Hecke Eigenvalue Property from SOV tranform

Based on a work to appear soon w/ J. Teschner

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New Perspectives in Conformal Field Theory and Gravity

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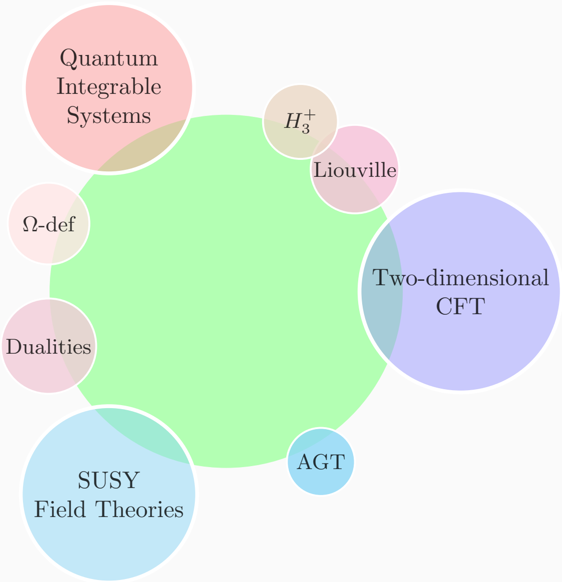
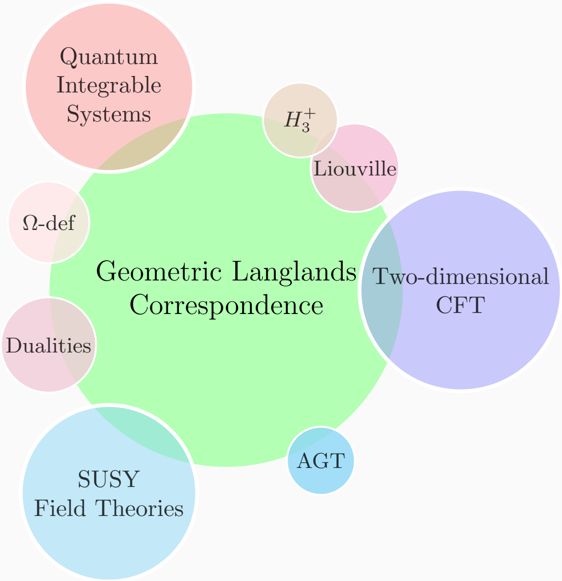
Introduction

Hidden connections and dualities



Hidden connections and dualities



The Geometric Langlands Correspondence

- (Punctured) Riemann Surface $\mathcal{C}$
- ${}^{\mathrm{L}}\mathfrak{g}$: Langlands-dual of complex Lie Algebra $\mathfrak{g}$

${}^{\mathrm{L}}\mathfrak{g}$ – local systems on $\mathcal{C}$	$\Longleftrightarrow$	$\mathcal{D}$ – modules on $\mathrm{Bun}_{\mathrm{G}}(\mathcal{C})$
$(\mathcal{E}, \nabla)$: holomorphic ${}^{\mathrm{L}}\mathrm{G}$-bundle $\mathcal{E}$ with connection ∇		Set of Differential Equations on $\mathrm{Bun}_{\mathrm{G}}(\mathcal{C})$

The Geometric Langlands Correspondence: Beilinson-Drinfeld

Beilinson-Drinfeld: connection in **oper** form

for $\mathfrak{g} = \mathfrak{sl}_2$: $\nabla \sim dz \left(\partial_z + \left(\begin{smallmatrix} 0 & t(z) \\ 1 & 0 \end{smallmatrix} \right) \right)$

$\boxed{{}^{\mathrm{L}}\mathfrak{g} - \text{local systems on } \mathcal{C}}$ $\Longleftrightarrow$ $\boxed{\mathcal{D} - \text{modules on } \mathrm{Bun}_{\mathrm{G}}(\mathcal{C})}$

determined by: $r = 1, \cdots, 3g - 3 + n$

$$\left(\epsilon^2 \partial_z^2 + t(z)\right) \psi(z) = 0 \qquad\qquad \mathcal{D}_r \mathbf{\Psi} = E_r \mathbf{\Psi}$$

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Key ingredient: existence of Hecke operator

$$\mathfrak{H}(z)\, \boldsymbol{\Psi} = \psi(z)\, \boldsymbol{\Psi}$$

In today's talk

$^{L}\mathfrak{g}$ – local systems on $\mathcal{C}$ $\Longleftrightarrow$ $\mathcal{D}$ – modules on $\mathrm{Bun}_{\mathrm{G}}(\mathcal{C})$

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Hecke operator: $\mathfrak{H}(z)\, \boldsymbol{\Psi} = \psi(z)\, \boldsymbol{\Psi}$

Today: A CFT way to Geometric Langlands

SOV makes Geometric Langlands computable!

$\rightarrow$ Explicitly construct diagonal action of Hecke operator

The CFT way to Quantum Geometrical Langlands

A tale of two integrable models: Sklyanin SOV

Deep relation to Integrable Models: Quantum $\mathrm{SL}(2,\mathbb{C})$ Gaudin Model:

$$\mathcal{H}_r := \sum_{s\neq r} \frac{\mathcal{J}_r^a \, \mathcal{J}_{a,r}}{z_r - z_s}\,, \qquad \mathcal{J}_r^+ = \partial_{x_r}\,, \;\; \mathcal{J}_r^0 = x_r\partial_{x_r} - j_r\,, \;\; \mathcal{J}_r^- = x_r^2\partial_{x_r} + 2j_r\, x_r$$

Sklyanin Separation of Variables:

$$\mathcal{H}_r\Psi_r = E_r\Psi_r \Longrightarrow \boldsymbol{\Psi} \equiv \prod_{k=1}^{n-2} \psi_k(y_k)\,, \qquad \left(\partial_{y_k}^2 + t(y_k)\right)\psi_k(y_k) = 0$$

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Realisation of Geometric Langlands!

$$\boxed{{}^{L}\mathfrak{g} - \text{local systems on } \mathcal{C}} \quad \Longleftrightarrow \quad \boxed{\mathcal{D} - \text{modules on } \mathrm{Bun}_{\mathrm{G}}(\mathcal{C})}$$

$$\left(\epsilon^2\partial_z^2 + t(z)\right)\ \boldsymbol{\psi}(z) = 0 \qquad\qquad \mathcal{H}_r\boldsymbol{\Psi} = E_r\boldsymbol{\Psi}$$

$$t(y_k) = -\sum_{r=1}^{n}\left(\frac{j_r(j_r+1)}{(y_k-z_r)^2} - \frac{E_r}{y_k-z_r}\right)$$

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Realisation of Geometric Langlands!

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$\left(\epsilon^2\partial_z^2 + t(z)\right)\,\boldsymbol{\psi}(z) = 0$		$\mathcal{H}_r\boldsymbol{\Psi} = E_r\boldsymbol{\Psi}$

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Hecke?

A tale of two CFT's: Quantum Geometric Langlands

Liouville : $\mathfrak{Vir}$ = H_3^+ WZNW : $\widehat{\mathfrak{sl}}_2$

$$\mathcal{Z}_{\mathrm{WZ}}(x,z) := \left\langle \Phi^{j_1}(x_1|z_1)\cdots\Phi^{j_n}(x_n|z_n)\right\rangle_{H_3^+}, \qquad (k+2)\partial_{z_r}\mathcal{Z}_{\mathrm{WZ}}(x,z) = \mathcal{H}_r\mathcal{Z}_{\mathrm{WZ}}(x,z)$$

$$\mathcal{Z}_{\mathrm{L}}(y,z) := \left\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)V_{\frac{-1}{2b}}(y_1)\cdots V_{\frac{-1}{2b}}(y_{n-2})\right\rangle_{\mathrm{L}}, \qquad \mathcal{D}^{BPZ}_{y_k}\mathcal{Z}_{\mathrm{L}}(y,z) = 0$$

Sklyanin SOV: classical limit of H_3^+ - Liouville correspondence

$$\mathcal{Z}_{\mathrm{WZ}}(x,z) = \int d\boldsymbol{y}\,\mathcal{K}^{SOV}(x,z,y)\,\mathcal{Z}_{\mathrm{L}}(y,z)$$

$k+2=b^{-2}\to 0$: KZ $\to (\mathcal{H}_r - E_r)\boldsymbol{\Psi} = 0$, BPZ $\to \left(\partial^2_{y_k} + t(y_k)\right)\chi_k(y_k) = 0$

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Quantum deformation of Geometric Langlands!

Emerging picture:

Liouville CFT $\mathcal{D}^{BPZ}_{y_k}\mathcal{Z}_{\mathrm{L}}(y,z)=0$	$\overset{\mathfrak{q}\text{-}\mathfrak{GL}}{\underset{H_3^+\text{-L}}{\Longleftrightarrow}}$	$\widehat{\mathfrak{sl}}_2$-WZNW $(k+2)\partial_{z_r}\mathcal{Z}_{\mathrm{WZ}}=\mathcal{H}_r\mathcal{Z}_{\mathrm{WZ}}$
$\Downarrow\; b^{-2}\to 0$		$\Downarrow\; k\to -2$
Whittaker Model $\left(\partial^2_{y_k}+t(y_k)\right)\chi_k(y_k)$	$\overset{\mathfrak{GL}}{\underset{\text{SOV}}{\Longleftrightarrow}}$	Gaudin Model $\mathcal{H}_r\boldsymbol{\Psi}=E_r\boldsymbol{\Psi}$

SOV makes $\mathfrak{GL}$ computable!

Emerging picture:

Liouville CFT
$\mathcal{D}_{y_k}^{BPZ}\mathcal{Z}_{\mathrm{L}}(y,z)=0$

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Whittaker Model
$\left(\partial_{y_k}^2+t(y_k)\right)\chi_k(y_k)$

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Gaudin Model
$\mathcal{H}_r\boldsymbol{\Psi}=E_r\boldsymbol{\Psi}$

SOV makes $\mathfrak{GL}$ computable!

Quantum Hecke operators?

Quantum Hecke Operator

Quantum Hecke Operator: our proposal

Recall: $\mathfrak{H}(y)\boldsymbol{\Psi}(x,z) = \psi(y)\boldsymbol{\Psi}(x,z)$

We propose the following quantum Hecke operator for $k \neq -2$:

$$\boxed{\mathfrak{H}(z_0)\, \mathcal{Z}^{(n)}_{\mathrm{WZ}}(\mathbf{x},\mathbf{z}|\mathbf{j}) = \int \frac{d^2x_0}{|x_0|^{2(k+2)}} \mathcal{Z}^{(n+1)}_{\mathrm{WZ}}\Big((\mathbf{x},x_0),(\mathbf{z},z_0)|(\mathbf{j},k/2)\Big)}$$

Field with spin $j = k/2$ is a $\widehat{\mathfrak{sl}}_2$ degenerate rep.

null-vector decoupling: $$\sum_{r=1}^{n} \frac{\mathcal{J}_r^-}{z_0 - z_r}\, \mathcal{Z}^{(n+1)}_{\mathrm{WZ}}\Big((\mathbf{x},x_0),(\mathbf{z},z_0)|(\mathbf{j},k/2)\Big) = 0$$

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Solved by , $\xi_r = -\dfrac{z_0-z_r}{x_0-x_r}$, $\mathbf{j}' = (j_1,\cdots,j_{n-1},\frac{k}{2}-j_n)$:

$$\mathcal{Z}^{(n+1)}_{\mathrm{WZ}}\Big((\mathbf{x},x_0),(\mathbf{z},z_0)|(\mathbf{j},k/2)\Big) = \prod_{r=1}^{n-1}\frac{|z_0-x_r|^{2j_r}}{|\xi_r|^{2j_r}}\,\mathcal{Z}^{(n)}_{\mathrm{WZ}}(\boldsymbol{\xi},\mathbf{z}|\mathbf{j}')$$

Quantum Hecke Operator: Hecke eigenvalue property

Reproduces Hecke Eigenvalue at critical level:

$$
\begin{aligned}
\mathfrak{H}(z_0)\, \mathcal{Z}^{(n)}_{\mathrm{WZ}}(\boldsymbol{x}, \boldsymbol{z}|\boldsymbol{j}) &= \mu_k \mathcal{Z}^{(n+1)}(\boldsymbol{x}, x_0), (\boldsymbol{z}, z_0)|(\boldsymbol{j}, -(k+2)/(2))) \\
&= \int \mathcal{K}^{(n+1)}_{\mathrm{SOV}}(\cdots)\mathcal{Z}^{(n+1)}_L(\cdots) \\
&\downarrow k \to -2 \\
\mathfrak{H}(z_0)\, \Psi^{(n)}(\boldsymbol{x}, \boldsymbol{z}|\boldsymbol{j}) &= \chi(z_0) \int \mathcal{K}^{(n)}_{\mathrm{SOV}}(\cdots) \prod_{j=1}^{n-2} \chi_j(y_j) \\
&= \chi(z_0)\, \Psi^{(n)}(\boldsymbol{x}, \boldsymbol{z}|\boldsymbol{j})
\end{aligned}
$$

Quantum Hecke Operator: Remarks

$$\boxed{\mathfrak{H}(z_0)\,\mathcal{Z}^{(n)}_{\mathrm{WZ}}(\mathbf{x},\mathbf{z}|\mathbf{j}) = \int \frac{d^2x_0}{|x_0|^{2(k+2)}}\mathcal{Z}^{(n+1)}_{\mathrm{WZ}}\Big((\mathbf{x},x_0),(\mathbf{z},z_0)|(\mathbf{j},k/2)\Big)}$$

Remarks:

- Eigenvalue property: $\mathfrak{H}(z_0)\,\Psi^{(n)}(\mathbf{z},\mathbf{x}) = \chi(z_0)\,\Psi^{(n)}(\mathbf{z},\mathbf{x})$,
- Commuting ops: $\mathfrak{H}$ **Non-local, integral ops.** $\leftrightarrow$ $\mathcal{H}_r$ **local, diff.** ops.
- Hecke $\sim$ Non-local Q-operator $(\partial_z^2 + t(z))\psi(z)$,
- Diagonalised directly through SOV,

Quantum Hecke Operator: Remarks

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Remarks:

- Eigenvalue property: $\mathfrak{H}(z_0)\,\Psi^{(n)}(\mathbf{z},\mathbf{x}) = \chi(z_0)\,\Psi^{(n)}(\mathbf{z},\mathbf{x})$,
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- Hecke $\sim$ Non-local Q-operator $(\partial_z^2 + t(z))\psi(z)$,
- Diagonalised directly through SOV,
- SOV makes $\mathfrak{GL}$ computable!
- Concrete ralisation in SUSY Gauge Theory (NS program)

Conclusions & Outlook

A complete story

$$\mathfrak{H}(z_0)\, \mathcal{Z}^{(n)}_{\mathrm{WZ}}(\mathbf{x}, \mathbf{z}|\mathbf{j}) = \int \frac{d^2x_0}{|x_0|^{2(k+2)}}\, \mathcal{Z}^{(n+1)}_{\mathrm{WZ}}\Big((\mathbf{x}, x_0), (\mathbf{z}, z_0)|(\mathbf{j}, k/2)\Big)$$

Liouville CFT: $\mathcal{D}^{BPZ}_{y_k} \mathcal{Z}_{\mathrm{L}}(y, z) = 0$

$\mathfrak{q}$-$\mathfrak{GL}$ $\Longleftrightarrow$ H_3^+-L

$\widehat{\mathfrak{sl}}_2$-WZNW: $(k+2)\partial_{z_r} \mathcal{Z}_{\mathrm{WZ}} = \mathcal{H}_r \mathcal{Z}_{\mathrm{WZ}}$

$\Downarrow\ b^{-2} \to 0$ $\qquad\qquad$ $\Downarrow\ k \to -2$

Whittaker Model: $\left(\partial^2_{y_k} + t(y_k)\right) \chi_k(y_k)$

$\mathfrak{GL}$ $\Longleftrightarrow$ SOV

Gaudin Model: $\mathcal{H}_r \mathbf{\Psi} = E_r \mathbf{\Psi}$

$$\mathfrak{H}(y)\mathbf{\Psi}(x, z) = \psi(y)\mathbf{\Psi}(x, z)$$

Outlook

What's more?

- Higher genus
- Higher rank (first SOV for $\mathfrak{sl}_3$)

Thank you for your attention!