Weak Universality
in Lattice Gauge Theories and Spin Systems

Debasish Banerjee

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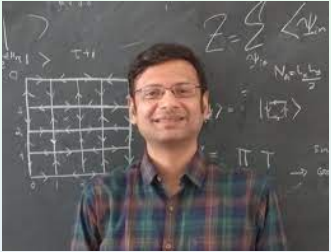
Saha Institute of Nuclear Physics, Kolkata, India

Work done with

(a) Indrajit Sau,
IACS, Kolkata



(b) Arnab Sen
IACS, Kolkata

Based on:

- Sau, Sen, DB; Phys. Rev. Lett. 130, 071901 (2022).
- Sau, Sen, DB (in preparation).

Outline

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Universality and Weak Universality

Lattice Gauge Theories

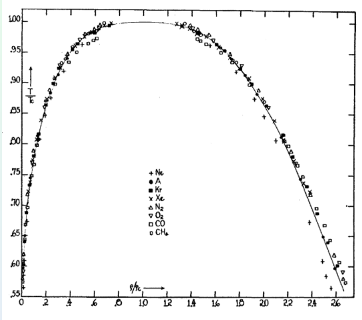
Confinement Deconfinement Transition

Results for Weak Universality

Conclusion

Phase Transitions and Universality

- Transitions between phases of matter are central to their microscopic understanding.
- Many physical systems show identical behaviour of physical quantities (like order parameters, susceptibilities, correlation functions, specific heats) close to a *continuous* phase transition.
- Liquid-gas transitions are the same as magnetic transitions!



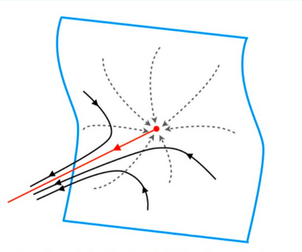
Guggenheim (1945): Universality in the gas-liquid transition of 8 different liquids.

Phase Transitions and Universality

- Wilson's Renormalization Group (1970) unveiled the connections between phase transitions and quantum field theories (QFTs).

- Demonstrated the existence of *fixed points* which govern the low-energy behaviour of QFTs/SFTs.

- Conformal Field Theories typically provide a description of physics close to the critical points.



Flow diagram of QFTs with couplings: relevant operators move the theory away from a fixed point.

Critical Exponents

Universality can be characterized by critical exponents:
($\tau = \frac{T - T_c}{T_c}$ is a reduced co-ordinate, r is spatial distance)

- Specific heat $C \sim |\tau|^{-\alpha}$
- Order Parameter (OP) $\Psi \sim |\tau|^{-\beta}$
- Susceptibility $\chi \sim |\tau|^{-\gamma}$
- Correlation Length $\xi \sim |\tau|^{-\nu}$
- Correlation function $\langle \Psi(r)\Psi(0) \rangle \sim r^{-(d-2+\eta)}$

Not all exponents are independent: $2 - \eta = \frac{\gamma}{\nu}$.

Critical exponents are unique to spatial dimensions, and the global symmetry breaking on either side of the transition.

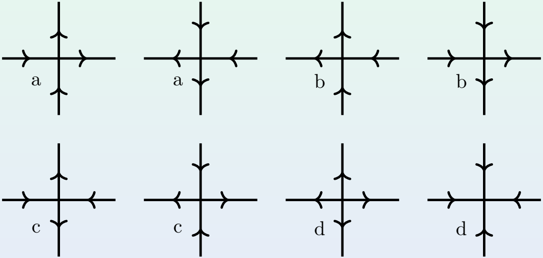
$\longrightarrow$ Universality Classes.

2d Ising model

- $\alpha = 0$
- $\beta = 1/8$
- $\gamma = 7/4$
- $\nu = 1$
- $\eta = 1/4$

Weak Universality

- Symmetry of OP and dimensionality of the system does not uniquely specify the effect of marginal operators on critical exponents.
- The eight-vertex model solved by Baxter (1971) has continuously varying critical exponents.



- Maps to the 2-layer Ising model:

$$H = -J_1 \sum_{\langle ij \rangle} \sigma_i \sigma_j - J_2 \sum_{\langle ij \rangle} \tau_i \tau_j - \lambda \sum_{\langle ij \rangle} \sigma_i \sigma_j \tau_i \tau_j$$

Weak Universality

- Kadanoff and Wegner (1971) attributed this to the existence of a marginal operator, and computed the critical exponents which depend on λ (q is a geometrical factor).

- $\alpha = 2\lambda q$; $\beta = \frac{1-q\lambda}{8}$; $\gamma = \frac{7}{4}(1-q\lambda)$; $\nu = 1-q\lambda$; $\eta = \frac{1}{4}$.

- $\frac{\beta}{\nu} = \frac{1}{8} = \frac{\beta_0}{\nu_0}$; $\frac{\gamma}{\nu} = \frac{7}{4} = \frac{\gamma_0}{\nu_0}$, η: same as the 2d Ising model.

- Suzuki (1974): According to renormalized perturbation theory critical exponents should be computed using renormalized Green's function $G_0(k, T) = (k^2 + \xi^{-2})^{-1}$, where the T dependence enters through ξ.

- Observed in many statistical mechanical and spin systems till date. We report a first occurrence of this phenomena in pure gauge theories.

Outline

Wilson $U(1)$ LGT

- Wilson-Kogut-Susskind: Quantum rotors.

  Wilson(PRD, 1974), Kogut-Susskind (PRD, 1975).

  $U = L^+;\ U^\dagger = L^-;\ E = L^z$

  $[E, U] = U;\ [E, U^\dagger] = -U^\dagger;\ [U, U^\dagger] = 0$

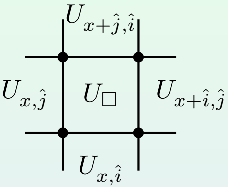
- $H = \frac{e^2}{2}\sum_{x,i} E^2_{x,x+i} - \frac{1}{2e^2}\sum_\Box (U_\Box + U_\Box^\dagger)$.

- States in flux E basis are labeled with (quantized) angular momenta $|0\rangle, |\pm1\rangle, |\pm2\rangle, \cdots$.

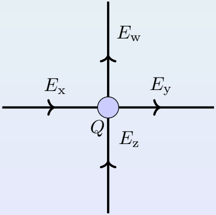
- Gauge fields act as off-diagonal operators: $U|m\rangle = |m+1\rangle, \cdots;\ U^\dagger|m\rangle = |m-1\rangle, \cdots$.

- Gauge Invariance: $[G_x, H] = 0$.

- $G_x = \sum_i (E_{x,\hat{i}} - E_{x-\hat{i},\hat{i}})$



$U_\Box = U_{x,\hat{i}} U_{x+\hat{i},\hat{j}} U^\dagger_{x+\hat{j},\hat{i}} U^\dagger_{x,\hat{j}}$



$E_y + E_w - E_x - E_z = Q$.

The $U(1)$ Gauss Law

- G_x generates local unitary transformations:
  $V = \prod_x \exp(-i\theta_x G_x)$, and $\widetilde{H} = V \cdot H \cdot V^\dagger = H$.
- Fock space splits into superselection sectors with $q_x \in \mathbb{Z}$ for each x.
- Gauge invariant states:

$$\prod_x \exp(-i\theta_x G_x) |\psi\rangle = |\psi\rangle ;$$

$$\prod_x \left(1 - i\theta_x G_x - \frac{\theta_x^2 G_x^2}{2} + \cdots \right) |\psi\rangle = |\psi\rangle .$$

- For a $U(1)$ GT, $0 \leq \theta_x < 2\pi$, $\longrightarrow$ $G_x |\psi\rangle = 0$.
- Total flux coming into a site = total flux leaving the site: typically used in particle physics contexts.

Breaking $U(1) \to \mathbb{Z}_2$ the Gauss Law

- Allow any even charge in the theory $q_x = 0, \pm 2, \pm 4, \cdots$. This is equivalent to modifying the Gauss' Law condition to $G_x |\psi\rangle = q_x |\psi\rangle = 2n_x |\psi\rangle$, where n_x are integers.

- What does this imply for the local gauge symmetry?

$$\prod_x \left(1 - i\theta_x G_x - \frac{\theta_x^2 G_x^2}{2} + \cdots\right) |\psi\rangle = |\psi\rangle \, ,$$

$$\prod_x \left(1 - i\theta_x (2n_x) - \frac{\theta_x^2 (2n_x)^2}{2} + \cdots\right) |\psi\rangle = |\psi\rangle \, ,$$

$$\prod_x \exp(-i2\theta_x \cdot n_x) |\psi\rangle = |\psi\rangle \, .$$

- $\theta_x = \{0, \pi\}$, effectively a $\mathbb{Z}_2$ gauge theory.

Quantum Link $U(1)$ Gauge Theory

- Preserve identical gauge invariance using finite dimensional Hilbert space for single gauge links.

  Horn (PLB, 1981), Orland-Rohrlich (NPB, 1990), Wiese-Chandrasekharan (NPB, 1997).

- Quantum Rotors $\longrightarrow$ Quantum Spins.
- The three operators $E, U, U^\dagger$ can be represented by the generators of a $SU(2)$ algebra: $E = S^z, \quad U = S^+, \quad U^\dagger = S^-$.
- Satisfies $[E, U] = U; \qquad [E, U^\dagger] = -U^\dagger$.
- $[U, U^\dagger] = 2E$ extends the scenarios from those in Wilson-type LGTs.
- Hamiltonian: $H = \frac{e^2}{2} \sum_{x,i} E^2_{x,x+i} - \frac{1}{2e^2} \sum_{\Box} (U_\Box + U^\dagger_\Box)$.
- Gauss' Law: $[G_x, H] = 0$.
- Identical Gauss' Law, Hamiltonian. Acts on different Hilbert space.

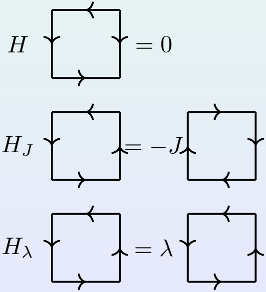
Quantum Links in $(2+1)$-d

Minimum spin representation $S = \frac{1}{2}$ has a two-dimensional local Hilbert space for gauge links.

$$E|\rightarrowtail\rangle = \tfrac{1}{2}|\rightarrowtail\rangle; \quad U|\rightarrowtail\rangle = 0; \quad U^\dagger|\rightarrowtail\rangle = |\leftarrowtail\rangle;$$

$$E|\leftarrowtail\rangle = -\tfrac{1}{2}|\leftarrowtail\rangle; \quad U|\leftarrowtail\rangle = |\rightarrowtail\rangle; \quad U^\dagger|\leftarrowtail\rangle = 0;$$

$$\mathrm{H} = -J\sum_\Box \left(U_\Box + U_\Box^\dagger\right) + \lambda J \sum_\Box \left(U_\Box + U_\Box^\dagger\right)^2$$

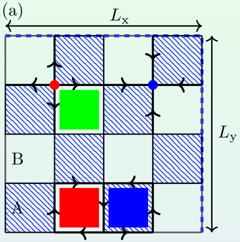


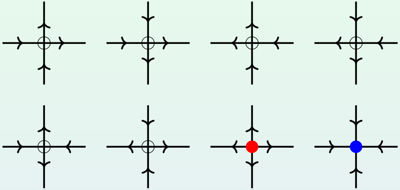
E_{xy}^2 is a constant: drops in H, but enters via G_x.

$Z = \mathrm{Tr}\ \left[e^{-\beta \mathrm{H}}\mathbb{P}_{\mathbb{G}}\right]$; $\mathbb{P}_G = \prod_x \frac{1}{8}\{6\delta(G_x) + \delta(G_x - 2) + \delta(G_x + 2)\}$

$$U_\Box = S_{xy}^+ S_{yz}^+ S_{zw}^- S_{wx}^-$$

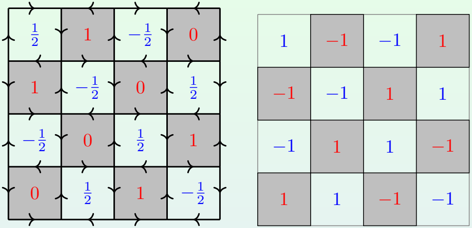
Gauss' Law for $U(1)$ and $\mathbb{Z}_2$ cases





- For the spin-$\frac{1}{2}$ QLM: $q_x = 0, \pm 1, \pm 2$.
- For the $\mathbb{Z}_2$ theory $q_x = \pm 1$ are not allowed.
- Only six states satisfy the $q_x = 0$, and two for $q_x = \pm 2$.
- Temperature controls the density of the $q_x = \pm 2$.
- Annealed disorder: impurities in thermal equilibirum.

Computational Methods

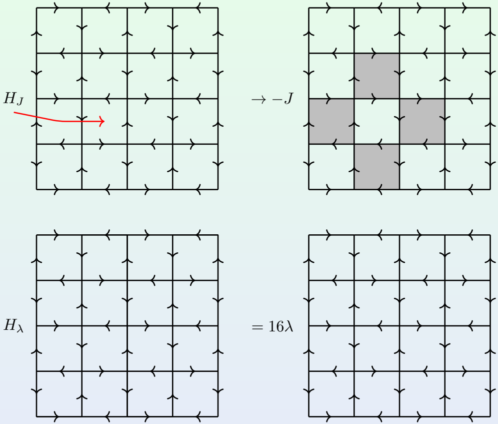


- Cluster Algorithm for simulating the Kramers-Wanner dualized version of the model.

  DB, Jiang, Widmer, Wiese. J. Stat. Mech. (2013) P12010.

- Pure Gauge Theory in $(2+1)$-d maps to a height model in 3d.
- The computation is done on a Euclidean system with $L \times L \times \beta$, where the β is varied, and $L \longrightarrow \infty$ for thermodynamic limit.
- Two-component order parameter (M_A, M_B) capture the ordering of the two sublattices.

Competing orders at $T = 0$



Charges absent at $T = 0$ and physics identical to $U(1)$ theory.

Confined Phase 1: Both sublattices order, both sublattices flippable

Confined Phase 2: One sublattice orders, as resonating plaquettes.

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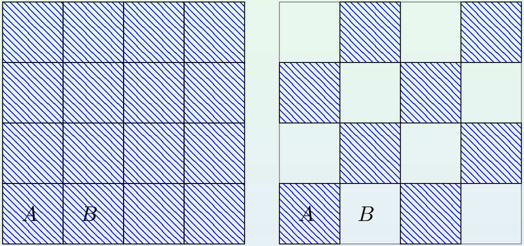

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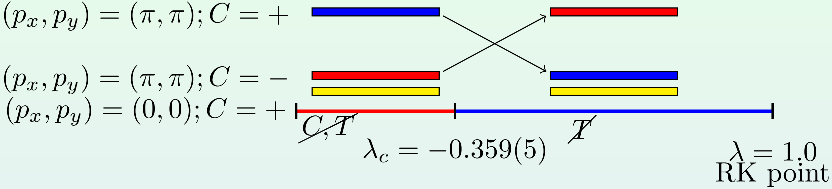
$T = 0$ Phase Diagram



- At $\lambda < \lambda_c$, Spontaneous Symmetry Breaking (SSB) of charge conjugation C and lattice translation T_x, T_y.
- At $\lambda > \lambda_c$, SSB of lattice translation symmetry T_x, T_y.
- At $\lambda \sim \lambda_c$, emergent $SO(2)$ symmetry with a pseudo-Goldstone boson. Weak first order phase transition.

DB, Jiang, Widmer, Wiese. J. Stat. Mech. (2013) P12010.

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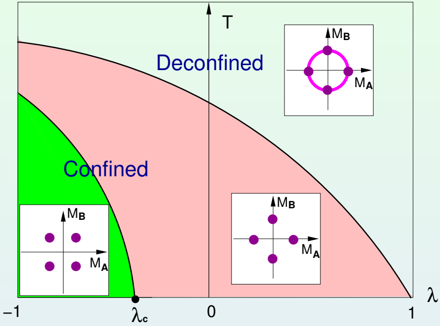

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Global Center Symmetry

- Use the action formulation: consider the $U(1)$ theory first.

$$Z = \mathrm{Tr}\left[\mathrm{e}^{-\beta H}\prod_{\vec{x}}\delta(G_{\vec{x}})\right] = \mathrm{Tr}\left[\mathrm{e}^{-\beta H}\int\prod_{\vec{x}} dA_0(\vec{x})\mathrm{e}^{iA_0(\vec{x})G_{\vec{x}}}\right],$$

where $A_0(\vec{x}) \rightarrow$ Lagrange multiplier for imposing the Gauss Law.

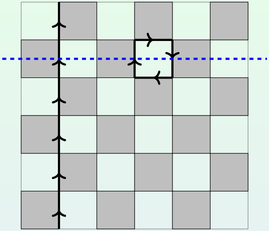
- Use transfer matrix formulation and gauge transformations to convert it to the action formulation:

$$Z = \int\prod_{x,\mu} dU_{x,\mu}\mathrm{e}^{-\beta w\sum_{\square}(U_{\square}+U_{\square}^{\dagger})},$$

where $U_{x,0} = \mathrm{e}^{-iA_0(x)}$ is the temporal gauge link.

- $U_{\square,\mu\nu} = \mathrm{e}^{i\phi_{x,\hat{\mu}}} \cdot \mathrm{e}^{i\phi_{x+\hat{\mu},\hat{\nu}}} \cdot \mathrm{e}^{-i\phi_{x+\hat{\nu},\hat{\mu}}} \cdot \mathrm{e}^{-i\phi_{x,\hat{\nu}}}$

Global Center Symmetry



- Plaquette action invariant under: $U_{\hat{\tau}}(\vec{x}, x_\tau) = zU_{\hat{\tau}}(\vec{x}, x_\tau)$, z is the center of the gauge group.
  $\rightarrow z \in U(1)$ for a $U(1)$ LGT,
  $\rightarrow z \in \mathbb{Z}_2$ for a $\mathbb{Z}_2$ LGT.

- Winding Loops transform non-trivially under center symmetry.
  Polyakov loop: $L = \prod_{\tau=0}^{L_\tau - 1} U_\tau(\vec{x}, x_\tau)$; $\quad L \rightarrow z \cdot L$.

- $\langle L \rangle = 0$, at small T; while $\langle L \rangle \neq 0$ at high T.

- Spontaneous breaking of center symmetry conventionally identified with the confinement-deconfinement transition.

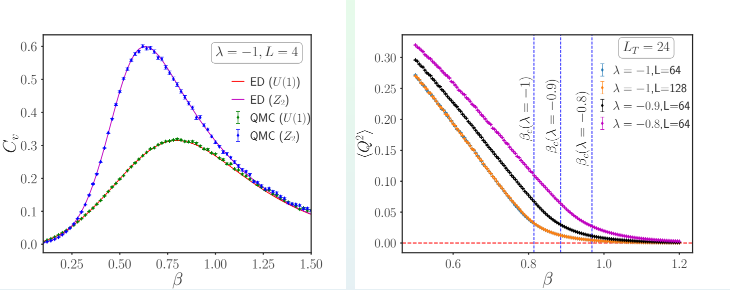
- (M_A, M_B) tracks the center symmetry, and can be measured very accurately via improved estimators $\rightarrow$ cluster sizes.

Svetitsky-Yaffe Conjecture (1982)

- L, and its fluctuations can be used to construct the free energy around the critical point (Landau-Ginzburg approach).
- Confined spatial directions: no long range correlations.
- EFT for the LGT in $(d+1)$-dim is a spin model with short range couplings in d-dim.
- Spins transform under the same symmetry as the center group.
- Using RG, relate the critical phenomena (if any) at the finite-T confinement-deconfinement transition to the critical phenomena of an appropriate spin model.
- Examples in $(2+1)$-d:
  Z_2 LGT $\rightarrow$ same critical exponents as the 2d Ising model,
  $U(1)$ LGT $\rightarrow$ BKT phase transition as the 2d XY model.

Effect of the Charges $q_x = \pm 2$

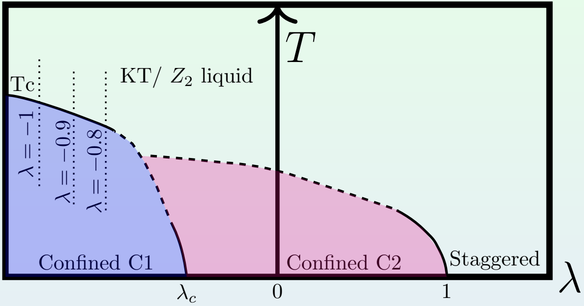
$U(1)$ LGT with $q_x = \pm 2$ cause deviations from the usual $\mathbb{Z}_2$ LGT.



- Energetically, mass gap of the charges $M \sim \lambda J$.
- Theory confining at $T = 0$, charges do not play a role.
- At temperatures for $T \sim \lambda J \sim M$, charges become relevant.
- At $T \to \infty$, $\langle Q^2 \rangle \to 1$.
- For $T \sim T_c$, $\langle Q^2 \rangle \sim \exp(-a|\lambda|/T)$.

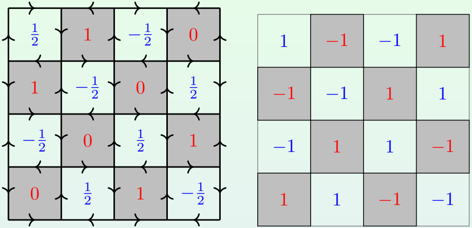
Outline

Finite Temperature Phase Diagram



- Scans in β for $L_t = 4, \cdots, 24$ for $\lambda J = -1.0, -0.9, -0.8$.
- Different L_t necessary for continuum limit (in progress).
- Most results will be on the finest lattice $L_t = 24a$; FSS on spatial lattices upto $L = 512a$.

Magnetization, Susceptibilities, Binder Ratios



- Magnetization $M_X = \frac{1}{L_T} \sum_{\tilde{x}} \eta_{\tilde{x}}^X h_{\tilde{x}}^X$, where, $X = A, B$.
- Susceptibility

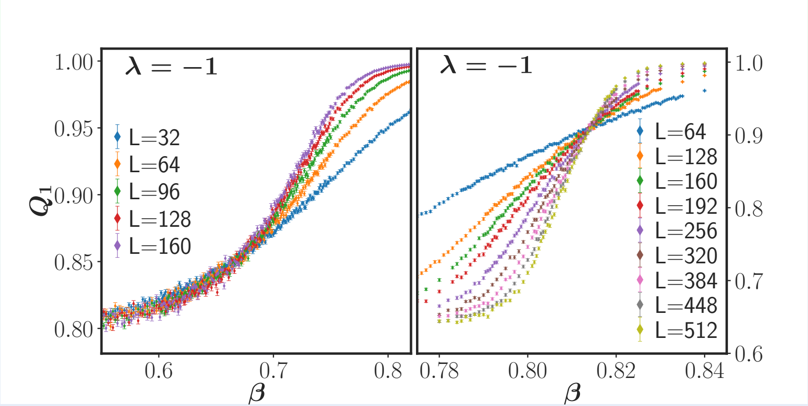
$$\chi_{\text{tot}} = \frac{1}{V} \langle M^2 \rangle \,, \; \chi_{\text{conn}} = \frac{\beta}{V} \sum_X (\langle M_X^2 \rangle - \langle |M_X| \rangle^2),$$

where $M^2 = \sum_{X=A,B} (M_X^2)$, $V = L^2$, and $\beta = \epsilon L_T$.

- Three different Binder cumulants to estimate the critical exponents:

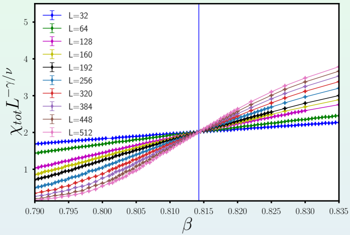
$$Q_1 = \frac{1}{2} \sum_X \frac{\langle |M_X| \rangle^2}{\langle M_X^2 \rangle}; \quad Q_{2a} = 2 - \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2}; Q_{2b} = \frac{3}{2} - \frac{1}{4} \sum_X \frac{\langle M_X^4 \rangle}{\langle M_X^2 \rangle^2}.$$

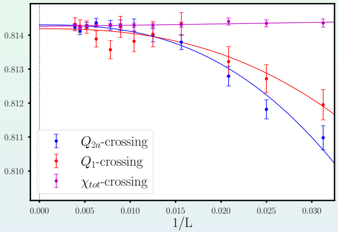
Charges change the critical behaviour



- Following the SY conjecture: the $U(1)$ LGT shows a BKT transition.
- The presence of $q_x = \pm 2$ changes the critical behaviour radically.

Estimating the critical coupling





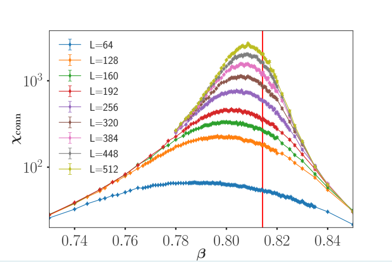
- Crossing points of $\chi_{tot} \cdot L^{-\frac{\gamma}{\nu}}$, Q_1, Q_{2a}, Q_{2b} to estimate $T_c = 1/\beta_c$.
- Fix $\frac{\gamma}{\nu} = \frac{7}{4}$, value for 2d Ising model.
- All observables give consistent estimates of β_c for $L > 100a$.

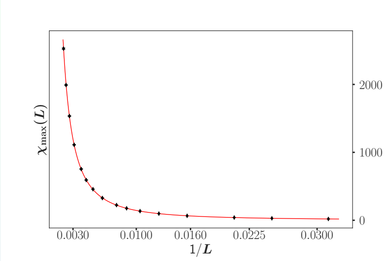
Precision of estimations

L_T	β_c	η	$\nu(Q_1)$	$\nu(Q_{2a})$	$\nu(Q_{2b})$
$\lambda = -1.0$					
24	0.814279(14)	0.2472(9)	1.35(2)	1.38(1)	1.38(2)
16	0.813783(15)	0.2479(9)	1.32(4)	1.34(2)	1.34(4)
8	0.811129(14)	0.2489(8)	1.33(3)	1.31(2)	1.34(3)
4	0.801059(12)	0.2509(8)	1.29(1)	1.31(1)	1.29(2)
2	0.767685(10)	0.2497(7)	1.19(1)	1.20(1)	1.20(1)
$\lambda = -0.9$					
24	0.885292(17)	0.2550 (18)	1.45(3)	1.47(4)	1.45(3)
$\lambda = -0.8$					
24	0.968196(26)	0.2511 (10)	1.64(9)	1.68(4)	1.64(8)

Table: Estimates of β_c, η, ν for different values of L_T, λ.
For 2d Ising, $\eta = \frac{1}{4}$, and $\nu = 1.0$.

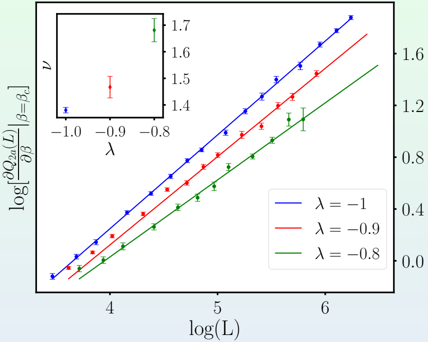
Estimating the critical exponent η





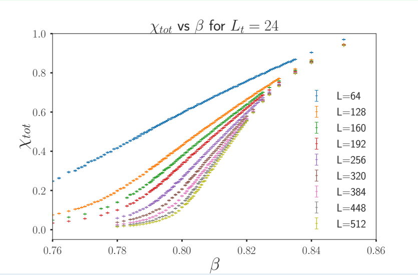
- Scaling of the peak of χ_{conn} to compute η:
  $\chi_{\mathrm{conn,max}}(L) = b \cdot L^{\gamma/\nu} = bL^{2-\eta}$; $\chi^2/DOF \sim 1.3$.
- Extracted from three different bare couplings, λ.
- Independent validation of the assumption $\frac{\gamma}{\nu} = \frac{7}{4}$.

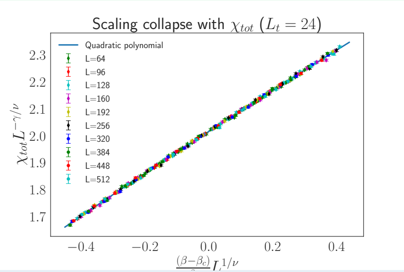
Weak Universality: floating ν



- For a dimensionless phenomenological coupling $R(\beta, L)$:
  $\frac{\partial R(L)}{\partial \beta}\Big|_{\beta_c} = aL^{1/\nu}(1 + bL^{-\omega})$
- Slope of log-log plot of the derivative vs lattice size gives $1/\nu$.
- Precise estimate of β_c essential.
- Consistent values of ν obtained from Q_1, Q_{2a}, Q_{2b}, all > 1.

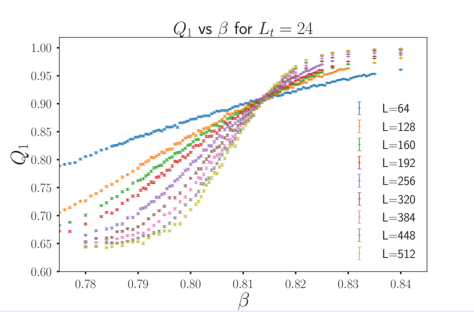
Scaling Collapse of χ_{tot}

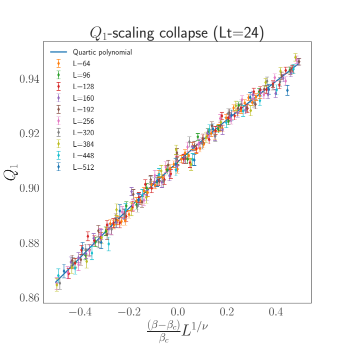




Consistent estimates of the critical exponents are also obtained from scaling collapse.

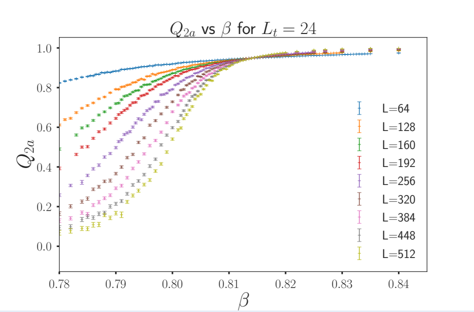
Scaling Collapse of Q_1

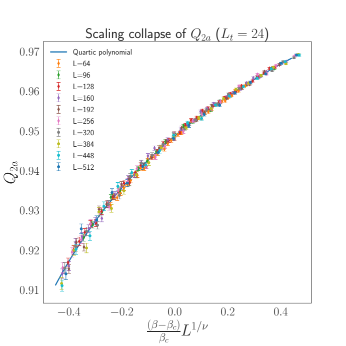




Consistent estimates of the critical exponents are also obtained from scaling collapse.

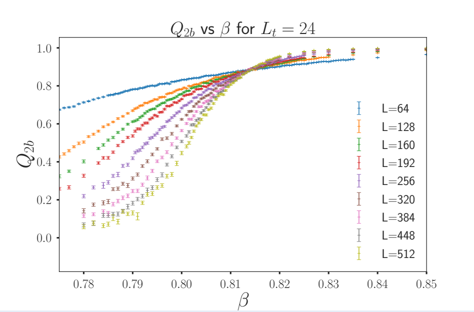
Scaling Collapse of Q_{2a}

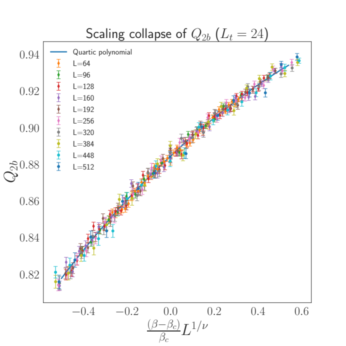




Consistent estimates of the critical exponents are also obtained from scaling collapse.

Scaling Collapse of Q_{2b}





Consistent estimates of the critical exponents are also obtained from scaling collapse.

Outline

Outlook

- Svetitsky-Yaffe conjecture has played a central role in understanding confinement-deconfinement transitions in gauge theories.
- Marginally relevant operators have interesting physics, can induce weak universality.
- We provide an example of this phenomenon from the gauge theory side, thereby validating the conjecture also for exotic scenarios.
- Analytic expression for the marginal operator in the EFT?
- Explore the thermal phase diagram for more negative λ to the Ising limit.
- Critical region around the tricritical point where three phases meet.
- Similar studies for Wilson-type gauge theories could be informative.

Stay tuned! Thank you for your attention!