# Quantum Chaos and Complexity in Triangular Billiard Systems

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**DESY Theory Workshop : 28/09/2023**

ct.qmat Complexity and Topology in Quantum Matter

Julius-Maximilians-UNIVERSITÄT WÜRZBURG

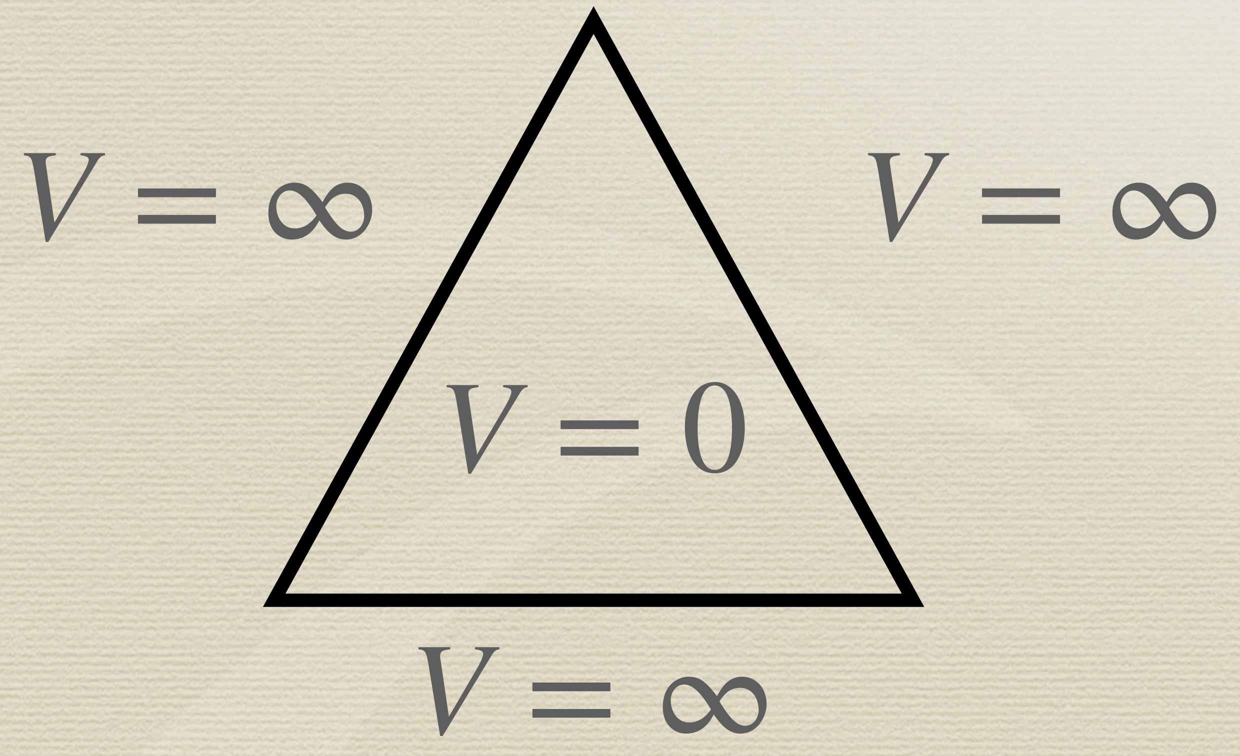
DAAD

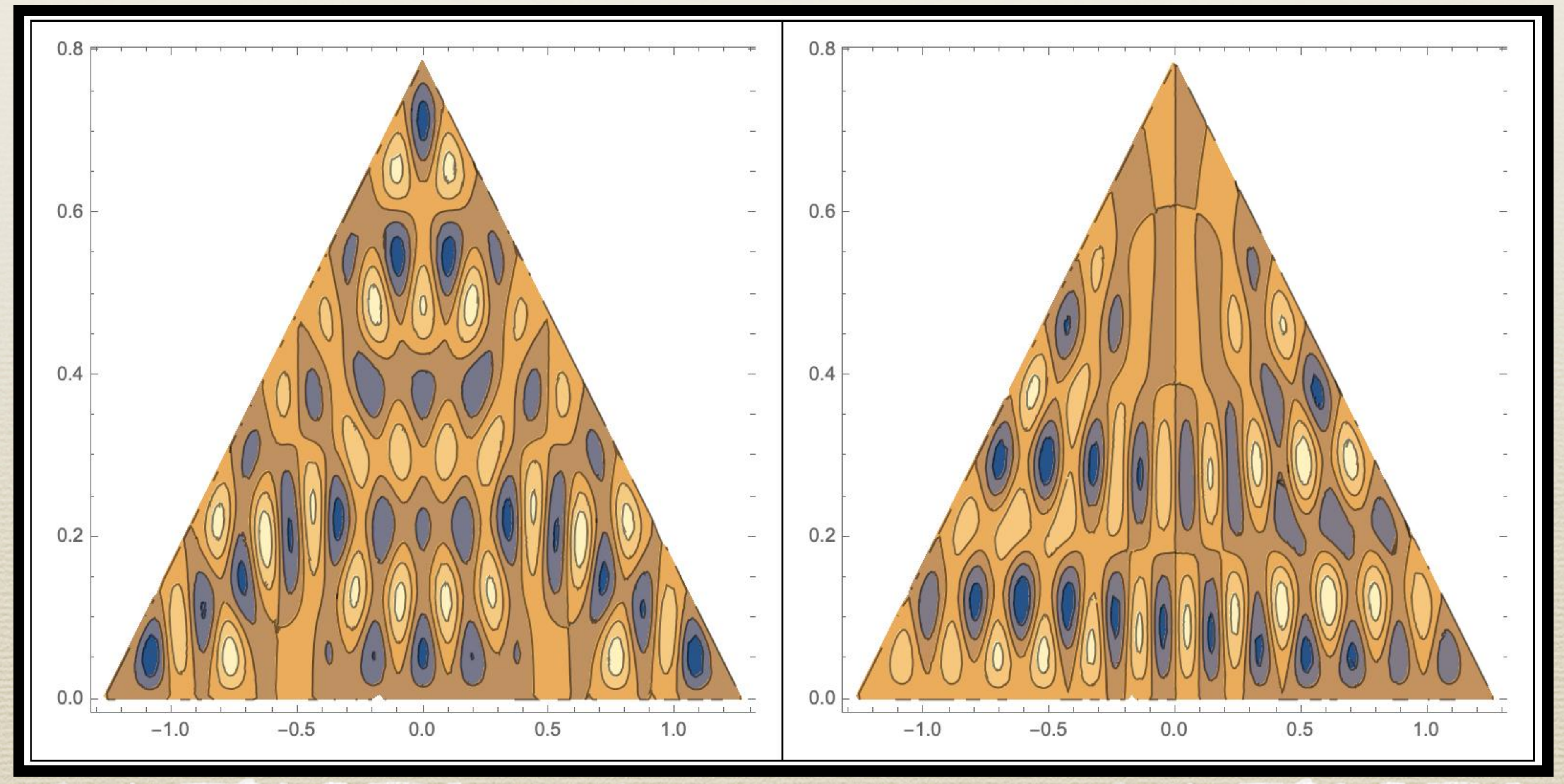
# Motivation

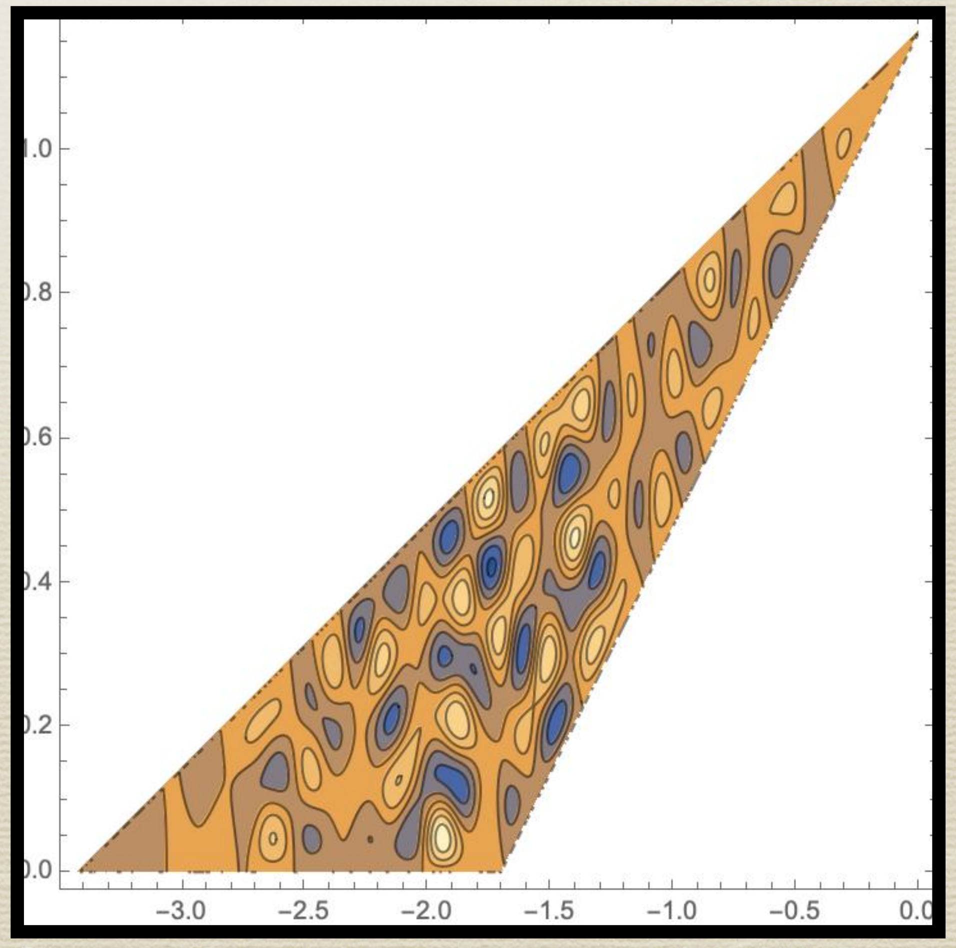
* Role of topology in quantum chaos
* Effects of symmetry in quantum chaos
* Hierarchy among the chaotic billiards
* Comments for generic billiard systems

# Set up

* Quantum triangular billiards with both rational and irrational angles
* Triangles with angles $\{\frac{p_1}{q_1}\pi, \frac{p_2}{q_2}\pi, \frac{p_3}{q_3}\pi\} \equiv \{\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\}$ and $\frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3} = 1.$
* Solve $\nabla^2\psi_n = E_n\psi_n$ for an infinite triangular potential well subject to the boundary condition, $\psi_n(x_\partial) = 0$ where $x_\partial$ marks the triangle's edges.





# Properties of the set-up

* Phase space of a particle inside a triangular billiard with

P. Richens and M. Berry. Physica D: Nonlinear Phenomena, 2(3):495–512, 1981

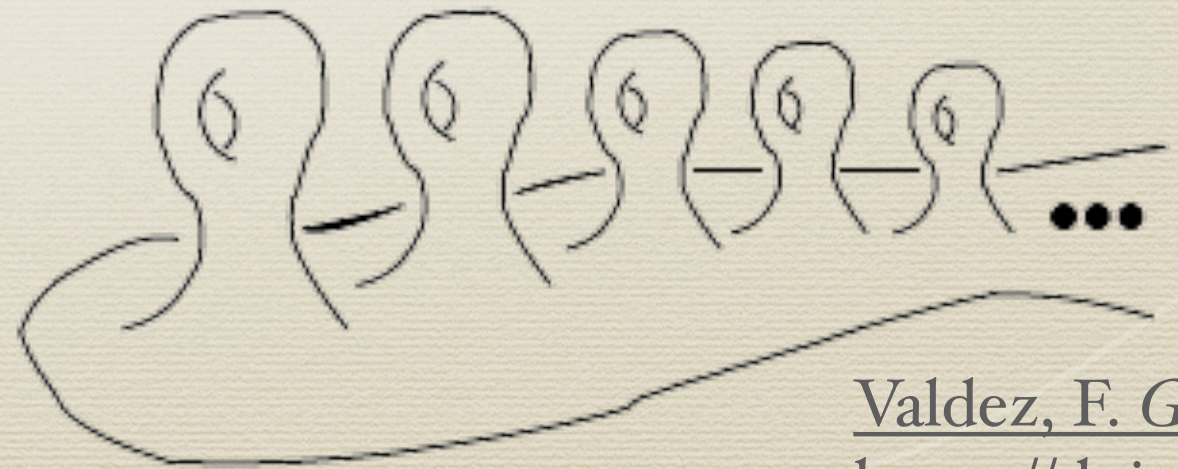
* a) rational angles → compact finite genus Riemann surface of genus, $g \geq 1$.

 ,  ,  , …

| | |
|---|---|
| {1/3,1/3,1/3} | → Genus 1 |
| {2/5,2/5,1/5} | → Genus 2 |
| {1/3,1/4,5/12} | → Genus 3 |

* b) irrational angles → non-compact real orientable infinite genus surfaces called **Loch Ness monster**!

$$\{(-1+\sqrt{5})/8,\ (5-\sqrt{5})/16,\ (13-\sqrt{5})/16\} \rightarrow$$



Valdez, F. *Geom Dedicata* **143**, 143–154 (2009). https://doi.org/10.1007/s10711-009-9378-x

# Integrability of the set-up

* Zero classical Kolmogorov-Sinai (KS) entropy and zero classical Lyapunov coefficient
* **Classical integrable system : a)** degrees of freedoms = constants of motion, **b)** non-singular vector fields in phase space from the constants of motion, **c)** Phase flow is confined to surfaces of genus 1. Example: Genus 1 billiards
* **Classical pseudointegrable system : a)** degrees of freedoms = constants of motion, **b)** Vector fields in phase space are singular, **c)** Phase flow is confined to surfaces of finite genus, $g \geq 2$. Example: Genus 2 and higher genus billiards
* **Classical nonintegrable : a)** Number of degrees of freedoms ≠ Number of constants of motion, **b)** Phase space is flow is non-compact. Example: Irrational angle billiards

# Quantities of our interest

* Eigenspectrum statistics
* Spectral complexity
* Krylov complexity

# Results : eigenspectrum statistics

* Integrable triangles → Poisson statistics for their energy levels.
* The non-integrable triangles → Gaussian orthogonal ensemble (GOE).
* Deviations due to → **a)** truncations within the Hilbert space, **b)** scarring and superscarring mechanisms.
* The pseudointegrable triangles → intermediate behaviour with stronger deviations from the GOE compared to nonintegrable triangles.

# Results : spectral complexity

* Spectral complexity (SC) for a $N$ dimensional Hilbert space: $C_S(t) = \sum_{p \neq q}^{N} \left[ \frac{\sin(t(E_p - E_q)/2)}{N(E_p - E_q)/2} \right]^2$

  where $E_p$ energy eigenvalues.

* We find for a chaotic billiard with the Kernel of GOE → $\frac{2}{3}\pi^2 N \log\left(\frac{t}{\pi N}\right)$

* We find for an integrable billiard (uncorrelated spectrum) → $\left(1 - \frac{1}{N}\right) 2\pi t$

* SC cannot distinguish between different genera, but it can clearly distinguish between integrable, isosceles, and generic triangles.

Hugo A. Camargo, Viktor Jahnke, Hyun-Sik Jeong, Keun-Young Kim, and Mitsuhiro Nishida, arXiv:2306.11632 [hep-th]

Triangles
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
$C_S$
t
1
100
$10^4$
$10^6$
$10^8$
$10^{10}$
100
$10^5$
9
10
11
2
22
12
25
13
19
1
16
18
6
7
17
26
8
20
14
4
5
15
21
23
27
24
3

# Results : Krylov complexity

- State Krylov Complexity (Spread Complexity):

- How much is the time evolved state $|\psi(t)\rangle = e^{-iHt}|\psi(t=0)\rangle$ "spread" over state Krylov basis?

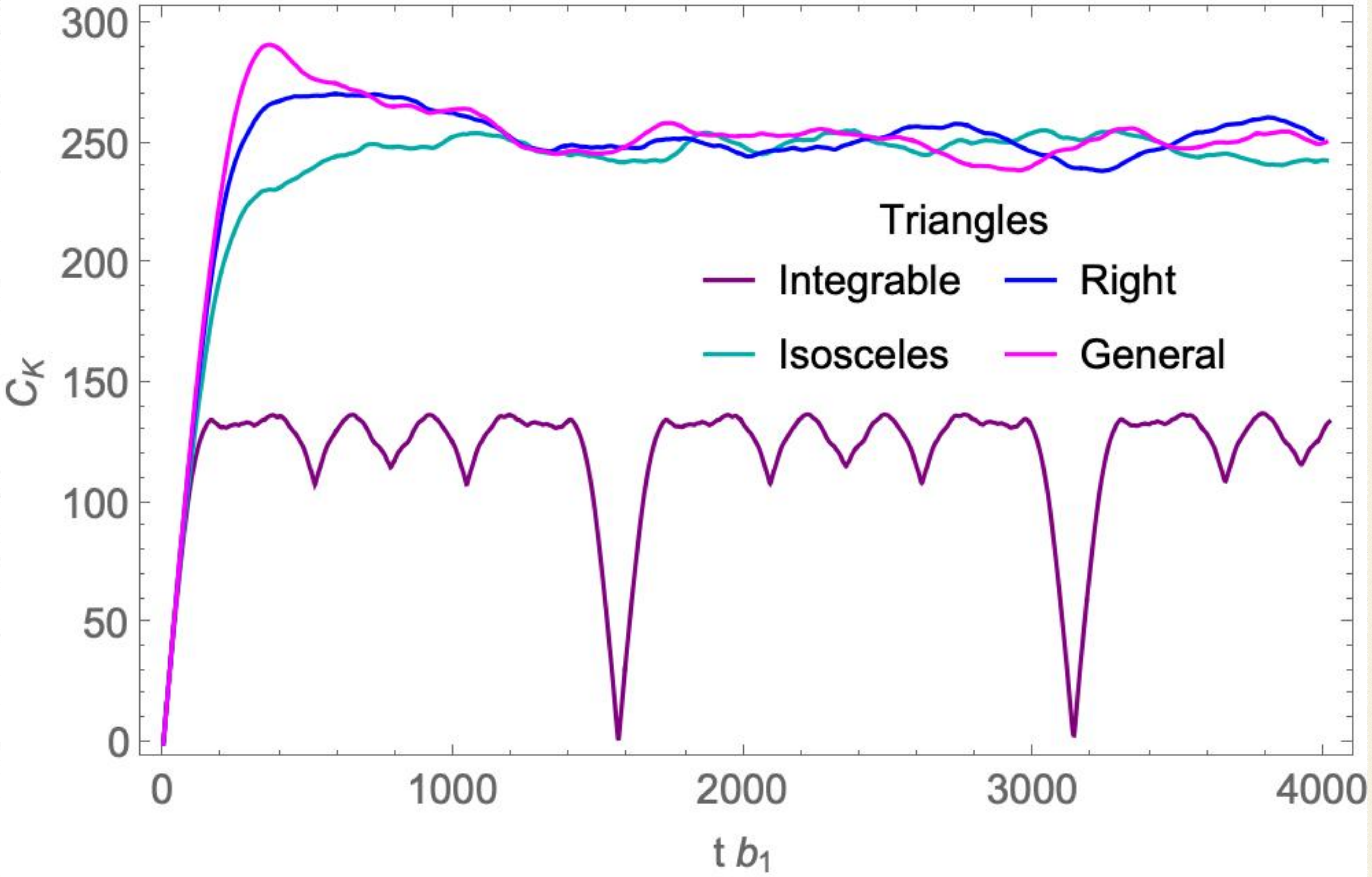
- $|\psi(t)\rangle = e^{-iHt}|\psi(t=0)\rangle = \sum_n \frac{(-it)^n}{n!} H^n |\psi(t=0)\rangle$

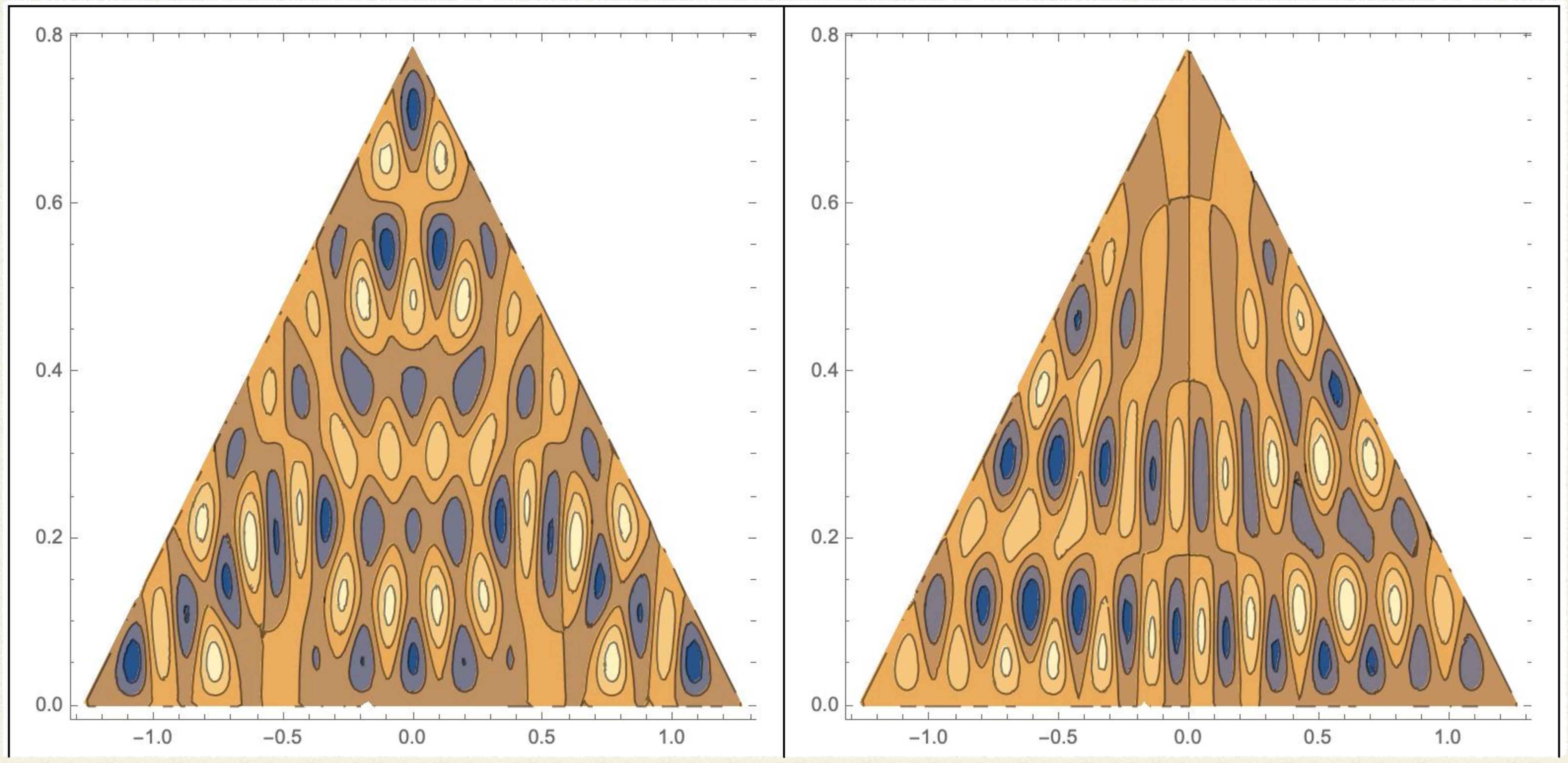
- Basis by orthonormalisation : Krylov basis (GNS construction)
  $(|\psi(t=0)\rangle, H|\psi(t=0)\rangle, H^2|\psi(t=0)\rangle, \dots H^n|\psi(t=0)\rangle \quad \rightarrow \quad |K_1\rangle, |K_2\rangle, \dots |K_n\rangle$

Vijay Balasubramanian, Pawel Caputa, Javier M. Magan, and Qingyue Wu Phys. Rev. D **106**, 046007

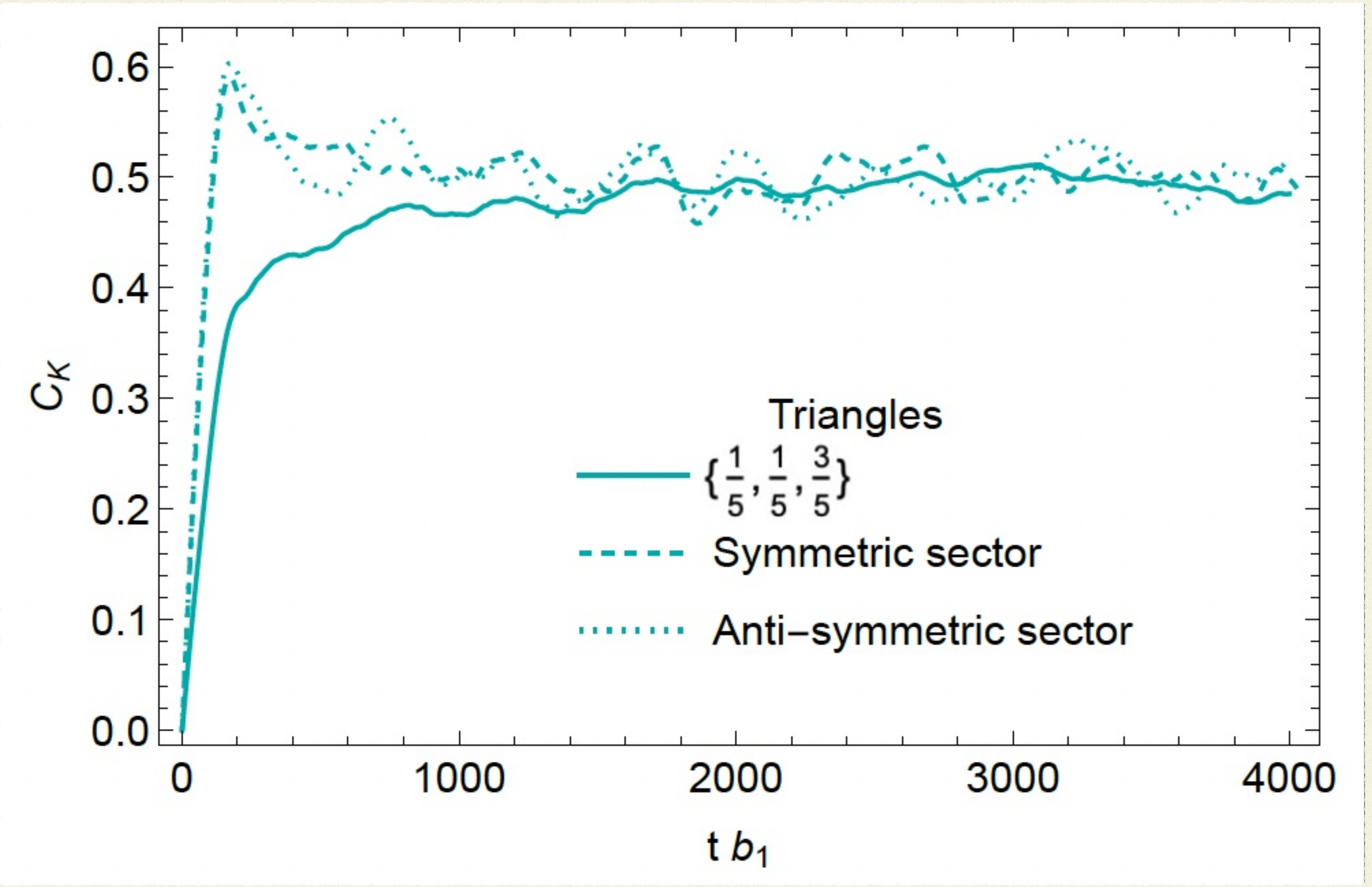
# Results : Krylov complexity

* $|\psi(t)\rangle = \sum_n \psi_n(t)\,|K_n\rangle$
* Krylov complexity (KC): $C_k = \sum_n n\,|\psi_n(t)|^2$
* Chaotic system: initial growth → peak → slight decay → saturation
* Non-chaotic system: initial growth → saturation



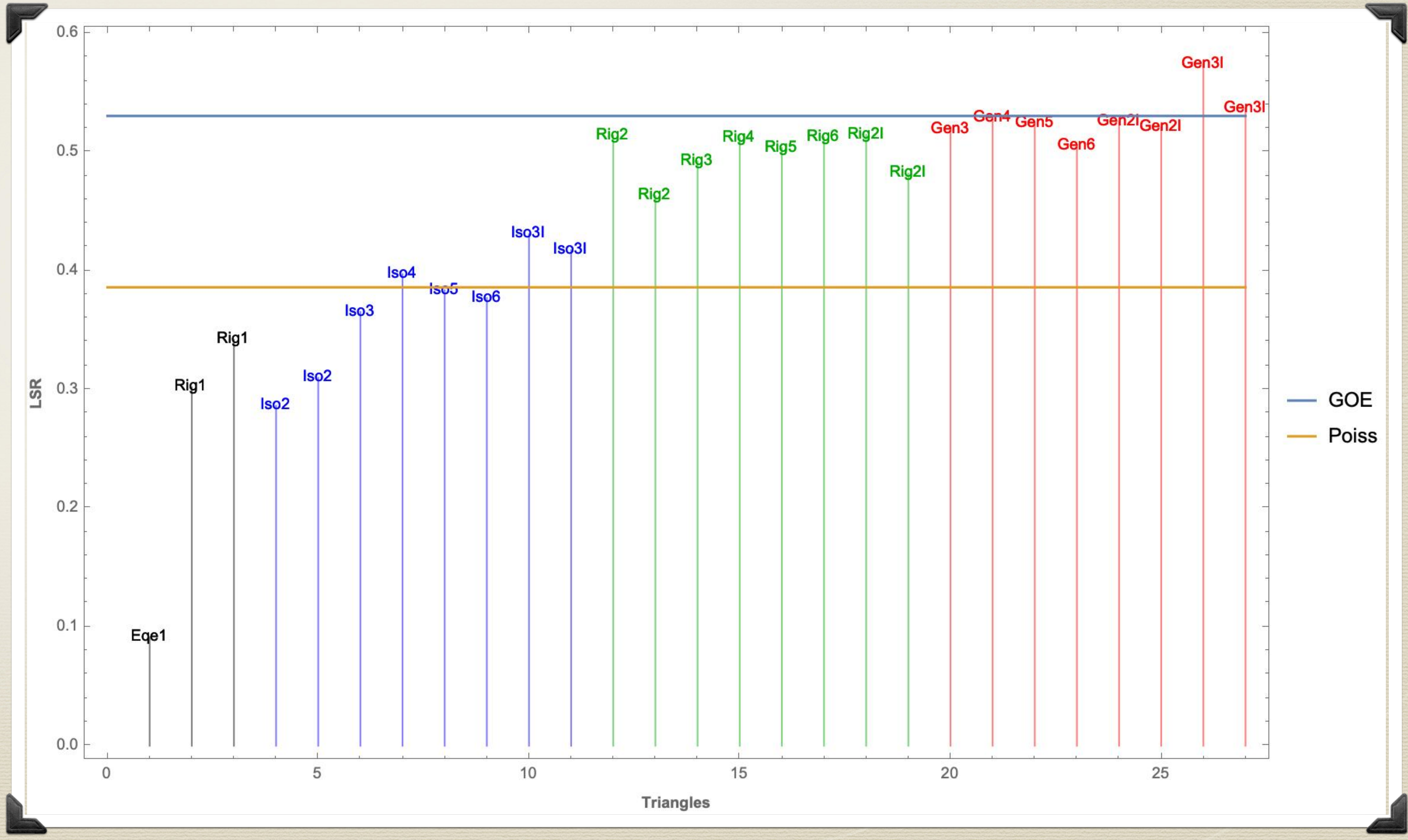
92nd and 95th wave function of $\{\frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, -\frac{1}{4}(-4+\sqrt{2})\}$




# Symmetry over topology

# Sum it up!

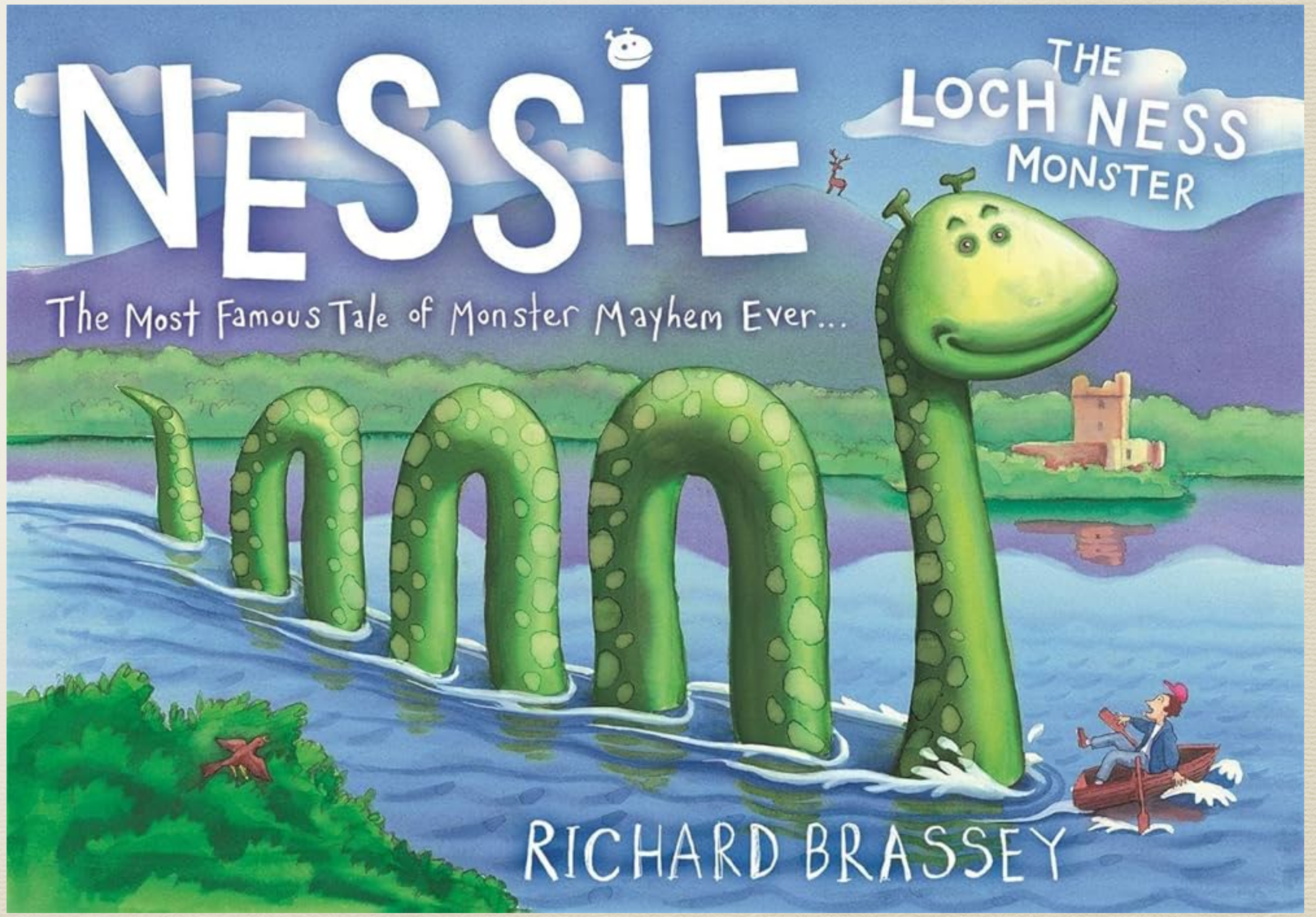
- Energy spacing : $s_i = E_{i+1} - E_i$
- Level spacing ratio (LSR) :
  $$\bar{r} = \frac{1}{N} \sum_{i=1}^{N} \frac{\min(s_i, s_{i-1})}{\max(s_i, s_{i-1})}$$



# Sum it up!

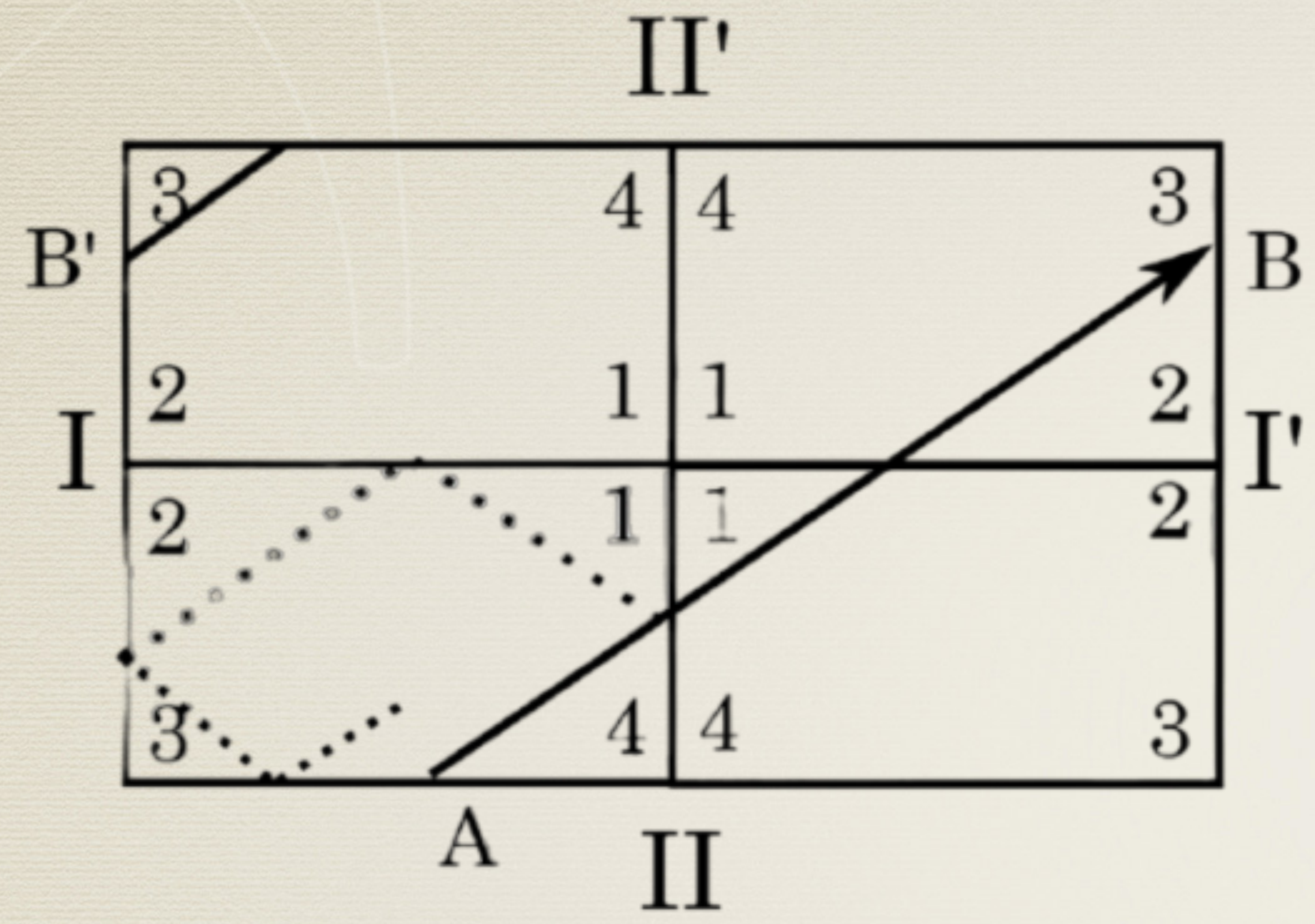
* Integrable, Isosceles, and general triangular billiards can be separated.
* Revival of complexity is only attainable for billiards corresponding to genus 1.
* Subtle effects of the topology are observed in the hierarchy.
* Symmetry resolving is essential for studying chaotic properties in billiard system.
* A more sensitive measure of chaotic properties is imperative for advancing toward a quantum ergodic hierarchy.

# For now we have the hierarchy from the Donut to THE LOCH NESS MONSTER!

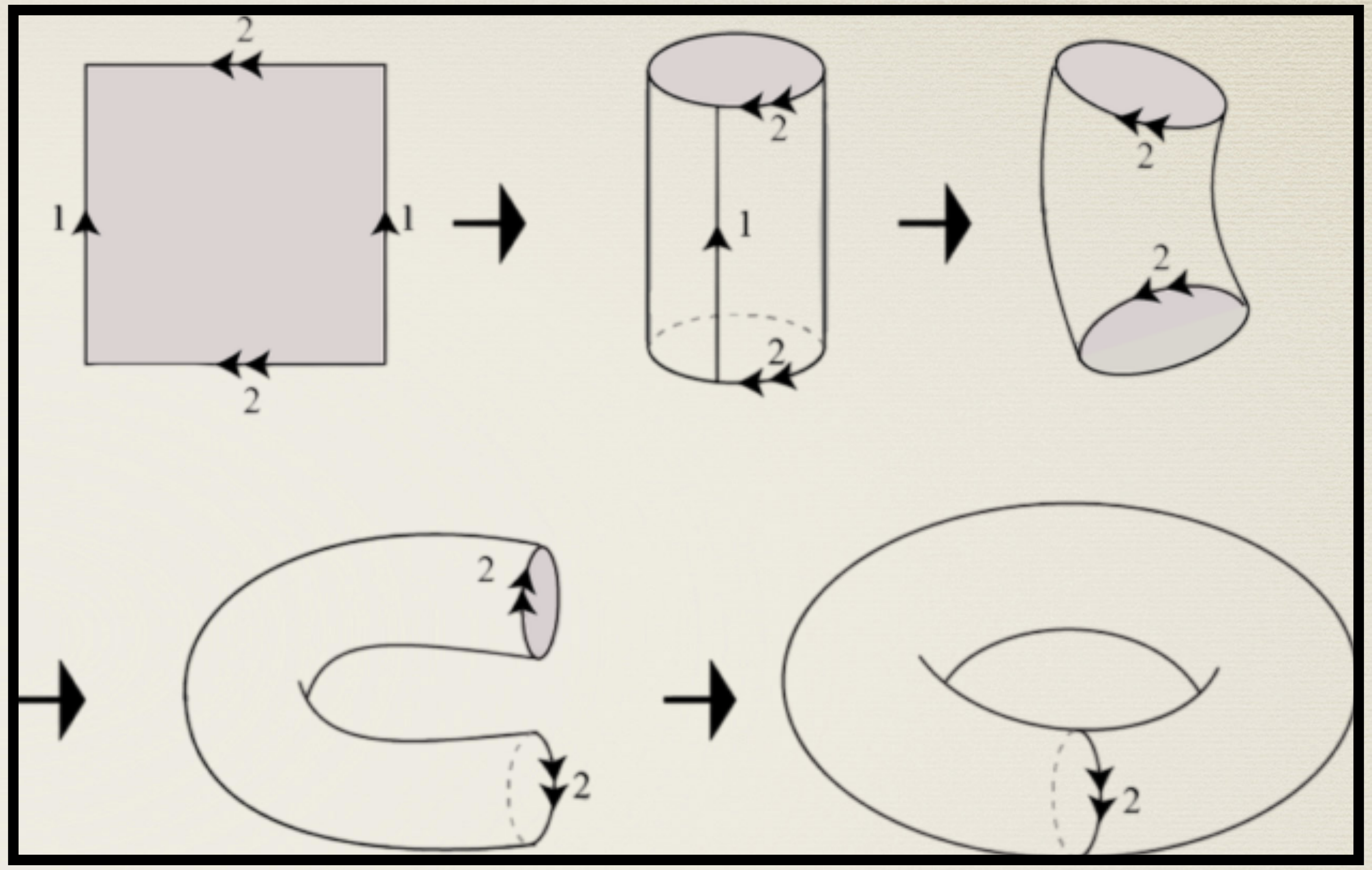



*Thank you for your attention!*

# Backup slide : Compactification




Sudhir Ranjan Jain and Rhine Samajdar Rev. Mod. Phys. 89, 045005



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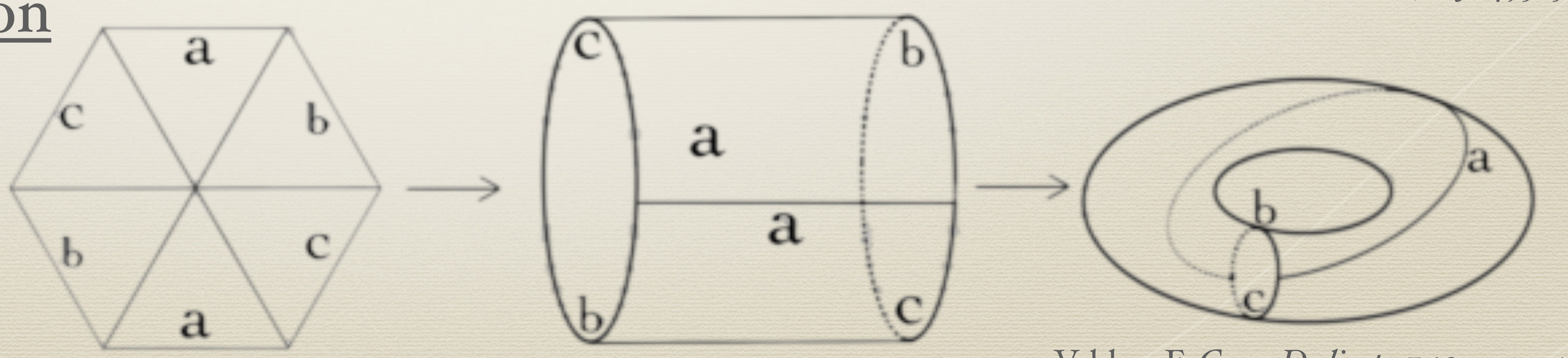
## Katok–Zemljakov construction

{1/3,1/3,1/3} → Genus 1 :

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