The rare hyperon decay $\Sigma^+ \to p\ell^+\ell^-$ on the lattice

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Outline

1. Introduction
2. Rare Hyperon Lattice Theory
3. Exploratory Calculation
4. Conclusions/Outlook

Introduction

Rare Decays

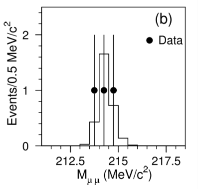
- One avenue to search for BSM physics in is via rare decay processes
- $s \to d$ quark transitions are FCNCs that are good probes for BSM physics:
  - $K^{+/0} \to \pi^{+/0}\ell^+\ell^-$ [hep-lat/2202.08795]
  - $K^{+/0} \to \pi^{+/0}\nu\bar{\nu}$ [hep-lat/1910.10644]
  - $\Sigma^+ \to p\ell^+\ell^-$ [hep-lat/2209.15460]
- To find new physics we need an experimental measurement and a SM prediction to compare with

Experimental Measurement

First observed by HyperCP: [hep-ex/0501014]

- 3 events. Possible new particle $\Sigma^+ \to pP^0, P^0 \to \mu^+\mu^-$ with $m_{P^0} \simeq 214$ MeV
- Known as the HyperCP anomaly
- Attempts to explain with SUSY as a sgoldstino or light pseudoscalar Higgs

  [hep-ph/0509147] [hep-ph/0610362]

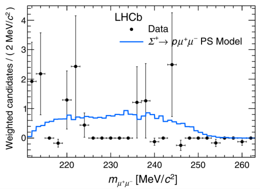


Recently observed at LHCb: [hep-ex/1712.08606]

- 10 events. No evidence of the HyperCP anomaly seen

$$\mathcal{B}(\Sigma^+ \to p\mu^+\mu^-)_{\text{LHCb}} = 2.2^{+1.8}_{-1.3} \times 10^{-8}$$

- Currently working on improved measurements
  + angular observables
  + e^+e^- mode



Phenomenological Calculation

- Existing SM prediction [hep-ph/0506067] [hep-ph/1806.08350] shows rare hyperon decay is long distance dominated by

$$\Sigma^+ \to p\gamma^* \,,\, \gamma^* \to \ell^+\ell^-$$

- Need to compute 4 hadronic form factors a, b, c, d (funcs of q^2) of the hadronic $\Sigma^+ \to p\gamma^*$ amplitude

$$\mathcal{A}_\mu^{rs} = \bar{u}_p^r(\mathbf{p}) \left[i\sigma_{\nu\mu}q^\mu(a + b\gamma_5) + (q^2\gamma_\mu - q_\mu \not{q})(c + d\gamma_5) \right] u_\Sigma^s(\mathbf{k})$$

$$q = k - p$$

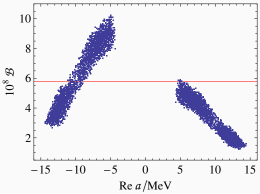
- $\mathrm{Re}\, a$ and $\mathrm{Re}\, b$ at $q^2 = 0$ from experimental measurement of $\Sigma^+ \to p\gamma$
- $\mathrm{Re}\, c$ and $\mathrm{Re}\, d$ from vector meson dominance
- All imaginary parts from ChPT

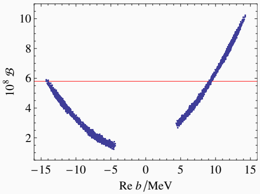
- $\Sigma^+ \to p\gamma$ B.F. and asym measurements constrain $|a|^2 + |b|^2$ and $\mathrm{Re}(ab^*)$
- Combine with $\mathrm{Im}\, a$ and $\mathrm{Im}\, b$ from ChPT gives 4-fold ambiguity

$$\mathrm{Re}\, a = \pm 13.3\mathrm{MeV} \quad , \quad \mathrm{Re}\, b = \mp 6.0\mathrm{MeV}$$
$$\text{or} \quad \mathrm{Re}\, a = \pm 6.0\mathrm{MeV} \quad , \quad \mathrm{Re}\, b = \mp 13.3\mathrm{MeV}$$

- Gives rise to large range in SM prediction

$$1.6 \times 10^{-8} < \mathcal{B}(\Sigma^+ \to p\mu^+\mu^-)_{\mathrm{SM}} < 9.0 \times 10^{-8}$$





[hep-ph/1806.08350]

- Recent 4.2σ discrepancy in $\mathcal{B}(\Sigma^+ \to p\gamma)$ from BESIII collaboration [hep-ex/2302.13568] (although rare hyperon prediction is not significantly affected)

Lattice Theory

Lattice Theory

- Based on the work [hep-lat/2209.15460]

Prospects for a lattice calculation of the rare decay $\Sigma^+ \to p\ell^+\ell^-$

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- Extend the methodology of the rare kaon decay $K \to \pi\ell^+\ell^-$ on the lattice [hep-lat/1507.03094] to apply to the rare hyperon decay
- Need to handle the additional spin degree of freedom, and the different spectrum involved
- In this section we assume a continuum space-time and infinite temporal extent (but finite volume)

Minkowski Amplitude

- Long distance part $\Sigma^+ \to p\gamma^*$ given by amplitude

$$\mathcal{A}_\mu^{rs} = \int d^4x \langle p(\boldsymbol{p}), r| \, T[H_W(x) J_\mu(0)] \, \big|\Sigma^+(\boldsymbol{k}), s\big\rangle$$

- J_μ is the Electromagnetic current
- H_w is the $s \to d$ effective weak Hamiltonian

$$H_W = \frac{G_f}{\sqrt{2}} V_{us} V_{ud}^* \left[C_1(Q_1^u - Q_1^c) + C_2(Q_2^u - Q_2^c) + \ldots \right]$$

with 4-quark operators

$$Q_1^q = (\bar{d}\gamma^{L\mu} s)(\bar{q}\gamma_\mu^L q) \qquad\qquad Q_2^q = (\bar{d}\gamma^{L\mu} q)(\bar{q}\gamma_\mu^L s)$$

- Wilson coefficients $C_{i>2}$ suppressed by factor $\frac{V_{ts}V_{td}}{V_{us}V_{ud}} \sim 10^{-3}$

Minkowski Amplitude

- Amplitude definition

$$\mathcal{A}_\mu^{rs} = \int d^4x \, \langle p(\boldsymbol{p}), r| \, T[H_W(x) J_\mu(0)] \left|\Sigma^+(\boldsymbol{k}), s\right\rangle$$

- Form factor decomposition

$$\mathcal{A}_\mu^{rs} = \bar{u}_p^r(\boldsymbol{p}) \left[i\sigma_{\nu\mu} q^\nu (a + b\gamma_5) + (q^2\gamma_\mu - q_\mu \not{q})(c + d\gamma_5) \right] u_\Sigma^s(\boldsymbol{k})$$

- Spectral representation

$$\mathcal{A}_\mu^{rs} = -i \int_0^\infty d\omega \left(\frac{\rho_\mu^{rs}(\omega)}{\omega - E_\Sigma(\boldsymbol{k}) - i\epsilon} + \frac{\sigma_\mu^{rs}(\omega)}{\omega - E_p(\boldsymbol{p}) - i\epsilon} \right)$$

- In finite volume spectral functions have the form

$$\rho_\mu^{rs}(\omega)_L = \sum_\alpha \frac{\delta(\omega - E_\alpha(\boldsymbol{k}))}{2E_\alpha(\boldsymbol{k})} \langle p(\boldsymbol{p}), r| \, J_\mu \, |E_\alpha(\boldsymbol{k})\rangle_L \, \langle E_\alpha(\boldsymbol{k})| \, H_W \, |\Sigma(\boldsymbol{k}), s\rangle_L$$

$$\sigma_\mu^{rs}(\omega)_L = \sum_\beta \frac{\delta(\omega - E_\beta(\boldsymbol{p}))}{2E_\beta(\boldsymbol{p})} \langle p(\boldsymbol{p}), r| \, H_W \, |E_\beta(\boldsymbol{p})\rangle_L \, \langle E_\beta(\boldsymbol{p})| \, J_\mu \, |\Sigma(\boldsymbol{k}), s\rangle_L$$

Euclidean Correlators

- In a finite Euclidean space-time have access to 4-point function

$$\Gamma^{(4)}_{\mu}(t_p, t_H, t_\Sigma) = \int d^3\mathbf{x} \, \langle \psi_p(t_p, \mathbf{p}) \, H_W(t_H, \mathbf{x}) J_\mu(0) \, \bar{\psi}_\Sigma(t_\Sigma, \mathbf{k}) \rangle$$

  with unpolarised interpolators ψ_p and ψ_Σ

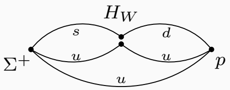
- Amputate external state creation, propagation and annihilation (assuming ground state dominance)

$$\hat{\Gamma}^{(4)}_{\mu}(t_H) = \frac{E_\Sigma E_p}{Z_p Z^*_\Sigma m_\Sigma m_p} e^{+E_p t_p} e^{-E_\Sigma t_\Sigma} \, \Gamma^{(4)}_{\mu}(t_p, t_H, t_\Sigma)$$
$$= \int_0^\infty d\omega \, \begin{cases} \widetilde{\rho}_\mu(\omega)_L \, e^{-(E_\Sigma - \omega) t_H} & \text{for} \quad t_H < 0 \\ \widetilde{\sigma}_\mu(\omega)_L \, e^{-(\omega - E_p) t_H} & \text{for} \quad t_H > 0 \end{cases}$$

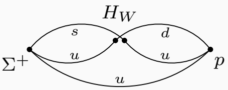
- Dirac matrix valued spectral densities

$$\widetilde{\rho}_\mu(\omega)_L = \frac{\sum_{rs} u^r_p \rho^{rs}_\mu(\omega)_L \bar{u}^s_\Sigma}{4 m_p m_\Sigma} \,, \text{ etc}$$

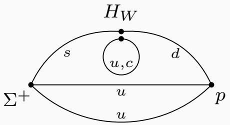
To compute these correlators need to compute Wick contractions topologies:



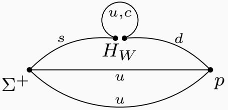
C_{sd}



C_{su}



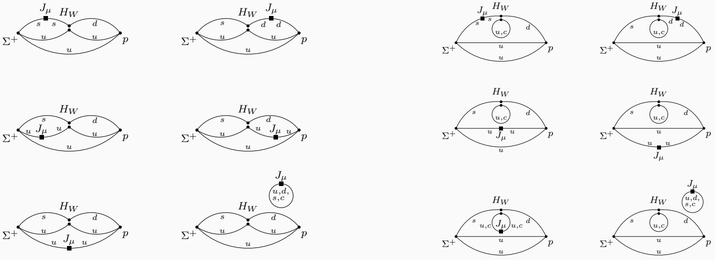
E



S

Referred to as the Non-Eye (top) and Eye (bottom) type diagrams

4-point function requires a current insertion on each leg (and disconnected diagram)



etc

Integrated Correlator

- Integrate amputated 4-point function within window $t_H \in [-T_a, T_b]$

$$
\begin{aligned}
I_\mu^{(4)}(T_a, T_b) &= -i \int_{-T_a}^{T_b} dt_H \, \hat{\Gamma}_\mu^{(4)}(t_H) \\
&= -i \int_0^\infty d\omega \left[\widetilde{\rho}_\mu(\omega)_L \, \frac{1 - e^{-(\omega - E_\Sigma)T_a}}{\omega - E_\Sigma} + \widetilde{\sigma}_\mu(\omega)_L \, \frac{1 - e^{-(\omega - E_p)T_b}}{\omega - E_p} \right] \\
&= \widetilde{F}_\mu + T_a, T_b \text{ exp terms}
\end{aligned}
$$

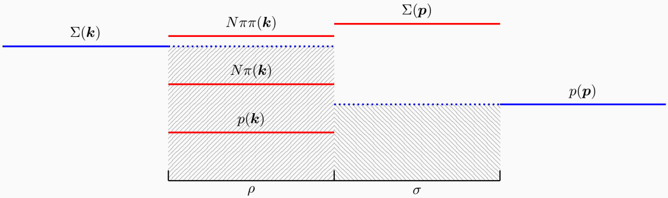
- F_μ is the FV estimator: $\mathcal{A}_\mu = F_\mu + \text{FV corrections}$ (see later)

$$
\widetilde{F}_\mu = -i \int_0^\infty d\omega \left[\frac{\widetilde{\rho}_\mu(\omega)_L}{\omega - E_\Sigma} + \frac{\widetilde{\sigma}_\mu(\omega)_L}{\omega - E_p} \right]
$$

- Remove T_b exp terms by taking $T_b \to \infty$
- $T_a \to \infty$ limit blows up for region of ρ_μ spectrum with $\omega < E_\Sigma$

Growing Exponentials

$$I^{(4)}_{\mu}(T_a, T_b) = -i\int_0^{\infty} d\omega \left[\widetilde{\rho}_{\mu}(\omega)_L \frac{1-e^{-(\omega-E_{\Sigma})T_a}}{\omega - E_{\Sigma}} + \widetilde{\sigma}_{\mu}(\omega)_L \frac{1-e^{-(\omega-E_p)T_b}}{\omega - E_p}\right]$$



Example physical point intermediate spectra with $\boldsymbol{k} = \boldsymbol{0}$ and $|\boldsymbol{p}| = 400$ MeV

- Must remove growing exponentials before taking the $T_a \to \infty$ limit
- In practice might also have to remove slowly decaying exponentials to improve convergence as $T_a,\ T_b \to \infty$

Intermediate state removal: (Pseudo-)scalar shift

- Can also remove a single growing exponential via a (pseudo-)scalar operator shift to H_w. Amplitude invariant due to chiral Ward identities [hep-lat/1212.5931]

$$H'_W = H_W - c_S \bar{d}s - c_P \bar{d}\gamma_5 s \quad \Rightarrow \quad \mathcal{A}'_\mu = \mathcal{A}_\mu$$

- Choose c_S and c_P such that a single matrix element vanishes

$$\langle p(\boldsymbol{k})| H'_W |\Sigma(\boldsymbol{k})\rangle = \bar{u}_p \left[(a_H - c_S a_S) + (b_H - c_P b_P)\gamma_5\right] u_\Sigma = 0$$

$$\therefore \qquad c_S = \frac{a_H}{a_S} \qquad \text{and} \qquad c_P = \frac{b_H}{b_P}$$

- In general, need both operators to remove intermediate state unlike in the rare Kaon decay [hep-lat/1507.03094])
- Σ at rest ($\boldsymbol{k} = \boldsymbol{0}$) is a special kinematic point as it removes need for pseudoscalar shift since

$$\bar{u}^r_p(\boldsymbol{0})\gamma_5 u^s_\Sigma(\boldsymbol{0}) = 0$$

Intermediate state removal: Explicit subtraction

- Can construct the problematic exponentials and explicitly remove them from the integrated 4-point function
- Requires energies and matrix elements from 3-point functions of H_W and J_μ
- Example: single proton intermediate state

$$\frac{1}{2E_p(\mathbf{k})} \langle p(\mathbf{p})| J_\mu |p(\mathbf{k})\rangle \langle p(\mathbf{k})| H_W |\Sigma(\mathbf{k})\rangle \frac{e^{-(E_\Sigma(\mathbf{k})-E_p(\mathbf{k}))T_a}}{E_\Sigma(\mathbf{k}) - E_p(\mathbf{k})}$$
$$\sim \hat{\Gamma}^{(3)}_{J_\mu} \cdot \hat{\Gamma}^{(3)}_{H_W} \frac{e^{-(E_\Sigma(\mathbf{k})-E_p(\mathbf{k}))T_a}}{E_\Sigma(\mathbf{k}) - E_p(\mathbf{k})}$$

- Can in theory be applied to all intermediate states
- Once all problematic states are removed and the $T_a, T_b \to \infty$ limit is taken we get the FV estimator F_μ

Finite Volume Corrections

- Get the ∞-volume amplitude with the FV correction

$$\mathcal{A}^{rs}_{\mu} = F^{rs}_{\mu} + \Delta F^{rs}_{\mu}$$

- Process similar to the $\pi\pi$ states in $K_L - K_S$ mass difference calculation

[hep-lat/1504.01170]

- Powerlike finite volume effects from multiparticle states with $E_{\alpha}(\boldsymbol{k}) < E_{\Sigma}(\boldsymbol{k})$ or $E_{\beta}(\boldsymbol{p}) < E_{p}(\boldsymbol{p})$.
- These are exactly the intermediate states with growing exponentials (aside from the single proton state)

- For the rare hyperon decay need to take care of:

Parity, $m_N \neq m_{\pi}$, non-zero spin

- The index space of ∞ volume states: ($S = \frac{1}{2}$ omitted)

$$|N\pi, J, l, \mu\rangle$$

Finite Volume Corrections: Parity

- States have good parity in the rest frame $\boldsymbol{k} = \mathbf{0}$ which significantly simplifies the FV corrections
- 4-quark operators with $(V - A)(V - A)$ structure can be separated into two parity sectors ($VV + AA$ and $VA + AV$)
- Weak Hamiltonian decomposes into parity conserving and changing parts

$$H_W = H_W^+ + H_W^-$$

- Gives rise to different parity intermediate states $|E_n, \mathbf{0}, \pm\rangle_L$ that overlap with the $H_W^\pm \left|\Sigma^+, \mathbf{0}\right\rangle_L$ states transforming in $G_1^\pm$ irreps
- Can therefore do separate FV corrections for each parity sector

Finite Volume Corrections

- From lattice 4-point func get finite volume estimator

$$F_\mu = -i \int_0^\infty \left(\frac{\rho(\omega)_L}{\omega - E_\Sigma} + \frac{\sigma(\omega)_L}{\omega - E_p} \right)$$

$$\int_0^\infty d\omega \frac{\rho(\omega)_L}{\omega - E_\Sigma} = \sum_n \frac{C_{n,\mu}}{2E_n(L)(E_n(L) - E_\Sigma)}$$

- Has poles when $E_n(L) = E_\Sigma$ that are not in the amplitude
- Get amplitude with finite volume corrections

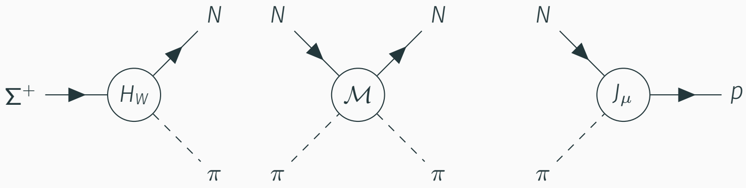
$$\mathcal{A}_\mu = F_\mu + \Delta F_\mu$$

- ΔF_μ must have **exactly** the same poles for them to cancel properly

- Correction is given by [hep-lat/1911.04036]

$$\Delta F^{rs}_{\mu}(\boldsymbol{k},\boldsymbol{p})_L = i\mathcal{A}^{r}_{J_\mu}(E_\Sigma,\boldsymbol{k},\boldsymbol{p})\cdot\mathcal{F}(E_\Sigma,\boldsymbol{k},L)\cdot\mathcal{A}^{s}_{H_W}(E_\Sigma,\boldsymbol{K})$$

$$\mathcal{F}(E,\boldsymbol{P},L) = \frac{1}{F(E,\boldsymbol{P},L)^{-1}+\mathcal{M}(E_{\mathrm{cm}})}$$



- Need to calculated these ∞ volume amplitudes and scattering
- $F(E,\boldsymbol{P},L)$ is known for non-degenerate masses and non-zero spin

- $N\pi$ scattering amplitude in CM frame in terms of phase shifts $\delta_{Jl}(p)$

$$\mathcal{M}(E_{\text{cm}})_{J'l'\mu',Jl\mu} = \delta_{J'J}\delta_{l'l}\delta_{\mu'\mu}\frac{8\pi E_{\text{cm}}}{p\cot\delta_{Jl}(p) - ip}$$

- Near the finite volume energies $E_n(L)$, $\mathcal{F}$ becomes rank 1 matrix with residues

$$\lim_{E\to E_n(L)} \frac{E - E_n(L)}{F(E,\boldsymbol{P},L)^{-1} + \mathcal{M}(E_{\text{cm}})} = \varepsilon^{(n),\text{in}} \otimes \varepsilon^{(n),\text{out}}$$

- Relating the finite and ∞ volume amplitudes

$$\langle E_n, \boldsymbol{k}| H_W \left|\boldsymbol{\Sigma}^+, \boldsymbol{k}, s\right\rangle_L = \varepsilon^{(n),\text{out}} \cdot \mathcal{A}^s_{H_W}$$

$$\langle p, \boldsymbol{p}, r| J_\mu \left|E_n, \boldsymbol{k}\right\rangle_L = \mathcal{A}^r_{J_\mu} \cdot \varepsilon^{(n),\text{in}}$$

Example: Single Channel $P = 0$

- If only a single channel contributes (and truncate at $l = 1$)

$$\mathcal{M}(E) = \frac{8\pi E}{p} \frac{1}{\cot \delta(E) - i}$$
$$F(E, L) = \frac{p}{8\pi E}(\cot \phi(E, L) + i)$$

- FV correction becomes

$$\Delta F_{\mu}^{rs} = -i \frac{p_{\Sigma}}{8\pi m_{\Sigma}} \mathcal{A}_{J_{\mu}}^{r}(m_{\Sigma}) e^{-2i\delta(m_{\Sigma})} \mathcal{A}_{H_W}^{s}(m_{\Sigma}) \times (\cot[\delta(m_{\Sigma}) + \phi(m_{\Sigma})] + i)$$

- cot corrects real contribution of amplitude
- $+i$ part gives the imaginary component

Expansion about the pole

- For volume tuned exactly at $E_{\bar{n}}(\bar{L}) = E_\Sigma$ we get result analogous to the $\pi\pi$ FV correction in $K_L - K_S$ mass difference calculation [hep-lat/1504.01170]
- Expanding away from this fine tuning by $\delta L = L - \bar{L}$ gives

$$\begin{aligned} F_\mu &= O(\delta L^{-1}) \\ \Delta F_\mu &= O(\delta L^{-1}) \\ F_\mu + \Delta F_\mu &= O(\delta L^{0}) \end{aligned}$$

- Poles cancel as expected
- Go on to calculate the $O(\delta L^0)$ and $O(\delta L^1)$ (see [hep-lat/2209.15460] for details)
- With this we now know how to extract the rare hyperon decay amplitude from the lattice

Exploratory Calculation

Exploratory Calculation: RBC-UKQCD Collaboration

The RBC & UKQCD collaborations

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University of Bern & Lund
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Peter Boyle (Edinburgh)
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Chulwoo Jung
Christopher Kelly
Meifeng Lin
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Stony Brook University
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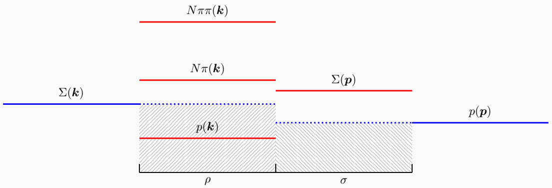
Exploratory Calculation: Ensemble details

- Iwasaki gauge action $\beta = 2.13$
- 2+1f Shamir domain-wall fermions
- $a \simeq 0.1$ fm $\simeq (1785\ MeV)^{-1}$
- Lattice size $24^3 \times 64\ (\times 16)_{L_s}$

[RBC-UKQCD 10.1103/PhysRevD.83.074508]

- $m_\pi \simeq 340\ MeV$
- $m_K \simeq 620\ MeV$
- $m_N \simeq 1200\ MeV$
- $m_\Sigma \simeq 1370\ MeV$



- No multiparticle states in this spectrum below the E_Σ threshold
- No power-like finite volume corrections: $\mathcal{A}_\mu = F_\mu + O(e^{-m_\pi L})$
- No imaginary component by optical theorem
- Only single proton intermediate state grows exponentially

Exploratory Calculation: Measurement Details

- Software Grid/Hadrons
- Kinematics $\boldsymbol{k} = \mathbf{0}$, $\boldsymbol{p} = \frac{2\pi}{L}(1, 0, 0)$
  $\rightarrow q^2 = -0.2\ GeV^2$
- Gauge fixed Gaussian sources
- Time translations (x32)

[github.com/paboyle/Grid]

[github.com/aportelli/Hadrons]

Source-Sink sampling

- Due to contraction method, full volume sum at source and sink for momentum projection requires $\sim 14,000$ solves
- Approximate with sum over N random position samples

[Y. Li et al. 10.1103/PhysRevD.103.014514]

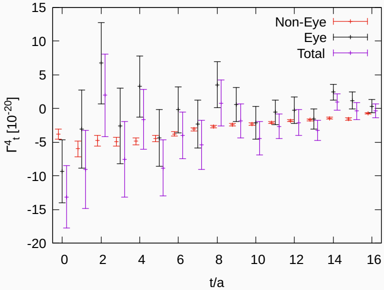
Exploratory Calculation: Eye Diagrams

- Eye diagrams require loop propagators: $S(x|x)\ \forall x$

- Stochastic estimator with $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ noise sources with spatial sparsening of 2 in each dimension
- Improve the signal-per-cost by 2x over full volume noise [hep-lat/2202.08795]
- So far have 1 hit of 16 noise sources measured, and are continuing to add additional hits using AMA approach

Exploratory Calculation: Preliminary Data

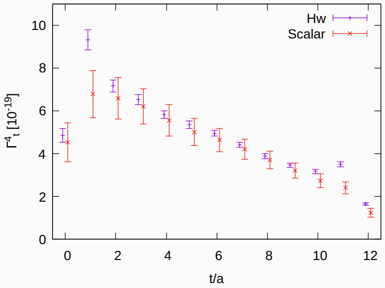
- Temporal component of the 4-point correlator with a source-sink separation $t_f/a = 16$ and e.m. current at $t_J/a = 8$.



- Observe good signal for the non-eye diagrams
- Stochastic estimation of eye diagrams give large errors dominating the total (Non-Eye $<$ Eye)

Exploratory Calculation: Scalar Shift Method

- Can perform the scalar shift to the non-eye diagrams with c_S measured from ratio of 3-point functions
- Comparison of weak Hamiltonian and scalar 4-point function ($\times c_S$)



- No signal observed in the difference with current statistics
- Must remove single proton intermediate state by other methods

Exploratory Calculation: Separation of Spectra

- Can separate the two spectral contributions by splitting the time integral

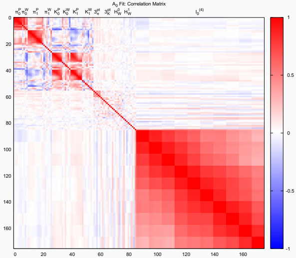
$$I^{\rho}_{\mu}(T_a) = -i\int_{-T_a}^{0} dt_H \hat{\Gamma}^{(4)}_{\mu}(t_H) = -i\int_{0}^{\infty} \widetilde{\rho}_{\mu}(\omega)_L \frac{1-e^{-(\omega-E_\Sigma)T_a}}{\omega - E_\Sigma}$$

$$I^{\sigma}_{\mu}(T_b) = -i\int_{0}^{T_b} dt_H \hat{\Gamma}^{(4)}_{\mu}(t_H) = -i\int_{0}^{\infty} \widetilde{\sigma}_{\mu}(\omega)_L \frac{1-e^{-(\omega-E_p)T_a}}{\omega - E_p}$$

- A similar separation can be made for the ∞ volume amplitude (and FV estimator)

$$\mathcal{A}_{\mu} = \mathcal{A}^{\rho}_{\mu} + \mathcal{A}^{\sigma}_{\mu}$$

- Non-separated integral has redundancy that causes large correlations that destabilise fits (example correlation matrix from rare kaon decay [hep-lat/2202.08795])



- We can extract the form factors directly

$$\widetilde{\mathcal{A}}_\mu = -i \int_0^\infty d\omega \left(\frac{\widetilde{\rho}(\omega)}{(\omega - E_\Sigma)} + \frac{\widetilde{\sigma}(\omega)}{(\omega - E_p)} \right) = \mathbb{P}_p \left[\sigma_{\nu\mu} q_\nu a - (q^2 \gamma_\mu + q_\mu \not{q}) c \right] \mathbb{P}_\Sigma$$

(in + parity sector)

with projector

$$\mathbb{P}_B = \sum_s \frac{u_B^s \bar{u}_B^s}{2m_B} = \frac{-i\not{p}_B + m_B}{2m_B}$$

- Euclidean 4-vectors and γ-matrix definitions used
- Traces give access to combinations of form factors f_μ

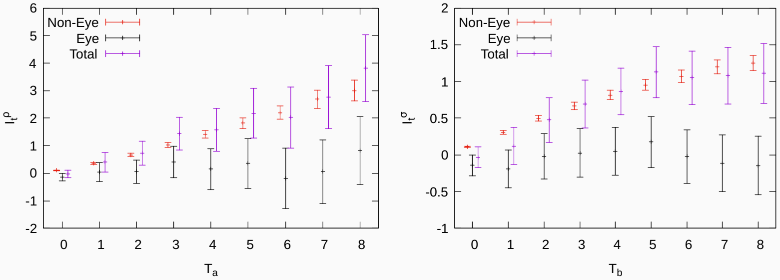
$$\mathrm{Tr}\left[\widetilde{\mathcal{A}}_\mu P^+ \gamma\right] = \zeta_{\mu,\gamma}\, f_\mu$$

- $P^+ = (1 + \gamma_t)/2$ projects positive parity external state
- $\zeta_{\mu,\gamma}$ accounts for artificial γ dependence
- We use the $\mu = t, z$ components

$$\begin{pmatrix} f_t \\ f_z \end{pmatrix} = \begin{pmatrix} 1 & m_\Sigma + m_p \\ m_\Sigma + m_p & q^2 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}$$

Exploratory Calculation: Integrated 4-point functions

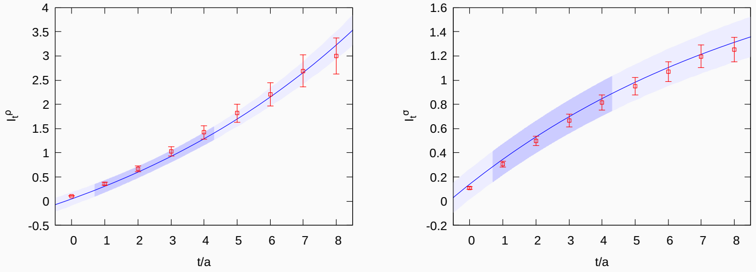
- Integrating (summing) in the two time orderings



- Large fluctuations in the eye diagrams cancel giving Eye $\lesssim$ Non-Eye
- Appears promising that with extra noise hits we can significantly improve results

- Perform a hierarchical fitting procedure with masses and overlap factors fit from 2-point functions
- These are passed as constants to the 4-point function fits

  Example Non-Eye 4-point fits for temporal component and $t_f/a = 16$



- Use fit ansatz with a single intermediate state exponential (energies fixed by m_p and m_Σ from 2-point functions)

Exploratory Calculation: Preliminary Results

Parameter	Result	P-value	$f_t = f_t^\rho + f_t^\sigma$
$f_t^{\rho,\text{NE}}$	2.16(31)	0.70	$-4.7(21.8) \times 10^{-2}$
$f_t^{\sigma,\text{NE}}$	−2.21(21)	0.40	
$f_t^{\rho,\text{Eye}}$	0.20(1.03)	0.72	−0.37(1.21)
$f_t^{\sigma,\text{Eye}}$	−0.57(71)	0.32	
$f_t^{\text{NE}} + f_t^{\text{Eye}}$	-	-	−0.42(1.21)
f_t^ρ	2.52(1.62)	0.69	−0.25(1.75)
f_t^σ	−2.78(92)	0.34	

- Eye and total contributions have very large errors due to stochastic noise
- Non-eye contribution has $10-15\%$ errors on separated spectral components, but has a large cancellation when combined giving large errors
- More investigation needed into the cause of this cancellation (approx. $SU(3)_F$ symmetry?)

- Inverting the linear relation between $f_{t,z}$ and a, b give form factors

Form Factor	Value	(Stat)	
a^{NE}	5	(16)	MeV
c^{NE}	0.009	(30)	
a^{Eye}	−58	(100)	MeV
c^{Eye}	0.034	(173)	
$a^{\mathrm{NE}} + a^{\mathrm{Eye}}$	−53	(100)	MeV
$c^{\mathrm{NE}} + c^{\mathrm{Eye}}$	0.043	(174)	
a	−53	(114)	MeV
c	0.018	(249)	

- For reference phenomenological values at $q^2 = 0$:

$$\mathrm{Re}\, a \sim 10\,\mathrm{MeV} \quad , \quad \mathrm{Re}\, c \sim 10^{-2}$$

- Note all fits made to data with $t_f = 16a \simeq 1.8$ fm

Exploratory Calculation: Preliminary Results

- If we also include data with source-sink separation $t_f = 12\,a \simeq 1.3$ fm (data only available for non-eye diagrams)

Form Factor	Value	(Stat)	
a^{NE}	4	(5)	MeV
c^{NE}	0.030	(9)	

- Start to observe result for the non-eye contribution to the c form factor
- Requires fitting approx 0.3 fm from the source/sink operators
- Will have large uncontrolled excited state contributions that must be addressed

Exploratory Calculation: Improvements

This calculation is currently limited by:

- Stochastic estimation of quark loops: (like the rare kaon decay)
  May be able to reduce loop noise with improved estimators
  e.g. frequency splitting
- Exponential signal-to-noise problem of baryons:
  Would benefit greatly from investigation into exponential variance reduction techniques e.g. multi-level algorithms
- Large cancellation between different intermediate spectra:
  Investigation into its origin may provide methodological improvements

May be possible to achieve a significant result with (excited states accounted for) at the unphysical point via $1/\sqrt{N}$ scaling of additional statistics

At the physical point this would likeley be prohibitaviy expensive. Need improved techniques

Conclusions/Outlook

Conclusions/Outlook

Conclusions

- Have theoretical framework to extract the RH amplitude from the lattice [hep-lat/2209.15460]
- Working towards an exploratory computation with $m_\pi \simeq 340\text{MeV}$
- Errors currently dominated by stochastic loop estimation and large cancellation between both intermediate spectra

Outlook

- RH and RK decays would both benefit from improved loop estimation
- Physical point calculation will likely require baryon variance reduction techniques, and will need to evaluate finite volume corrections

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