# Pure spinor superfield techniques in (twisted) supergravity

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This talk is based on joint work with Ingmar Saberi (arXiv:2304.12371).

It builds on previous work with Richard Eager, Chris Elliott, Simone Noja, Johannes Walcher and Brian R. Williams.

# Introduction

One interesting thing about supersymmetric field theories is the presence of protected subsectors which are sensitive to topological and holomorphic structures on spacetime.

These subsectors are extracted by *twisting*: Let $Q \in \mathfrak{g}_{\mathrm{odd}}$ with $[Q, Q] = 0$ and take invariants with respect to the odd abelian algebra spanned by $Q$.

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The twisted theories have many desirable properties:

- They are topological-holomorphic field theories (and thus much simpler than the full theory).
- They can typically be formulated in terms of geometric moduli problems on spacetime.
- Good behavior under quantization, nice results on symmetry enhancements...

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1. How can we compute twists efficiently?
2. What can we learn from the twists about the full theory?

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*In superspace, the supersymmetries act geometric. Twisting just means taking invariants in some odd direction.*

# Plan

I. How to produce universal superspace descriptions which are compatible with twisting?

$\longrightarrow$ Pure spinor superfields

II. What can we learn about the full theory and its twists?

$\longrightarrow$ Today: Eleven-dimensional supergravity

# I. Pure spinor superfields and twisting

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Let $\mathfrak{p} = \mathfrak{p}_0 \ltimes \mathfrak{t}$ be a super Poincaré algebra. We call $\mathfrak{t}$ the supertranslation algebra and denote the associated super Lie group by $T$. ($T \sim$ superspace)

There are two actions on the free superfield $C^\infty(T)$

$$\mathscr{L}, \mathscr{R} : \mathfrak{p} \longrightarrow \mathrm{Vect}(T)$$

given by the usual vector fields

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + \gamma^\mu_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\mu} \qquad D_\alpha = \frac{\partial}{\partial \theta^\alpha} - \gamma^\mu_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\mu}.$$

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Think of the superspace $T$ as a supermanifold equipped with a distribution.

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The pure spinor superfield formalism constructs multiplets from $\mathcal{O}_Y$-modules.

$$\begin{tikzcd}
\mathsf{Mod}^{\mathfrak{p}_0}_{\mathcal{O}_Y} \arrow[r, "A^\bullet"] \arrow[d] & \mathsf{Mult}_{\mathfrak{p}} \arrow[dl, shift left] \\
\mathsf{Mod}^{\mathfrak{p}_0}_{C^\bullet(\mathfrak{t})} \arrow[ur, shift left] &
\end{tikzcd}$$

*The pure spinor construction is compatible with twisting.*

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Choose $Q \in Y$. We can twist the multiplet by deforming the differential.

On the other hand, we can twist the input data for the formalism.

$$\mathfrak{p} \longrightarrow \mathfrak{p}_Q = H^\bullet(\mathfrak{p}, [Q, -])$$

$\mathfrak{p}_Q$ is the residual symmetry algebra and has a new nilpotence variety $Y_Q$ controlling further twists.

Both procedures are compatible [Saberi–Williams]:

$$A^\bullet(\mathcal{O}_Y)^Q \cong A^\bullet(\mathcal{O}_{Y_Q})$$

# II. (Twisted) eleven-dimensional supergravity

*Eleven-dimensional supergravity has two distinct twists.*

$$\begin{tikzcd}
\boxed{\text{Full (untwisted) theory}} \arrow[dd] \arrow[dr] & \\
& \boxed{\begin{array}{c}\text{Minimal twist}\\ U(5) \curvearrowright \mathbb{C}^5 \times \mathbb{R}\end{array}} \arrow[dl] \\
\boxed{\begin{array}{c}\text{Maximal twist}\\ G_2 \times SU(2) \curvearrowright \mathbb{R}^7 \times \mathbb{C}^2\end{array}} &
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$$

The maximal twist is Poisson–Chern–Simons theory.

$$\left(\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^{\bullet}(\mathbb{R}^7)\ ,\ \bar{\partial}_{\mathbb{C}^2} + \mathrm{d}_{\mathbb{R}^7}\ ,\ \{-,-\}_{PB}\right)$$

# Poisson–Chern–Simons theory

Let $X$ be a Calabi–Yau 2-fold. Think of the complex structure as an involutive distribution. Recall that the de Rham differential splits as $\mathrm{d}_{\mathrm{dR}} = \bar{\partial} + \partial$.

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Poisson–Chern-Simons theory on $X \times M$ is modeled by

$$(\Omega^{0,\bullet}(X)\,,\ \bar{\partial}\,,\ \{-,-\}_{PB}) \otimes (\Omega^{\bullet}(M), \mathrm{d}_{\mathrm{dR}}),$$

where the Poisson bracket is constructed by

$$\{\alpha, \beta\}_{PB} = \pi(\partial\alpha \wedge \partial\beta).$$

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*We can think about Poisson–Chern–Simons theory as modeling certain deformations of the complex structure, i.e. of a distribution.*

# Back to superspace

In eleven dimensions, we have three different superspaces:

$$\mathfrak{p} \rightsquigarrow T, \qquad \mathfrak{p}_{Q_{\min}} \rightsquigarrow T_{Q_{\min}}, \qquad \mathfrak{p}_{Q_{\max}} \rightsquigarrow T_{Q_{\max}}$$

All three are equipped with distributions spanned by odd left invariant vector fields. Except for the maximal twist, these are non-involutive.

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Consider differential forms $(\Omega^\bullet(T)\,,\, \mathrm{d}_{\mathrm{dR}}) \sim \mathbb{C}[x, \theta, \mathrm{d}\theta, \mathrm{d}x]$.

In a left-invariant basis $\lambda = \mathrm{d}\theta$ and $v = \mathrm{d}x + \lambda\theta$:

$$\mathrm{d}_{\mathrm{dR}} = \underbrace{\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta \frac{\partial}{\partial v^\mu}}_{\gamma} + \underbrace{\lambda^\alpha \left( \frac{\partial}{\partial \theta^\alpha} - \theta^\beta \gamma^\mu_{\alpha\beta} \frac{\partial}{\partial x^\mu} \right)}_{\bar{\partial}} + \underbrace{v^\mu \frac{\partial}{\partial x^\mu}}_{\partial}$$

The differential $\gamma$ reflects the non-involutivness of the distribution.

Taking cohomology with respect to $\gamma$ gives the pure spinor multiplet $A^\bullet(\mathcal{O}_Y) = H^0(\Omega^\bullet(T), \gamma)$.

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Applying this procedure in eleven-dimensions constructs:

- Poisson–Chern–Simons theory on $\mathbb{R}^7 \times \mathbb{C}^2$ from $T_{Q_{\max}}$.
- A quartic action functional for the minimal twist on $\mathbb{C}^5 \times \mathbb{R}$ from $T_{Q_{\min}}$.
- Cederwall's quartic action of eleven-dimensional supergravity in the pure spinor formalism from $T$.

Thank you!