\alpha'}{2}\pi t\,s_{12}\right]$$
$$= 16\pi^3 g_{10}^2(\alpha')^{\beta-3}(-1)^{\frac{4}{3}(1+\beta)+i\left(\lambda_2+\lambda_3\right)}\,K\left(h_i,\bar{h}_i\right)\delta(r-\bar{r})\theta(r-1)r^{\frac{2}{3}(1+\beta)-m}\,(r-1)^{\frac{2-\beta}{3}}$$
$$\times\,(2\pi t)^{\beta-2-m-n}\,\Gamma\left(2+m+n-\beta\right)$$

Non-analytic part

Logarithmic terms

$$\mathscr{A}^{1-loop}_{String}\Big|_{disc} \propto f\left(s_{12}, s_{23}\right) \log\left(\frac{s_{12}}{\mu^2}\right) + f\left(s_{23}, s_{12}\right) \log\left(\frac{s_{23}}{\mu^2}\right)$$

[2107.08009, Edison, Guillen, Johansson, Schlotterer, Teng]

$$\int_0^\infty d\omega\, \omega^{\Delta-1}\left[\omega^n \log\left(\gamma\omega^m\right)\right] = 2\pi m \gamma^{-\frac{1}{m}(\Delta+n)} \partial_\Delta\, \delta\left(i\left(\Delta+n\right)\right)$$

Thank You!

Dimensional regularization of eMZVs

Using small ϵ

For instance

$$\sum_{n,m\geq 1}\int_0^\infty dt\, t^\epsilon \frac{1}{m^a} q^{mn} = \frac{1}{2\pi}\left\{\frac{1}{\epsilon}\zeta_{1+a} - \log(2\pi)\zeta_{1+a} + \zeta'_{1+a}\right\} + \mathcal{O}\left(\epsilon^1\right)$$