The gradient flow formulation of the electroweak Hamiltonian

“Field Theory on the Lattice” seminar series of HU Berlin and DESY Zeuthen

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based on work with Robert V. Harlander | July 3, 2023

The effective electroweak Hamiltonian

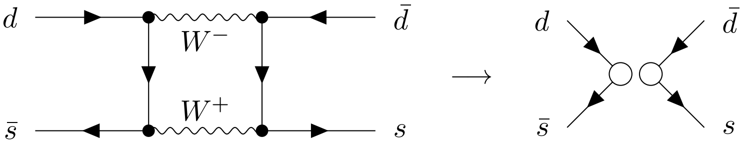
- Observables in flavor physics often computed with effective Hamiltonian of electroweak interactions

$$\mathcal{H}_{\rm eff} = -\left(\frac{4G_{\rm F}}{\sqrt{2}}\right)^x V_{\rm CKM} \sum_i C_i \mathcal{O}_i$$

with four-fermion operators like

$$\mathcal{O}^{|\Delta S|=2} = (\bar{s}\gamma_\mu(1-\gamma_5)d)\,(\bar{s}\gamma_\mu(1-\gamma_5)d)$$

for $K^0 - \bar{K}^0$ mixing:



- Wilson coefficients $C_i(\mu)$ obtained from perturbative matching to Standard Model at $\mu = \mu_W \sim M_W$
- $V_{\rm CKM}$: relevant entries of the CKM matrix, e.g. $V_{is}^* V_{id} V_{js}^* V_{jd}$ with $i, j = c, t$

Computing observables

- Flavor observables mostly at low energies

$\Rightarrow$ Use renormalization group equations to evolve down to appropriate scale to avoid large logarithms

- Schematically for Kaon mixing:

$$\langle \bar{K}^0 | \mathcal{H}_{\text{eff}}^{|\Delta S|=2} | K^0 \rangle \approx C(\mu_W) U(\mu_W, \mu_K) \langle \bar{K}^0 | \mathcal{O}^{|\Delta S|=2}(\mu_K) | K^0 \rangle$$

- Running with $U(\mu_W, \mu_K)$ determined by anomalous dimension γ of $\mathcal{O}^{|\Delta S|=2}$
- Matrix element $\langle \bar{K}^0 | \mathcal{O}^{|\Delta S|=2}(\mu_K) | K^0 \rangle$ nonperturbative

$\Rightarrow$ Compute on lattice

Complications

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i$$

- While $\mathcal{H}_{\text{eff}}$ is scheme independent, C_i and $\mathcal{O}_i$ are not:

Perturbative C_i:

- Dimensional regularization with $D = 4 - 2\epsilon$
- Operators mix through renormalization, also with evanescent operators (vanish in $D = 4$):

$$\mathcal{O}^{\text{R}} = Z_{\mathcal{O}\mathcal{O}}\mathcal{O} + Z_{\mathcal{O}\text{E}}E$$

- C_i scheme dependent:
  1. Explicit dependence on μ
  2. Scheme for γ_5
  3. Choice of evanescent operators

Lattice $\langle \mathcal{O}_i \rangle$:

- Lattice spacing a as UV regulator
- Have to take continuum limit $a \to 0$ in the end
- Operators mix through renormalization:

$$\mathcal{O}^{\text{R}} = Z_{11}\mathcal{O}_1 + Z_{12}\mathcal{O}_2$$

- $\langle \mathcal{O}_i \rangle$ scheme dependent

$\Rightarrow$ Scheme matching between lattice and perturbation theory additional source of uncertainty

Gradient flow

- Introduce parameter *flow time* $t \geq 0$ [Narayanan, Neuberger 2006; Lüscher 2009; Lüscher 2010]
- *Flowed fields* in $D+1$ dimensions obey differential *flow equations*:

$$\partial_t \Phi(t,x) = -\left.\frac{\delta S[\phi(x)]}{\delta \phi(x)}\right|_{\Phi(t,x)} \sim D_x \Phi(t,x) \quad \text{with} \quad \Phi(t,x)|_{t=0} = \phi(x)$$

- Flow equation drives flowed fields to minimum of action
- Flow equation similar to the heat equation (thermodynamics)

$$\partial_t u(t,\vec{x}) = \alpha \Delta u(t,\vec{x}) \quad \text{with} \quad \Delta = \sum_i \partial_{x_i}^2$$

- Fields at positive flow time smeared out with smearing radius $\sqrt{8t}$

$\Rightarrow$ Intuition: Regulates divergencies

Gradient flow for QCD

Gluon flow equation [Narayanan, Neuberger 2006; Lüscher 2010]

$$\partial_t B^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu} \quad \text{with} \quad B^a_\mu(t,x)\big|_{t=0} = A^a_\mu(x)$$

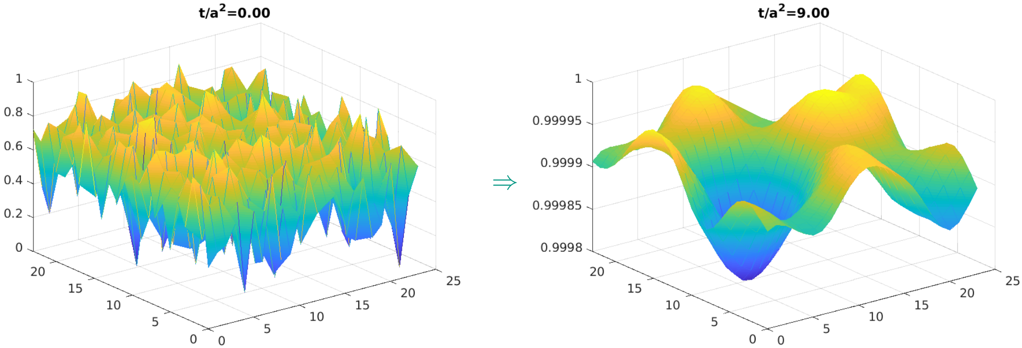
$$\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu - f^{abc}B^c_\mu, \qquad G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{abc}B^b_\mu B^c_\nu$$

Quark flow equation [Lüscher 2013]

$$\partial_t \chi = \Delta\chi \quad \text{with} \quad \chi(t,x)|_{t=0} = \psi(x)\,,$$
$$\partial_t \bar{\chi} = \bar{\chi}\overleftarrow{\Delta} \quad \text{with} \quad \overline{\chi}(t,x)|_{t=0} = \overline{\psi}(x)$$

$$\Delta = (\partial_\mu + B^a_\mu T^a)(\partial_\mu + B^b_\mu T^b), \qquad \overleftarrow{\Delta} = (\overleftarrow{\partial}_\mu - B^a_\mu T^a)(\overleftarrow{\partial}_\mu - B^b_\mu T^b)$$

Smearing



[Courtesy of Oliver Witzel]

Applications of the gradient flow in lattice QCD

- Inherent smearing to remove small-distance fluctuations [Narayanan, Neuberger 2006; Lüscher 2010; ...]
- Scale setting with the gradient flow extremely precise and cheap [Lüscher 2010; Borsányi et al. 2012; ...]
- Composite operators do not require renormalization [Lüscher, Weisz 2011]

$\Rightarrow$ Define gradient-flow scheme which is valid both on the lattice and perturbatively:

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- Define (renormalized) gradient-flow coupling [Lüscher 2010] :
  - Extract α_s from lattice simulations [Fodor et al. 2012; Fritsch, Ramos 2013; ...]
  - Study gradient-flow β function [Fodor et al. 2012; Fritsch, Ramos 2013; ...]
- Might help computing the QCD static force on the lattice [Brambilla, Chung, Vairo, Wang 2022]
- *Flowed operator product expansion* [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015] :

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  - *Flowed operator product expansion* [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015] :
    - Define the energy-momentum tensor of QCD on the lattice [Suzuki 2013; Makino, Suzuki 2014; Harlander, Kluth, **FL** 2018]
      $\Rightarrow$ Studies of thermodynamics [FlowQCD and WHOT-QCD since 2014]
    - Apply to quark dipole operators to (potentially) study hadronic CP-violation on the lattice [Rizik, Monahan, Shindler 2020; Kim, Luu, Rizik, Shindler 2021; Mereghetti, Monahan, Rizik, Shindler, Stoffer 2021; Bühler, Stoffer 2023]
    - Apply to electroweak Hamiltonian [Suzuki, Taniguchi, Suzuki, Kanaya 2020; Harlander, **FL** 2022]
    - ...
  - ...

Lattice simulation with gradient flow

Without gradient flow:

1. Produce gauge configurations, i.e. simulate dynamics of gluons and sea quarks
2. Measurement: Calculate correlation functions
3. Extract bare matrix elements
4. Continuum limit
5. Match lattice scheme to continuum scheme like $\overline{\text{MS}}$

- Scale setting, e.g. with hadron masses or gradient flow

With gradient flow:

1. Produce gauge configurations, i.e. simulate dynamics of gluons and sea quarks
2. Evolve to flow time t by numerically solving flow equations for each configuration
3. Measurement: Calculate correlation functions for each t
4. Extract gradient-flow matrix elements for each t
5. Continuum limit
6. Match gradient-flow scheme to continuum scheme like $\overline{\text{MS}}$ $\Rightarrow$ better behaved?

- Scale setting, e.g. with hadron masses or gradient flow

Energy-momentum tensor (EMT)

$$T_{\mu\nu} \equiv \frac{1}{g_0^2}\left[\mathcal{O}_{1,\mu\nu} - \frac{1}{4}\mathcal{O}_{2,\mu\nu}\right] + \frac{1}{4}\mathcal{O}_{3,\mu\nu} - \frac{1}{2}\mathcal{O}_{4,\mu\nu} - \mathcal{O}_{5,\mu\nu}$$

$$\mathcal{O}_{1,\mu\nu} \equiv F^a_{\mu\rho}F^a_{\nu\rho}, \qquad \mathcal{O}_{2,\mu\nu} \equiv \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma},$$

$$\mathcal{O}_{3,\mu\nu} \equiv \sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi_f, \qquad \mathcal{O}_{4,\mu\nu} \equiv \delta_{\mu\nu}\sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f \overleftrightarrow{\not{D}}\,\psi_f,$$

$$\mathcal{O}_{5,\mu\nu} \equiv \delta_{\mu\nu}\sum_{f=1}^{n_\mathrm{f}} m_{f,0}\bar{\psi}_f\psi_f$$

- Theoretically interesting: Contains the conserved currents under space-time translations
- Contains thermodynamical information at finite temperature
- EMT finite in the continuum, but needs renormalization on the lattice due to broken translational invariance

$\Rightarrow$ Non-trivial to define on the lattice

Flowed operator product expansion

- Flowed composite operators $\tilde{\mathcal{O}}_i(t, x)$ finite [Lüscher, Weisz 2011]
- Small flow-time expansion [Lüscher, Weisz 2011]:

$$\tilde{\mathcal{O}}_i(t, x) = \sum_j \zeta_{ij}(t)\mathcal{O}_j(x) + O(t)$$

- Invert to express operators through flowed operators [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:

Flowed OPE

$$T = \sum_i C_i \mathcal{O}_i = \sum_{i,j} C_i \zeta_{ij}^{-1}(t)\tilde{\mathcal{O}}_j(t) \equiv \sum_j \tilde{C}_j(t)\tilde{\mathcal{O}}_j(t)$$

- T defined in regular QCD expressed through finite flowed operators $\tilde{\mathcal{O}}_j(t)$
- Gradient-flow definition of T valid both on the lattice and perturbatively

Gradient-flow definition of the EMT

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⇒ Define EMT through flowed OPE [Suzuki 2013; Makino, Suzuki 2014] :

$$T_{\mu\nu} = \sum_i c_i \mathcal{O}_{i,\mu\nu} = \sum_{i,j} c_i \zeta_{ij}^{-1}(t)\tilde{\mathcal{O}}_j(t) = \sum_j \tilde{c}_j(t)\tilde{\mathcal{O}}_{j,\mu\nu}(t)$$

- Flowed operators:

$$\begin{aligned}
\mathcal{O}_{1,\mu\nu} &= F^a_{\mu\rho}F^a_{\nu\rho} & \Rightarrow \quad & \tilde{\mathcal{O}}_{1,\mu\nu}(t) = G^a_{\mu\rho}(t)G^a_{\nu\rho}(t), \\
\mathcal{O}_{2,\mu\nu} &= \delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma} & \Rightarrow \quad & \tilde{\mathcal{O}}_{2,\mu\nu}(t) = \delta_{\mu\nu}G^a_{\rho\sigma}(t)G^a_{\rho\sigma}(t), \\
& & \vdots &
\end{aligned}$$

- Gradient-flow definition does not require renormalization even on the lattice
- Flowed Wilson coefficients $\tilde{c}_i(t)$ can be calculated perturbatively:

$$\tilde{c}_i(t) = \frac{1}{g_0^2}\left(\zeta_{1i}^{-1}(t) - \frac{1}{4}\zeta_{2i}^{-1}(t)\right) + \frac{1}{4}\zeta_{3i}^{-1}(t)$$

  - NLO: [Suzuki 2013; Makino, Suzuki 2014] , NNLO: [Harlander, Kluth, **FL** 2018]

Method of projectors

- Define projectors [Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1987]

$$P_k[\mathcal{O}_i] \equiv D_k \langle 0 | \mathcal{O}_i | k \rangle \overset{!}{=} \delta_{ik} + O(\alpha_s)$$

- Apply to small flow-time expansion:

$$P_k[\tilde{\mathcal{O}}_i(t)] = \sum_j \zeta_{ij}(t) P_k[\mathcal{O}_j]$$

- $\zeta_{ij}(t)$ only depend on t

$\Rightarrow$ Set all other scales to zero

$\Rightarrow$ No perturbative corrections to $P_k[\mathcal{O}_j]$, because all loop integrals scaleless

"Master formula"

$$\zeta_{ij}(t) = P_j[\tilde{\mathcal{O}}_i(t)]\Big|_{p=m=0} = D_k \langle 0 | \tilde{\mathcal{O}}_i | k \rangle \Big|_{p=m=0}$$

Lagrangian

- Write Lagrangian for the gradient flow as [Lüscher, Weisz 2011; Lüscher 2013]

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi,$$

$$\mathcal{L}_{\text{QCD}} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_f} \bar{\psi}_f (\not{D}^{\mathrm{F}} + m_f)\psi_f + \dots$$

- Construct flowed Lagrangian using Lagrange multiplier fields $L^a_\mu(t,x)$ and $\lambda_f(t,x)$:

$$\mathcal{L}_B = -2 \int_0^\infty \mathrm{d}t\, \mathrm{Tr}\left[L^a_\mu T^a \left(\partial_t B^b_\mu T^b - \mathcal{D}^{bc}_\nu G^c_{\nu\mu} T^b \right) \right], \qquad \partial_t B^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu}$$

$$\mathcal{L}_\chi = \sum_{f=1}^{n_f} \int_0^\infty \mathrm{d}t \left(\bar{\lambda}_f (\partial_t - \Delta)\chi_f + \bar{\chi}_f \left(\overleftarrow{\partial_t} - \overleftarrow{\Delta} \right) \lambda_f \right), \qquad \partial_t \chi = \Delta\chi, \;\; \partial_t \bar{\chi} = \bar{\chi} \overleftarrow{\Delta}$$

⇒ Flow equations automatically fulfilled

⇒ QCD Feynman rules + gradient-flow Feynman rules (complete list in [Artz, Harlander, **FL**, Neumann, Prausa 2019])

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Solving the flow equations

- Split flow equation into linear part and remainder [Lüscher 2010]

$$\partial_t B^a_\mu = \partial_\nu \partial_\nu B^a_\mu + R^a_\mu \quad \text{with} \quad B^a_\mu(t,x)\big|_{t=0} = A^a_\mu(x)$$

- Solved by

$$B^a_\mu(t,x) = \int_y K_{\mu\nu}(t, x-y) A^a_\nu(y) + \int_y \int_0^t \mathrm{d}s\, K_{\mu\nu}(t-s, x-y) R^a_\nu(s,y)$$

with integration kernel

$$K_{\mu\nu}(t,x) = \int_p e^{\mathrm{i}p\cdot x} \delta_{\mu\nu} e^{-tp^2} \equiv \int_p e^{\mathrm{i}p\cdot x} \widetilde{K}_{\mu\nu}(t,p)$$

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Propagators

$$B^a_\mu(t,x) = \int_y K_{\mu\nu}(t, x-y) A^a_\nu(y) + \int_y \int_0^t \mathrm{d}s\, K_{\mu\nu}(t-s, x-y) R^a_\nu(s,y)$$

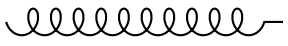
$$K_{\mu\nu}(t,x) = \int_p e^{ip\cdot x} \delta_{\mu\nu} e^{-tp^2} \equiv \int_p e^{ip\cdot x} \widetilde{K}_{\mu\nu}(t,p)$$

- Flowed gluon propagator contains fundamental gluon propagator:

$$\left\langle \widetilde{B}^a_\mu(t,p) \widetilde{B}^b_\nu(s,q) \right\rangle \Big|_{\mathrm{LO}} = \widetilde{K}_{\mu\rho}(t,p) \widetilde{K}_{\nu\sigma}(s,q) \left\langle \widetilde{A}^a_\rho(p) \widetilde{A}^b_\sigma(q) \right\rangle$$

⇒ Can express both by same Feynman rule

p

s, ν, b  t, μ, a $= \delta^{ab} \dfrac{1}{p^2} \delta_{\mu\nu}\, e^{-(t+s)p^2}$

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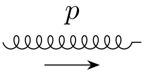
Flow lines

- Flowed gluon Lagrangian:

$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\,\mathrm{Tr}\left[L_\mu^a T^a\left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b\right)\right]$$

$\Rightarrow$ No squared L_μ^a in $\mathcal{L}_B \Rightarrow$ no propagator

- Instead mixed propagator $\left\langle \widetilde{B}_\mu^a(t,p)\widetilde{L}_\nu^b(s,q)\right\rangle$ called *flow line*:

s,ν,b  t,μ,a $= \delta^{ab}\,\theta(t-s)\delta_{\mu\nu}\,e^{-(t-s)p^2}$

- Directed towards increasing flow time

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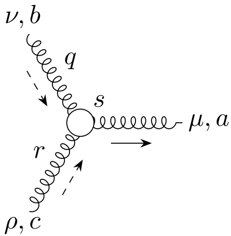
Flow vertices

$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\,\mathrm{Tr}\left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b\right)\right]$$

- Example:



$$= -igf^{abc}\int_0^\infty \mathrm{d}s\left(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu\right)$$

- Integral restricted by $\theta(t-s)$ from outgoing flow line
- Incoming lines can be both flow lines and flowed propagators

Renormalization

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- QCD renormalization of QCD parameters like α_s and quark masses
- Flowed gluon fields do not require renormalization [Lüscher 2010; Lüscher, Weisz 2011]
- Flowed quark fields have to be renormalized: $\chi^\mathrm{R} = Z_\chi^{1/2} \chi^\mathrm{B}$ [Lüscher 2013]

$\Rightarrow$ χ acquire anomalous dimension and not scheme independent

- "Physical" scheme: Ringed fermions $\mathring{\chi} = \mathring{Z}_\chi^{1/2} \chi^\mathrm{B}$ [Makino, Suzuki 2014]:

$$\mathring{Z}_\chi = -\frac{2N_\mathrm{c}}{(4\pi t)^2 \left.\langle \bar{\chi}^\mathrm{B} \overleftrightarrow{\not{D}} \chi^\mathrm{B} \rangle\right|_{m=0}}$$

$\Rightarrow$ $\mathring{\chi}$ formally independent of renormalization scale μ

- $\mathring{Z}_\chi$ available through NNLO [Artz, Harlander, **FL**, Neumann, Prausa 2019]
- Composite operators do not require renormalization [Lüscher, Weisz 2011]

$\Rightarrow$ Gradient-flow scheme without operator mixing

Example for projector

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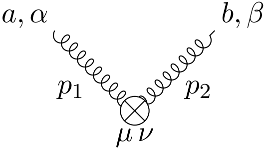
"Master formula"

$$\zeta_{ij}(t) = P_j[\tilde{\mathcal{O}}_i(t)]\Big|_{p=m=0} = D_k\langle 0|\tilde{\mathcal{O}}_i|k\rangle\Big|_{p=m=0}$$

- Feynman rule for operator:

$$\mathcal{O}_{1,\mu\nu} = F^a_{\mu\rho}F^a_{\nu\rho} = \partial_\mu A^c_\rho \partial_\nu A^c_\rho + \cdots \quad \Rightarrow$$



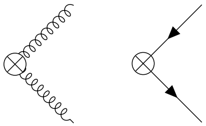
$$= -p_{1,\mu}p_{2,\nu}\delta_{\alpha\beta}\delta^{ab}$$

$\Rightarrow$ Projector for $\mathcal{O}_{1,\mu\nu}$:

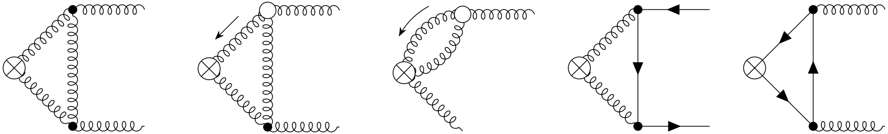
$$P_1[\mathcal{O}_{\mu\nu}] = -\frac{\delta^{ab}}{N_A}P_{\alpha\beta|\rho\mu|\sigma\nu}\frac{\partial}{\partial p_{1,\rho}}\frac{\partial}{\partial p_{2,\sigma}}\langle 0|A^a_\alpha(p_1)A^b_\beta(p_2)\mathcal{O}_{\mu\nu}|0\rangle$$

Sample diagrams

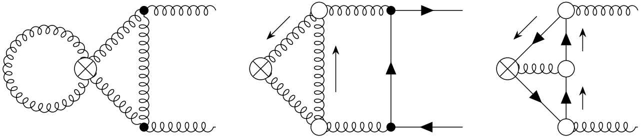
- LO:



- NLO:



- NNLO:



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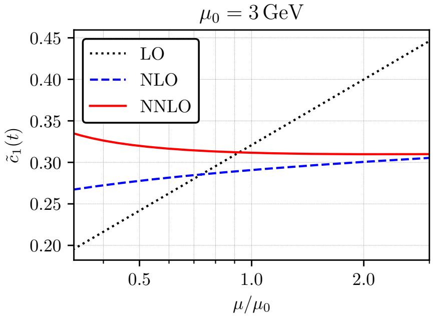
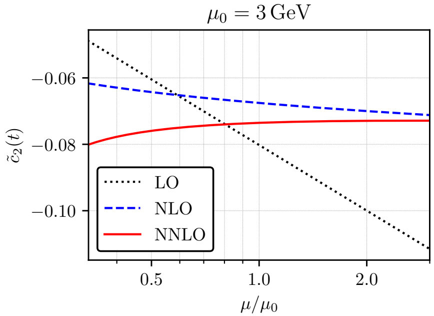
Automatized calculation

- qgraf [Nogueira 1991]: Generate Feynman diagrams
- q2e and exp [Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999]: Assign diagrams to topologies and prepare FORM code
- FORM [Vermaseren 2000; Kuipers, Ueda, Vermaseren, Vollinga 2013]: Insert Feynman rules, perform tensor reduction, Dirac traces, color algebra [van Ritbergen, Schellekens, Vermaseren 1998] , and expansions
- Generate system of equations employing integration-by-parts-like relations [Tkachov 1981; Chetyrkin, Tkachov 1981] with in-house Mathematica code
- Kira [Maierhöfer, Usovitsch, Uwer 2017; Klappert, **FL**, Maierhöfer, Usovitsch 2020] $\oplus$ FireFly [Klappert, **FL** 2019; Klappert, Klein, **FL** 2020]: Solve system to express all integrals through master integrals with Laporta algorithm [Laporta 2000]
- Calculation of master integrals:
  - Direct integration with Mathematica
  - Expansion employing HyperInt [Panzer 2014]
  - Numerical integration following sector decomposition strategy [Binoth, Heinrich 2000 + 2003] with FIESTA [Smirnov, Tentyukov 2008; Smirnov, Smirnov, Tentyukov 2009; Smirnov 2013] and in-house integration routines [Harlander, Neumann 2016] or pySecDec [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke 2017; Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk 2018; Heinrich, Jahn, Jones, Kerner, Langer, Magerya, Poldaru, Schlenk, Villa 2021]

Results

$$T_{\mu\nu} = \sum_i c_i \mathcal{O}_{i,\mu\nu} = \sum_{i,j} c_i \zeta_{ij}^{-1}(t)\tilde{\mathcal{O}}_j(t) = \sum_j \tilde{c}_j(t)\tilde{\mathcal{O}}_{j,\mu\nu}(t)$$





- Uncertainty through scale variation reduces from 3.9 % and 4.2 % to 1.6 % and 2.0 % for $\tilde{c}_1$ and $\tilde{c}_2$
- Similar for $\tilde{c}_3$, but still 33 % for subdominant $\tilde{c}_4$

Extracting thermodynamical quantities (I)

[Iritani, Kitazawa, Suzuki, Takaura 2018]

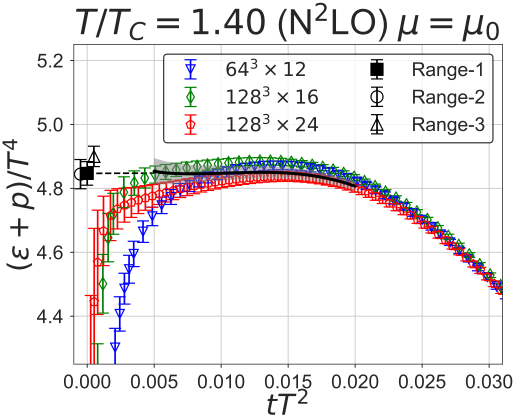
- Study of thermodynamics in pure Yang-Mills theory on the lattice to compute energy density and pressure at finite temperature:

$$\epsilon + p = -\frac{4}{3}\left\langle T_{00} - \frac{1}{4}T_{\mu\mu}\right\rangle ,$$
$$\epsilon - 3p = -\langle T_{\mu\mu}\rangle$$

- Grey band: continuum limit $a \to 0$ at fixed tT^2
- Extrapolation $tT^2 \to 0$ gives physical result
- Divergent behavior for $t \lesssim a^2$ due to finite lattice spacing effects

$\Rightarrow$ Small $t \sim \frac{1}{\mu^2}$ for precise perturbative calculations and extrapolation, large t for stable lattice simulations

$\Rightarrow$ *Window*



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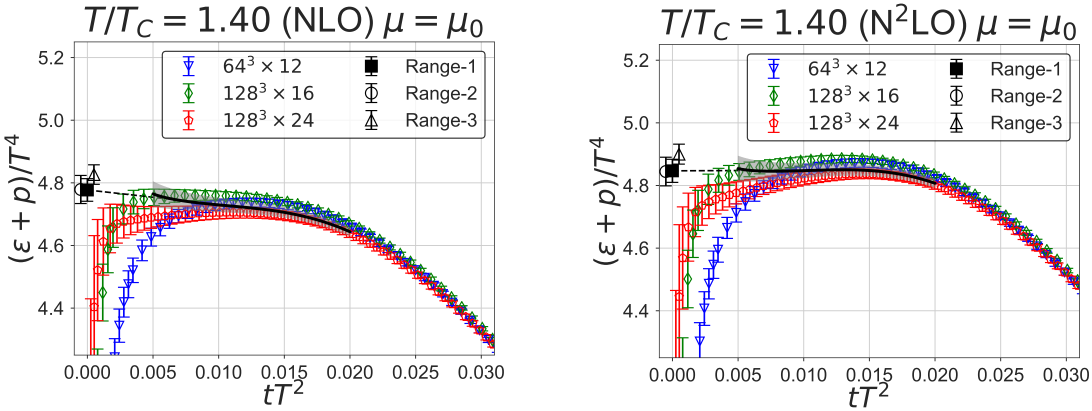
Extracting thermodynamical quantities (II)

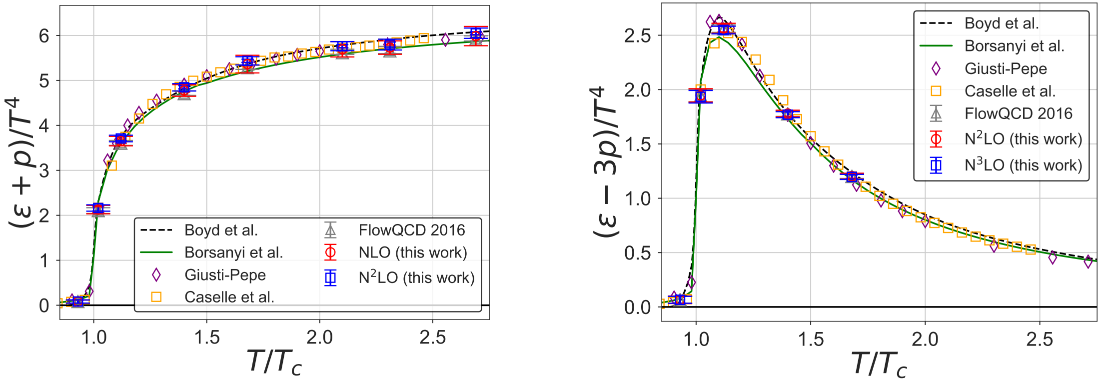
[Iritani, Kitazawa, Suzuki, Takaura 2018]



- NNLO coefficients reduce scale uncertainty to be negligible
- NNLO coefficients reduce t dependence (flatter curve) and thus improve extrapolation $tT^2 \to 0$

Comparison to other calculations

[Iritani, Kitazawa, Suzuki, Takaura 2018]



- Uncertainties of [Giusti, Pepe 2016] and [Caselle, Nada, Panero 2018] (not shown in plots) $\sim 1 - 6$ times smaller, depending on T/T_c
- Studying $t \to 0$ extrapolation more carefully reduces uncertainties related to extrapolation to be negligible, uncertainty then dominated by statistics [Suzuki, Takaura 2021]

Flowed OPE for the electroweak Hamiltonian

- Write electroweak Hamiltonian as

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

- Gradient-flow scheme valid both on the lattice and perturbatively:

Perturbative $\tilde{C}_j$:

- Dimensional regularization with $D = 4 - 2\epsilon$
- Finite and scheme independent:
  1. No explicit dependence on μ
  2. No dependence on scheme for γ_5
  3. Independent of evanescent operators

Lattice $\langle \tilde{\mathcal{O}}_j \rangle$:

- Lattice spacing a as UV regulator
- Finite for $a \to 0$
- No operator mixing

Flowed OPE for the electroweak Hamiltonian

- Write electroweak Hamiltonian as

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

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Lattice $\langle \tilde{\mathcal{O}}_j \rangle$:

  - Lattice spacing a as UV regulator
  - Finite for $a \to 0$
  - No operator mixing

- Three ingredients:
  - C_i known perturbatively through (N)NLO (depending on process)
  - ζ_{ij}^{-1} has to be computed, some first results in [Suzuki, Taniguchi, Suzuki, Kanaya 2020; Harlander, **FL** 2022]
  - $\langle \tilde{\mathcal{O}}_j \rangle$ to be computed on the lattice

Operator basis

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

- Operator basis depends on process
- We focus on the current-current operators relevant for $|\Delta F| = 2$ processes
- Operator basis not unique even for the same process, but different bases related by basis transformations
- CMM basis [Chetyrkin, Misiak, Münz 1997]:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\text{L}}\gamma_\mu T^a \psi_{2,\text{L}}\right)\left(\bar{\psi}_{3,\text{L}}\gamma_\mu T^a \psi_{4,\text{L}}\right),$$
$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\text{L}}\gamma_\mu \psi_{2,\text{L}}\right)\left(\bar{\psi}_{3,\text{L}}\gamma_\mu \psi_{4,\text{L}}\right)$$

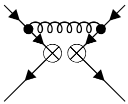
with

$$\psi_{\text{R/L}} = P_\pm \psi = \tfrac{1}{2}(1 \pm \gamma_5)\psi$$

Evanescent operators

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$$

- In dimensional regularization, loop corrections produce additional non-reducible γ structures:



$$\Rightarrow \qquad (\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}) \otimes (\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3})$$

- These contributions have to be attributed to *evanescent* operators like [Buras, Weisz 1990]

$$E_2^{(1)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_2 \qquad \text{with} \qquad \gamma_{\mu_1\cdots\mu_n} \equiv \gamma_{\mu_1}\cdots\gamma_{\mu_n}$$

- Algebraically they are of $O(\epsilon)$ and vanish for $D \to 4$
- Nonetheless required to renormalize the physical operators
- Renormalization has to take care of finite pieces from $\frac{1}{\epsilon}$ (poles) $\times\ \epsilon$ (operators)
- Every loop order introduces more evanescent operators

Complete operator basis

- Physical operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right),$$
$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right)$$

- Evanescent operators through NNLO:

$$E_1^{(1)} = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_1,$$
$$E_2^{(1)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_2,$$
$$E_1^{(2)} = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \psi_{4,\mathrm{L}}\right) - 20E_1^{(1)} - 256\mathcal{O}_1,$$
$$E_2^{(2)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \psi_{4,\mathrm{L}}\right) - 20E_2^{(1)} - 256\mathcal{O}_2$$

Flowed operator basis

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- Flowed physical operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_1 = -\mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu T^a \chi_{4,\mathrm{L}}\right)$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_2 = \mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu \chi_{4,\mathrm{L}}\right)$$

- Flowed evanescent operators:

$$\tilde{E}_1^{(1)} = -\mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \chi_{4,\mathrm{L}}\right) - 16\tilde{\mathcal{O}}_1,$$

$$\tilde{E}_2^{(1)} = \mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \chi_{4,\mathrm{L}}\right) - 16\tilde{\mathcal{O}}_2,$$

$$\tilde{E}_1^{(2)} = -\mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \chi_{4,\mathrm{L}}\right) - 20\tilde{E}_1^{(1)} - 256\tilde{\mathcal{O}}_1,$$

$$\tilde{E}_2^{(2)} = \mathring{Z}_\chi^2\left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \chi_{4,\mathrm{L}}\right) - 20\tilde{E}_2^{(1)} - 256\tilde{\mathcal{O}}_2$$

- Since flowed operators do not have to be renormalized, the flowed evanescent operators actually vanish and could be dropped
- Keeping them allows us to check our results

Renormalization (I)

- Small-flow-time expansion for operators of electroweak Hamiltonian:

$$\begin{pmatrix} \tilde{\mathcal{O}}(t) \\ \tilde{E}(t) \end{pmatrix} \asymp \zeta^{\mathrm{B}}(t) \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}$$

with $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)^{\mathrm{T}}, \qquad E = (E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)})^{\mathrm{T}}$

- Since regular operators divergent, $\zeta^{\mathrm{B}}(t)$ divergent as well
- Regular operators renormalized through

$$\begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} = Z \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix} \equiv \begin{pmatrix} Z_{\mathrm{PP}} & Z_{\mathrm{PE}} \\ Z_{\mathrm{EP}} & Z_{\mathrm{EE}} \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}$$

- Renormalized $\zeta(t)$:

$$\begin{pmatrix} \tilde{\mathcal{O}}(t) \\ \tilde{E}(t) \end{pmatrix} \asymp \zeta^{\mathrm{B}}(t) Z^{-1} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} \equiv \zeta(t) \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} \equiv \begin{pmatrix} \zeta_{\mathrm{PP}}(t) & \zeta_{\mathrm{PE}}(t) \\ \zeta_{\mathrm{EP}}(t) & \zeta_{\mathrm{EE}}(t) \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}}$$

Renormalization (II)

- Renormalization matrix Z includes finite renormalization:

$$Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k Z_{ij}^{(k)} \quad \text{with} \quad Z_{ij}^{(k)} = \sum_{l=0}^{k} \frac{1}{\epsilon^l} Z_{ij}^{(k,l)}$$

- Related to anomalous dimension of operators and Wilson coefficients:

$$\mu \frac{\mathrm{d}\mathcal{O}_i(\mu)}{\mathrm{d}\mu} \equiv \gamma_{ij}\mathcal{O}_j(\mu) \qquad \text{and} \qquad \mu \frac{\mathrm{d}C_i(\mu)}{\mathrm{d}\mu} \equiv \gamma_{ji}C_j(\mu) \qquad \Rightarrow \qquad \gamma_{ij} = 2\alpha_s\beta_\epsilon Z_{ik}\frac{\partial Z_{kj}^{-1}}{\partial \alpha_s}$$

- Block form [Buras, Weisz 1990; Dugan, Grinstein 1991; Herrlich, Nierste 1995]:

$$\gamma^{(k)} = \begin{pmatrix} \gamma_{PP}^{(k)} & \gamma_{PE}^{(k)} \\ 0 & \gamma_{EE}^{(k)} \end{pmatrix} \qquad \text{and} \qquad Z^{(k,0)} = \begin{pmatrix} 0 & 0 \\ Z_{EP}^{(k,0)} & 0 \end{pmatrix}$$

- Ensures that matrix elements of renormalized evanescent operators vanish:

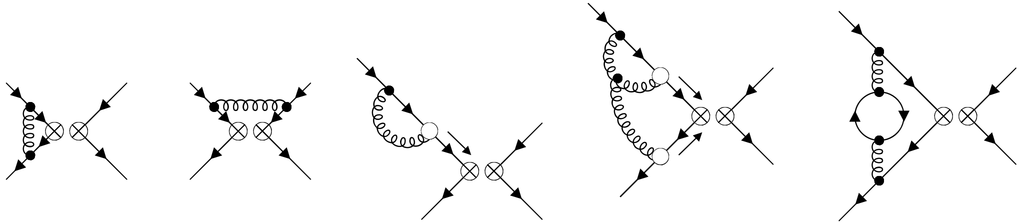
$$\langle E^R \rangle = Z_{EP}\langle \mathcal{O} \rangle + Z_{EE}\langle E \rangle \overset{!}{=} O(\epsilon)$$

Projectors and example diagrams

- Schematic projector for $\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$:

$$P_2[\mathcal{O}] = \frac{1}{16N_c^2}\,\mathrm{Tr}_{\text{line 1}}\mathrm{Tr}_{\text{line 2}}\,\langle 0|\left(\psi_{4,\mathrm{L}}\gamma_\nu\bar{\psi}_{3,\mathrm{L}}\right)\left(\psi_{2,\mathrm{L}}\gamma_\nu\bar{\psi}_{1,\mathrm{L}}\right)\mathcal{O}|0\rangle\Big|_{p=m=0}$$

- Sample diagrams:



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Results in CMM basis

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- Physical matching matrix $(\zeta^{-1})_{\mathrm{PP}}$:

$$(\zeta^{-1})_{11}(t) = 1 + a_s\left(4.212 + \frac{1}{2}L_{\mu t}\right) + a_s^2\left[22.72 - 0.7218\, n_f + L_{\mu t}\,(16.45 - 0.7576\, n_f) + L_{\mu t}^2\left(\frac{17}{16} - \frac{1}{24}\, n_f\right)\right],$$

$$(\zeta^{-1})_{12}(t) = a_s\left(-\frac{5}{6} - \frac{1}{3}L_{\mu t}\right) + a_s^2\left[-4.531 + 0.1576\, n_f + L_{\mu t}\left(-3.133 + \frac{5}{54}\, n_f\right) + L_{\mu t}^2\left(-\frac{13}{24} + \frac{1}{36} n_f\right)\right],$$

$$(\zeta^{-1})_{21}(t) = a_s\left(-\frac{15}{4} - \frac{3}{2}L_{\mu t}\right) + a_s^2\left[-23.20 + 0.7091\, n_f + L_{\mu t}\left(-15.22 + \frac{5}{12}\, n_f\right) + L_{\mu t}^2\left(-\frac{39}{16} + \frac{1}{8}\, n_f\right)\right],$$

$$(\zeta^{-1})_{22}(t) = 1 + a_s\, 3.712 + a_s^2\left[19.47 - 0.4334\, n_f + L_{\mu t}\,(11.75 - 0.6187\, n_f) + \frac{1}{4}L_{\mu t}^2\right]$$

- $a_s = \alpha_s(\mu)/\pi$ renormalized in $\overline{\mathrm{MS}}$ scheme and $L_{\mu t} = \ln 2\mu^2 t + \gamma_E$
- Set $N_c = 3$, $T_R = \frac{1}{2}$, and transcendental coefficients replaced by floating-point numbers

Checks

$$\zeta^{-1} = Z(\zeta^{\mathrm{B}})^{-1} = \begin{pmatrix} (\zeta^{-1})_{\mathrm{PP}} & (\zeta^{-1})_{\mathrm{PE}} \\ (\zeta^{-1})_{\mathrm{EP}} & (\zeta^{-1})_{\mathrm{EE}} \end{pmatrix}$$

- Finite after α_s + field renormalization and with Z from [Chetyrkin, Misiak, Münz 1997; Gambino, Gorbahn, Haisch 2003; Gorbahn, Haisch 2004]
- $(\zeta^{-1})_{\mathrm{EP}} = O(\epsilon)$
- Independent of QCD gauge parameter

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Basis transformations

- Different operator bases related by

$$\vec{\mathcal{O}}' = R(\vec{\mathcal{O}} + W\vec{E}) \qquad \text{and} \qquad \vec{E}' = M(\epsilon U\vec{\mathcal{O}} + [1 + \epsilon V]\vec{E})$$

- Not sufficient to simply rotate the physical submatrix with R: $\zeta'_{\mathrm{PP}} \neq R\zeta_{\mathrm{PP}}R^{-1}$
- 1. possibility:
  - Transform whole ζ^{B}
  - Perform renormalization in the same way as before with a different Z
- 2. possibility:
  - Rotate renormalized ζ_{PP}
  - But: basis transformation also changes the scheme of Z!
  - $\Rightarrow$ Restore the scheme by an additional finite renormalization [Chetyrkin, Misiak, Münz 1997; Gambino, Gorbahn, Haisch 2003; Gorbahn, Haisch 2004]:

$$\zeta'_{\mathrm{PP}} = R\zeta_{\mathrm{PP}}R^{-1}Z_{\mathrm{fin}}^{-1}$$

Transformation to non-mixing basis

- Physical operators:

$$\mathcal{O}_\pm = \frac{1}{2}\left[\left(\bar\psi_1^\alpha \gamma_\mu^{\mathrm{L}} \psi_2^\alpha\right)\left(\bar\psi_3^\beta \gamma_\mu^{\mathrm{L}} \psi_4^\beta\right) \pm \left(\bar\psi_1^\alpha \gamma_\mu^{\mathrm{L}} \psi_2^\beta\right)\left(\bar\psi_3^\beta \gamma_\mu^{\mathrm{L}} \psi_4^\alpha\right)\right]$$

- Evanescent operators and transformation matrices through NNLO defined in [Buras, Gorbahn, Haisch, Nierste 2006]
- Anomalous dimension diagonal, i.e. operators do not mix under RGE running
- We did the transformation in both ways and find agreement as well as diagonal form:

$$\zeta_{++}^{-1} = 1 + a_s\left(2.796 - \frac{1}{2}L_{\mu t}\right) + a_s^2\left[14.15 - 0.1739\, n_f + L_{\mu t}\,(6.509 - 0.4798\, n_f) + L_{\mu t}^2\left(-\frac{9}{16} + \frac{1}{24}\, n_f\right)\right],$$

$$\zeta_{--}^{-1} = 1 + a_s\,(5.546 + L_{\mu t}) + a_s^2\left[32.01 - 0.9524\, n_f + L_{\mu t}(21.23 - 0.8965\, n_f) + L_{\mu t}^2\left(\frac{15}{8} - \frac{1}{12}\, n_f\right)\right]$$

Status and outlook for flavor physics

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j :$$

- Kaon mixing ($|\Delta S| = 2$):
  - C_i: NLO [Buchalla, Buras, Lautenbacher 1995 and references therein] with two of three contributions known through NNLO [Brod, Gorbahn 2010 + 2012]
  - ζ_{ij}^{-1}: NNLO [Harlander, **FL** 2022] (NLO in different basis and scheme in [Suzuki, Taniguchi, Suzuki, Kanaya 2020])
  - $\langle\tilde{\mathcal{O}}_j\rangle$: ?
- Neutral B-meson mixing ($|\Delta B| = 2$):
  - C_i: NLO [Buchalla, Buras, Lautenbacher 1995 and references therein; Kirk, Lenz, Rauh 2017]
  - ζ_{ij}^{-1}: NNLO for mass difference [Harlander, **FL** 2022], calculation of remaining matching matrix planned
  - $\langle\tilde{\mathcal{O}}_j\rangle$ exploratory study in progress
- Non-leptonic $|\Delta F| = 1$ decays:
  - C_i: NNLO [Bobeth, Misiak, Urban 2000; Gorbahn, Haisch 2004]
  - ζ_{ij}^{-1}: NNLO, but without penguin operators [Harlander, **FL** 2022], extension to QCD penguin operators planned
  - $\langle\tilde{\mathcal{O}}_j\rangle$: ?

Summary and outlook

Summary:

- Introduced gradient-flow scheme which is valid both on the lattice and perturbatively
- Flowed Wilson coefficients and flowed matrix elements can be computed in different regularization schemes, e.g. perturbatively and on the lattice
- Discussed automatized setup for perturbative calculations in gradient-flow formalism
- Sketched extraction of observables with example of EMT
- Applied gradient-flow scheme to electroweak Hamiltonian

Outlook:

- Extend to more processes
- Compare to traditional approaches with schemes like RI-SMOM once matrix elements available
- Study extrapolation $t \to 0$, including effects of higher-dimensional operators of $O(t)$ (first study for EMT in [Suzuki, Takaura 2021])

Final words

While the gradient flow might not provide the most precise prediction for every lattice computation, it can be applied (almost?) everywhere, i.e. especially to problems inaccessible by lattice calculations so far

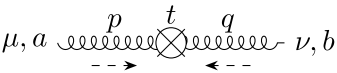
Gluon action density

- Simple first observable: vacuum expectation value of gluon action density

$$E(t,x) \equiv \frac{1}{4} G^a_{\mu\nu}(t,x) G^a_{\mu\nu}(t,x),$$

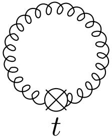
$$G^a_{\mu\nu}(t,x) = \partial_\mu B^a_\nu(t,x) - \partial_\nu B^a_\mu(t,x) + f^{abc} B^b_\mu(t,x) B^c_\nu(t,x)$$

- Feynman rules like



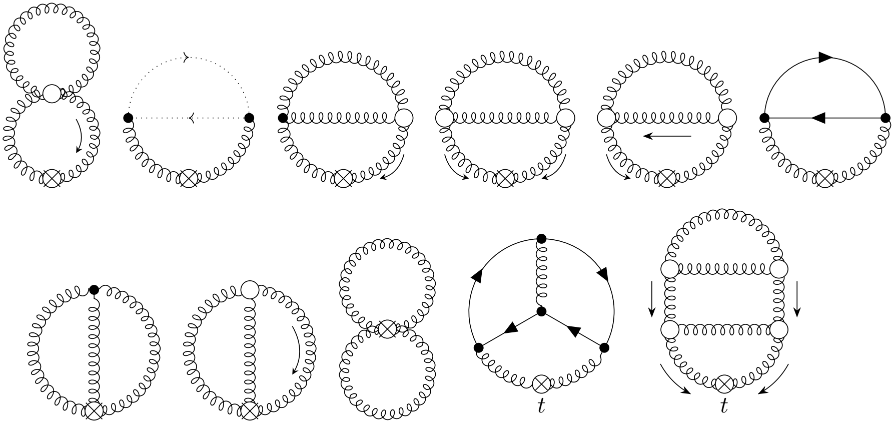
$$= -g^2 \delta^{ab} \big(\delta_{\mu\nu} p \cdot q - p_\mu q_\nu\big)$$

$$\langle E(t) \rangle|_{\mathrm{LO}} =$$



$$= \frac{3\alpha_s}{4\pi t^2} \frac{N_A}{8}$$

Sample Feynman diagrams for $\langle E(t) \rangle$ at higher orders



Integration-by-parts relations

- After tensor reduction, we end up with many scalar integrals of the form

$$I(\{t_f^{\mathrm{up}}\}, \{T_i\}, \{a_i\}) = \left(\prod_{f=1}^{F} \int_0^{t_f^{\mathrm{up}}} \mathrm{d}t_f\right) \int_{k_1,\ldots,k_L} \frac{\exp[-(T_1 q_1^2 + \cdots + T_N q_N^2)]}{q_1^{2a_1} \cdots q_N^{2a_N}}$$

with q_i linear combinations of k_j and T_i linear combinations of t_j, e.g. $q_1 = k_1 - k_2$ and $T_1 = t + 2t_1 - t_3$

Integration-by-parts relations

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- After tensor reduction, we end up with many scalar integrals of the form

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  with q_i linear combinations of k_j and T_i linear combinations of t_j, e.g. $q_1 = k_1 - k_2$ and $T_1 = t + 2t_1 - t_3$
- Chetyrkin and Tkachov observed [Tkachov 1981; Chetyrkin, Tkachov 1981]

$$\int_{k_1,\dots k_L}\frac{\partial}{\partial k_i^{\mu}}\left(\tilde{q}_j^{\mu}\frac{1}{P_1^{a_1}\dots P_N^{a_N}}\right) = 0$$

$\Rightarrow$ Linear relations between Feynman integrals

- Can easily be adopted to gradient-flow integrals
- Additional new relations for gradient-flow integrals:

$$\int_0^{t_f^{\mathrm{up}}}\mathrm{d}t_f\,\partial_{t_f}F(t_f,\dots) = F(t_f^{\mathrm{up}},\dots) - F(0,\dots)$$

Laporta algorithm

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- Schematically integration-by-parts read

$$0 = (d - a_1) I(a_1, a_2, a_3) + (a_1 - a_2) I(a_1 + 1, a_2 - 1, a_3) + (2a_3 + a_1 - a_2) I(a_1 + 1, a_2, a_3 - 1)$$

- Rarely possible to find general solution like

$$I(a_1, a_2, a_3) = a_1 I(a_1 - 1, a_2, a_3) + (d + a_1 - a_2) I(a_1, a_2 - 1, a_3) + 2a_3 I(a_1, a_2, a_3 - 1)$$

- Instead set up system of equations and solve it [Laporta 2000] :
  - Insert seeds $\{a_1 = 1, a_2 = 1, a_3 = 1\}, \{a_1 = 2, a_2 = 1, a_3 = 1\}, \ldots$:

$$\begin{aligned} 0 &= (d-1) I(1,1,1) + I(2,1,0), \\ 0 &= (d-2) I(2,1,1) + I(3,0,1) - I(3,1,0), \\ &\vdots \end{aligned}$$

  - Solve with Gaussian elimination

⇒ Express integrals through significantly smaller number of master integrals

$\langle E(t)\rangle$ through NNLO (I)

$$\langle E(t)\rangle = \frac{3\alpha_s}{4\pi t^2}\frac{N_A}{8}\left[1 + \frac{\alpha_s}{4\pi}e_1 + \left(\frac{\alpha_s}{4\pi}\right)^2 e_2 + O(\alpha_s^3)\right] + O(m^2)$$

$$e_1 = e_{1,0} + \beta_0\, L(\mu^2 t)$$

$$e_2 = e_{2,0} + (2\beta_0\, e_{1,0} + \beta_1)\, L(\mu^2 t) + \beta_0^2\, L^2(\mu^2 t)$$

$$L(z) \equiv \ln(2z) + \gamma_E$$

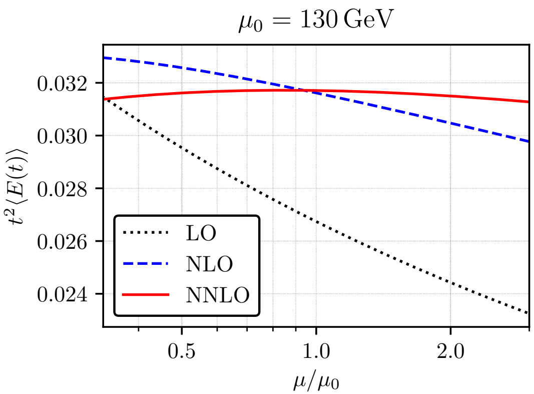
- $\langle E(t)\rangle$ finite after QCD renormalization of α_s only!
- NLO [Lüscher 2010] :

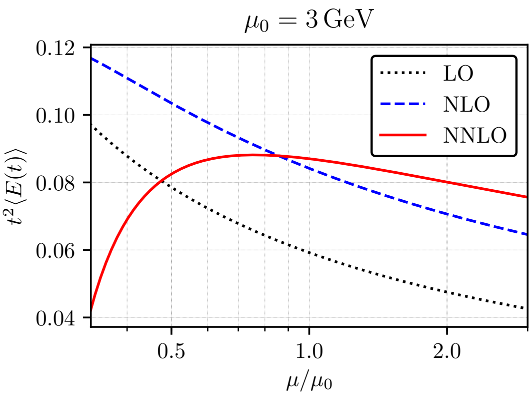
$$e_{1,0} = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right) C_A - \frac{8}{9} n_f T_R$$

- NNLO [Harlander, Neumann 2016; Artz, Harlander, **FL**, Neumann, Prausa 2019] :

$$e_{2,0} = 27.9786\, C_A^2 - (31.5652\ldots)\, n_f T_R C_A + \left(16\zeta(3) - \frac{43}{3}\right) n_f T_R C_F + \left(\frac{8\pi^2}{27} - \frac{80}{81}\right) n_f^2 T_R^2$$

$\langle E(t) \rangle$ through NNLO (II)





- Uncertainty through scale variation reduces from 3.3 % to 0.29 % at 130 GeV and from 19 % to 3.4 % at 3 GeV

Treatment of γ_5 (I)

- In dimensional regularization,

$$\{\gamma_\mu, \gamma_5\} = 0$$

  is incompatible with the trace requirement

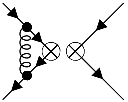
$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5) \neq 0 \xrightarrow[D \to 4]{} 4i\epsilon_{\mu\nu\rho\sigma}$$

- Different prescriptions for γ_5 (NDR, 't Hooft-Veltmann, DREG) lead to different results for scheme-dependent quantities like Wilson coefficients!

Treatment of γ_5 (II)

$$P_2[\mathcal{O}] = \frac{1}{16N_c^2}\,\mathrm{Tr}_{\text{line 1}}\mathrm{Tr}_{\text{line 2}}\,\langle 0|\left(\psi_{4,\mathrm{L}}\gamma_\nu\bar{\psi}_{3,\mathrm{L}}\right)\left(\psi_{2,\mathrm{L}}\gamma_\nu\bar{\psi}_{1,\mathrm{L}}\right)\mathcal{O}|0\rangle\Big|_{p=m=0}$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$$



- The quarks in our operators cannot annihilate due to different flavors

⇒ No γ_5 in traces produced by loop corrections

- Define external quarks in projectors to be left-handed, anticommute γ_5 from operator, and use $P_L^2 = P_L = \frac{1}{2}(1-\gamma_5)$

⇒ No traces with γ_5, simply use naively anticommuting γ_5

- Note: CMM basis avoids γ_5 in traces also for penguin operators ($|\Delta F| = 1$) [Chetyrkin, Misiak, Münz 1997]