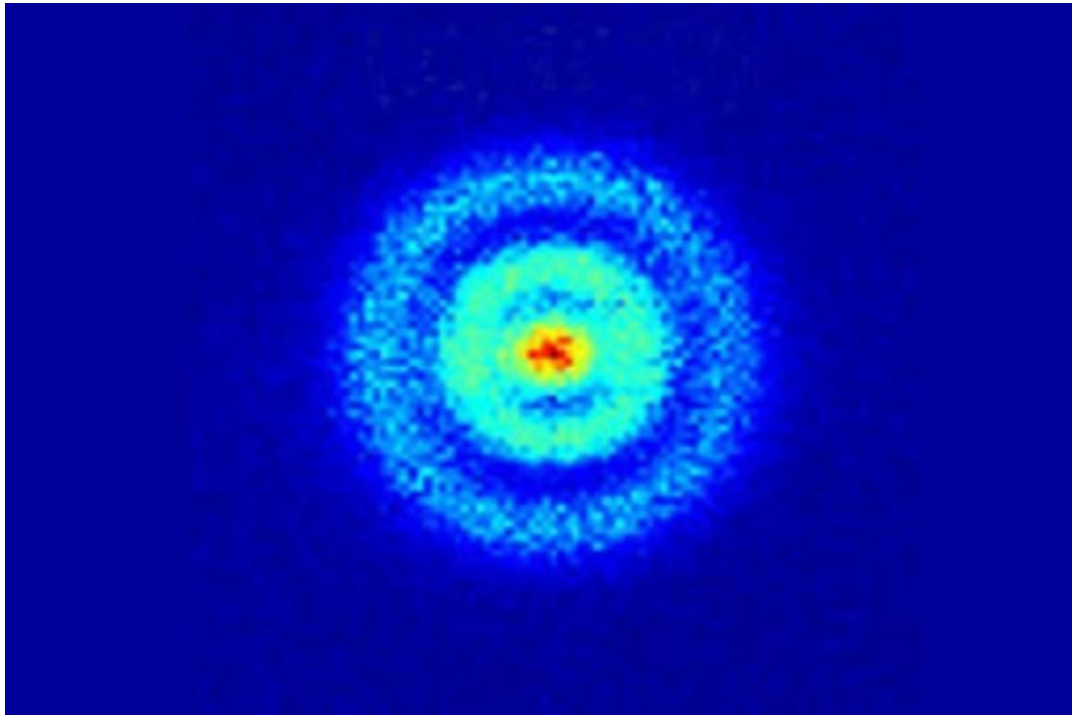
HU Berlin / NIC DESY Zeuthen joint lattice seminar

Lattice QCD calculation of electroweak processes by designing specific weight functions

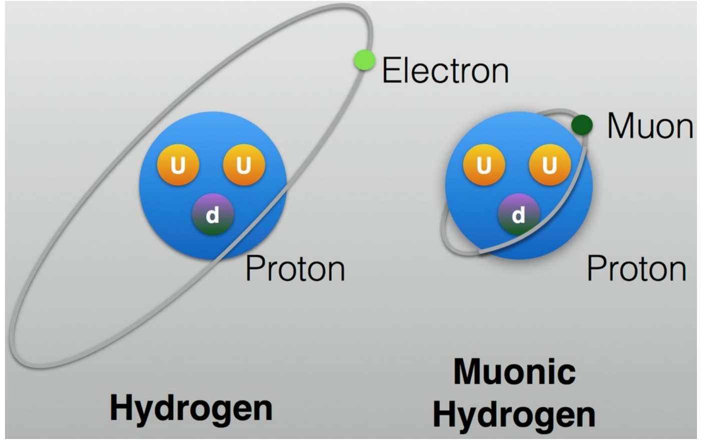
Xu Feng

2023.07.17

Starting from hadron charge radius



Muonic hydrogen



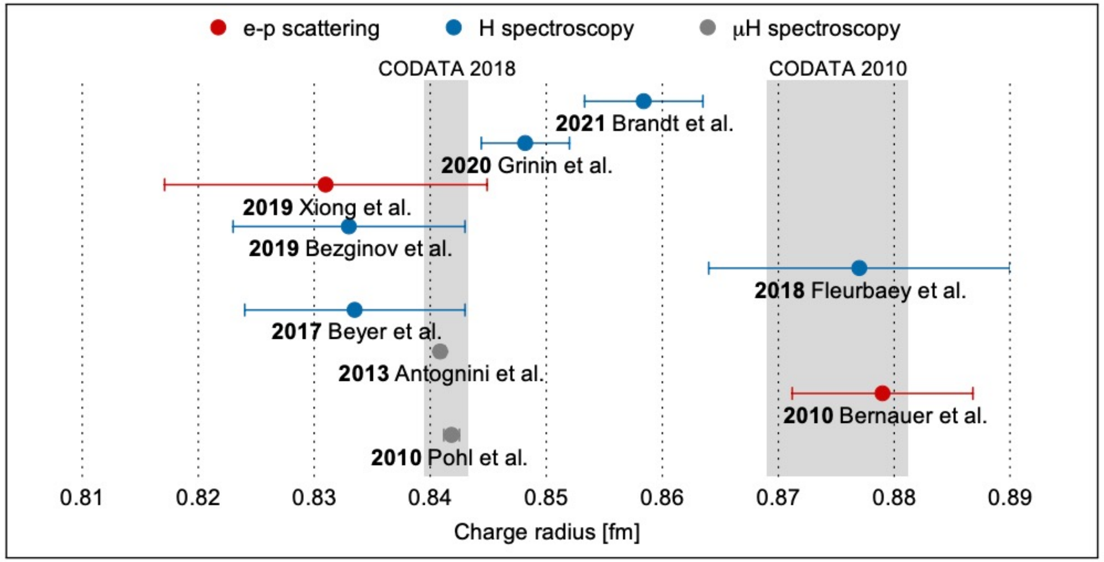
- Muon mass is about 200 times of electron
- Bohr radius for μH is 200 times smaller than H

➢ 2010, proton charge radius from μH

【Nature 466 (2010) 213】

- Precision 10 times better than before
- 4% smaller radius

5σ deviation → Proton size puzzle



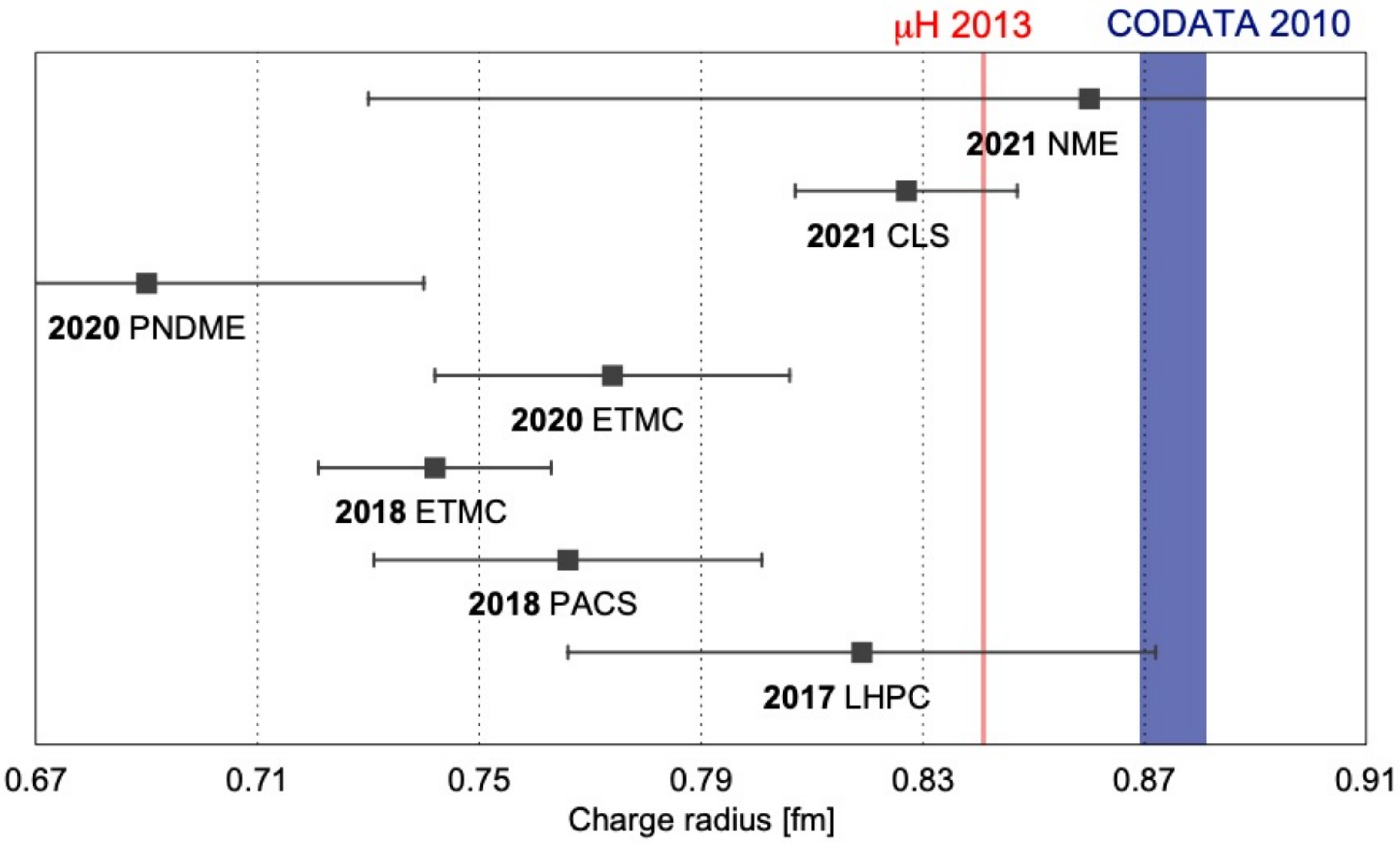
New experimental progress

➢ Still some discrepancies
➢ Consistently shrink the proton size

Puzzle possibly originates from experiments

However, as a fundamental quantity, the size of proton charge radius plays an important role in the theoretical prediction in spectroscopy

Direct lattice QCD calculation of charge radius



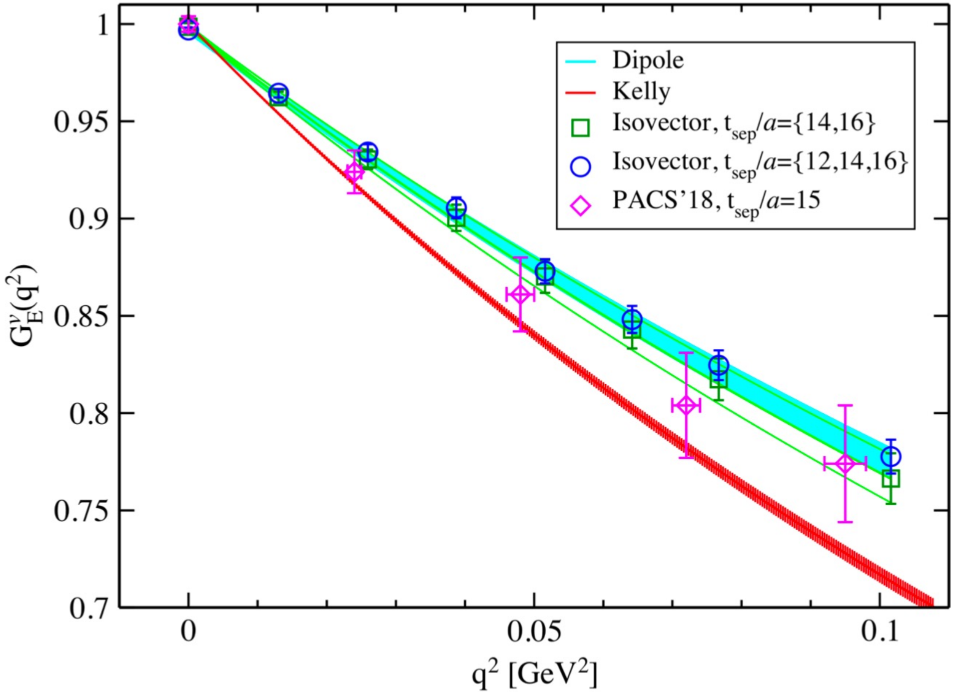
- Charge radius is the derivative of form factor → still hard to achieve ~1% accuracy

Direct lattice QCD calculation of charge radius

➢ Various systematic effects, especially the model dependence

$$\langle H(p_f)|J_\mu|H(p_i)\rangle = \bar{u}(p_f)\left[\gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\right]u(p_i), \quad q^2 = (p_f - p_i)^2$$

- Momenta p_i and p_f on the lattice are always discrete: $\frac{2\pi}{L}n$ ➡ modeling of q^2-dependence to extract charge radius



Fit type	$q^2_{\rm cut}$ [GeV^2]	$t_{\rm sep}/a$	Isovector $\sqrt{\langle r_E^2\rangle}$ [fm]	Isovector χ^2/dof
Linear	0.013	{12, 14, 16}	0.764(26)	⋯
		{14, 16}	0.806(35)	⋯
Dipole	0.102	{12, 14, 16}	0.785(17)	1.2(8)
		{14, 16}	0.806(26)	0.6(6)
Quadrature	0.102	{12, 14, 16}	0.785(19)	1.0(8)
		{14, 16}	0.783(30)	0.7(7)
z-exp ($k_{\rm max} = 3$)	0.102	{12, 14, 16}	0.776(28)	1.2(9)
		{14, 16}	0.796(37)	0.8(8)

[PACS Collaboration used a (10.8 fm)^4 lattice, PRD 2020]

- Model dependence could cause a 3% shift in r_p, e.g. 0.806(26) → 0.783(30)
- Twisted boundary condition can help, but requires more computational resources and still a fit functional form
- Why not calculate the charge radius directly at q^2=0 ➡ A model-independent approach to extract charge radius

XF, Y. Fu, L. Jin, PRD 101 (2020) 051502

Charge radius in the infinite volume

We start with a Euclidean hadronic function in the infinite volume

$$H(x) = \langle 0|A_4(x)J_4(0)|\pi(\vec{0})\rangle$$

The spatial Fourier transform of $H(x)$ yields

$$\tilde{H}(t,\vec{p}) \approx \frac{f_\pi}{2}(E+m_\pi)F_\pi(q^2)e^{-Et}$$

- The derivative of $\tilde{H}(t,\vec{p})$ at $|\vec{p}|^2 = 0$ leads to

$$D(t) \equiv m_\pi^2 \frac{\partial \tilde{H}(t,\vec{p})}{\partial |\vec{p}|^2}\bigg|_{|\vec{p}|^2=0} = -\frac{m_\pi^2}{3!}\int d^3\vec{x}\, |\vec{x}|^2 H(x)$$

- The derivative of the R.H.S. yields

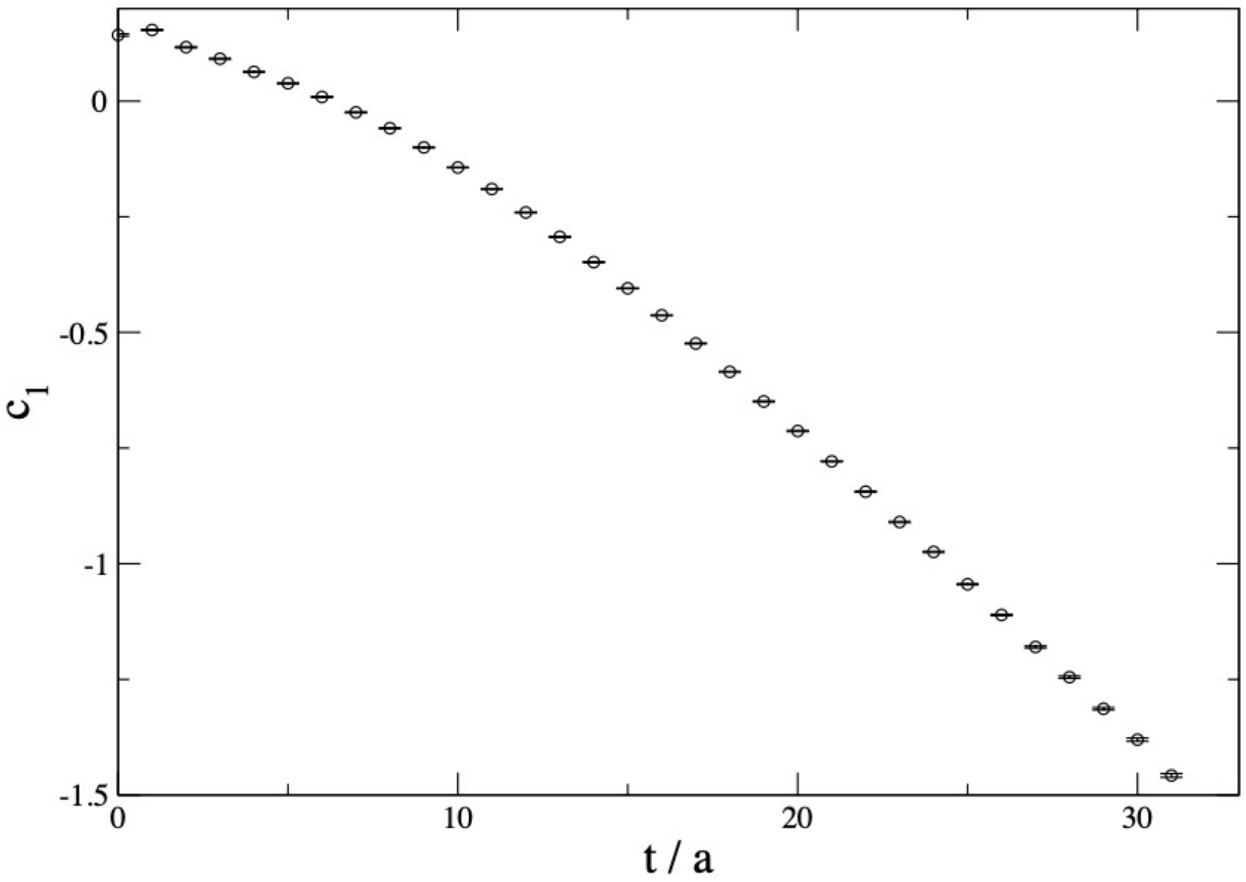
$$\frac{D(t)}{\tilde{H}(t,\vec{0})} = \frac{1}{4} - \frac{m_\pi t}{2} - c_1$$

with $c_1 = \frac{m_\pi^2}{6}\langle r_\pi^2\rangle$ from

$$F_\pi(q^2) = \sum_{n=0}^{\infty} c_n \left(\frac{q^2}{m_\pi^2}\right)^n$$

Significant finite-volume effects

$$c_1 = -\frac{D(t)}{\tilde{H}(t,\vec{0})} + \frac{1}{4} - \frac{m_\pi t}{2}$$



Large finite-volume effects:

$$m_\pi^2|\vec{x}|^2 H(x) \sim m_\pi^2|\vec{x}|^2 \exp(-m_\pi\sqrt{\vec{x}^2+t^2}) \sim 0.53$$

with $\sqrt{\vec{x}^2+t^2} \approx |\vec{x}| \sim L/2 = 2.5$ fm

Treat with finite-volume effects

In a finite box with a size L

$$
\begin{aligned}
D^{(L)}(t) &\equiv -\frac{m_\pi^2}{3!}\sum_{\vec{x}\in\mathbb{L}^3}|\vec{x}|^2 H^{\mathrm{latt}}(x),\\
\tilde{H}^{(L)}(t,\vec{0}) &\equiv \sum_{\vec{x}\in\mathbb{L}^3} H^{\mathrm{latt}}(x)
\end{aligned}
$$

The ratio is now written as

$$
\frac{D^{(L)}(t)}{\tilde{H}^{(L)}(t,\vec{0})} = \sum_{n=0}^{\infty}\beta_n^{(L)}(t)\, c_n
$$

with the coefficients $\beta_n^{(L)}(t)$ known explicitly through

$$
\beta_n^{(L)}(t) = -\frac{m_\pi^2}{3!}\sum_{\vec{x}\in\mathbb{L}^3}|\vec{x}|^2 I_n(x)
$$

$$
I_n(x) = \frac{1}{L^3}\sum_{\vec{p}\in\Gamma}\frac{E+m}{2m}\left(\frac{q^2}{m_\pi^2}\right)^n e^{-(E-m_\pi)t}\cos(\vec{p}\cdot\vec{x})
$$

FV effects incorporated in the weight function

Systematic effects

The ratio is now written as

$$\frac{D^{(L)}(t)}{\tilde{H}^{(L)}(t,\vec{0})} = \sum_{n=0}^{\infty} \beta_n^{(L)}(t)\, c_n$$

with $c_1 = \frac{m_\pi^2}{6}\langle r_\pi^2\rangle$

Note that when $a \to 0$ and $L \to \infty$, all the coefficients $\beta_n^{(L)}$ for $n \geq 2$ vanish

We can consider the contamination from $c_{n\geq 2}$ terms as the systematic effects

In the vector meson dominance model, c_n is given by $(\frac{m_\pi}{m_\rho})^{2n}$

For $n \geq 3$, c_n is estimated to be less than 0.1% of c_1

Error reduction

Define

$$D_k^{(L,\xi)}(t) \equiv (-1)^k \frac{m_\pi^{2k}}{(2k+1)!} \sum_{|\vec{x}|\le \xi L} |\vec{x}|^{2k} H^{\mathrm{latt}}(x)$$

- Choose the integral range ξL to reduce the statistical error

$D_k^{(L,\xi)}(t)$ are related to c_n through

$$\frac{D_k^{(L,\xi)}(t)}{\tilde{H}^{(L)}(t,\vec{0})} \doteq \sum_{n=0}^{\infty} \beta_{k,n}^{(L,\xi)}(t)\, c_n$$

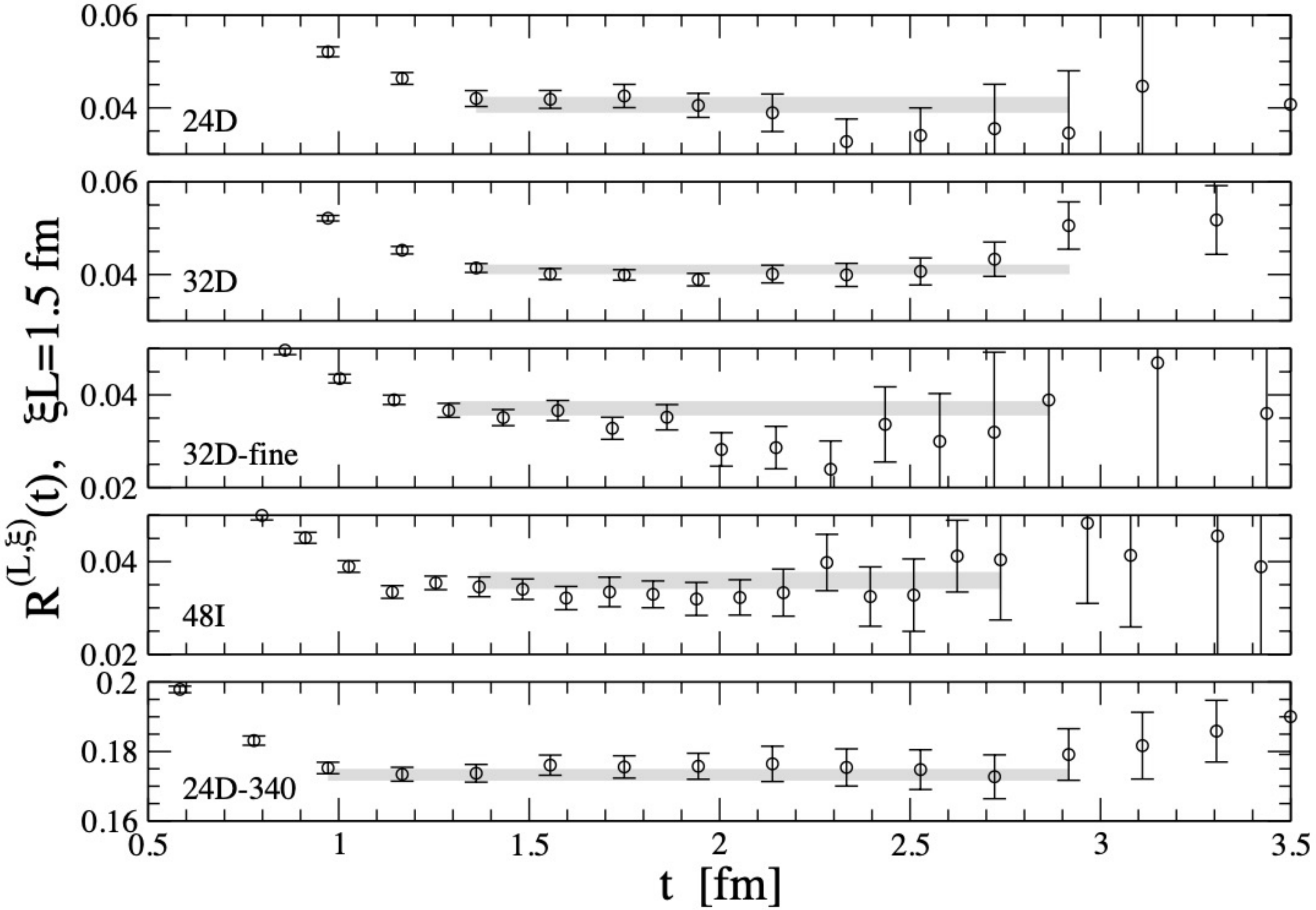
- Use $D_1^{(L,\xi)}$ and $D_2^{(L,\xi)}$ to remove the contamination from c_2

$$R^{(L,\xi)}(t) = \frac{f_1 D_1^{(L,\xi)}(t) + f_2 D_2^{(L,\xi)}(t)}{\tilde{H}^{(L)}(t,\vec{0})} + h$$

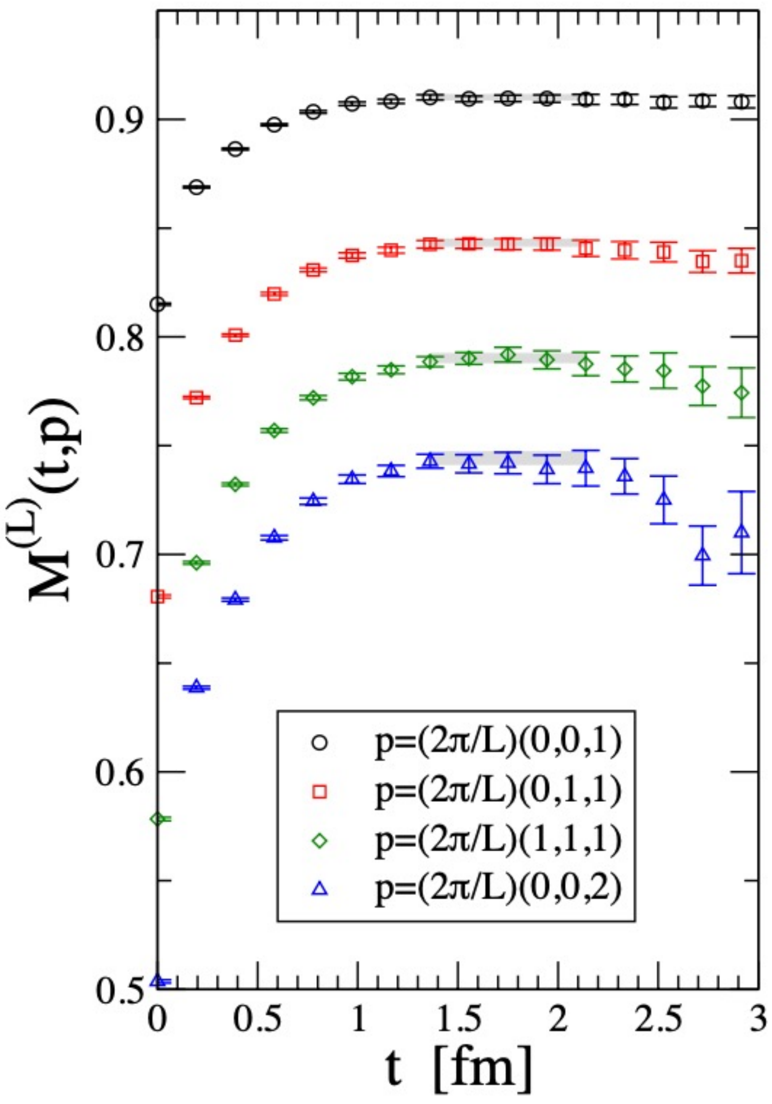
f_1, f_2 and h are chosen to remove the c_0 and c_2 terms

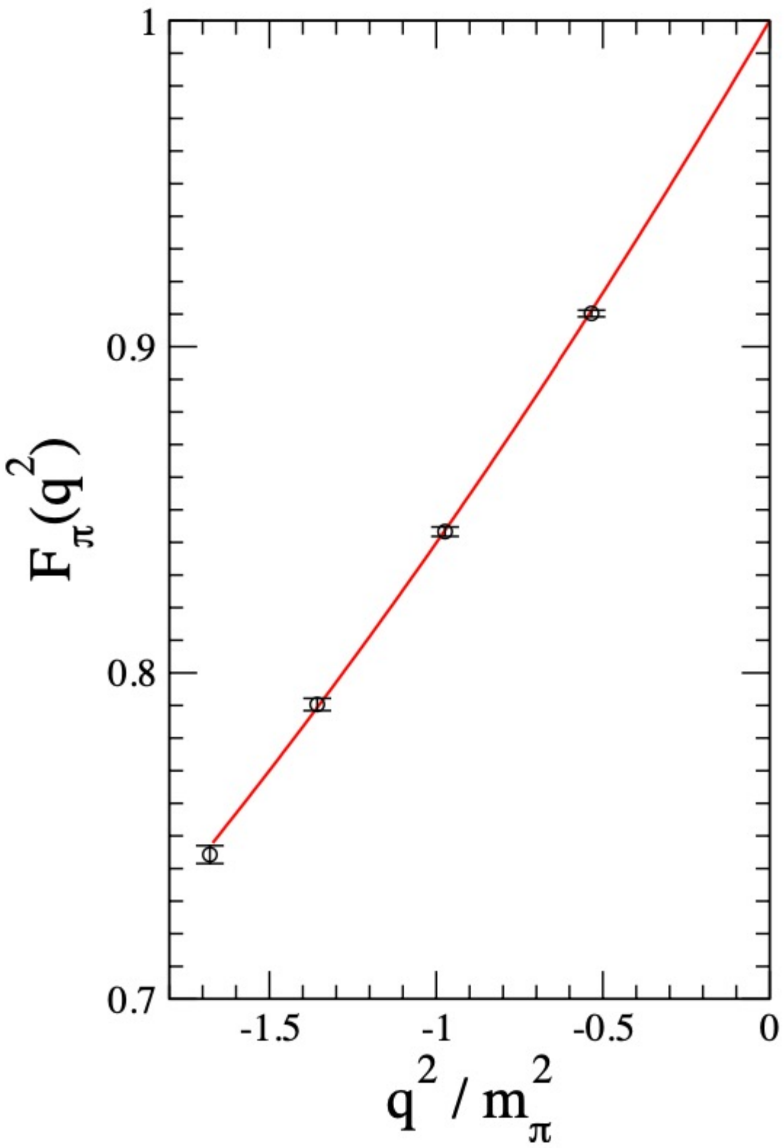
Results

XF, Y. Fu, L. Jin, PRD 101 (2020) 051502



Traditional method





Comparison

Ensemble	New	Traditional	
	$\langle r_\pi^2 \rangle$ [fm^2]	$\langle r_\pi^2 \rangle$ [fm^2]	c_V [fm^4]
$m_\pi = 141$ MeV, $a = 0.19$ fm, $L = 4.7$ fm	0.476(18)	0.466(30)	−0.002(2)
$m_\pi = 141$ MeV, $a = 0.19$ fm, $L = 6.2$ fm	0.480(10)	0.479(15)	0.001(1)
$m_\pi = 143$ MeV, $a = 0.14$ fm, $L = 4.6$ fm	0.423(15)	0.409(28)	0.001(2)
$m_\pi = 139$ MeV, $a = 0.11$ fm, $L = 5.5$ fm	0.434(20)	0.395(32)	−0.002(3)
$m_\pi = 341$ MeV, $a = 0.19$ fm, $L = 4.7$ fm	0.3485(27)	0.3495(44)	0.0015(2)
PDG	0.434(5)		

- At $m_\pi \approx 140$ MeV, the statistical errors of $\langle r_\pi^2 \rangle$ range from 2.1% to 4.6%
- At $m_\pi \approx 340$ MeV, the statistical uncertainty is 0.8%

Short summary

➢ We propose to calculate a physical quantity through the summation

$$O = \sum_{\vec{x}} \omega(\vec{x}, t) H^{(L)}(\vec{x}, t)$$

- Hadronic function $H^{(L)}(x)$ from lattice QCD

- Weight function $\omega(\vec{x}, t)$ is analytically known and contains all the non-QCD information

  - In the infinite volume, the weight function is as simple as $|x^2|$

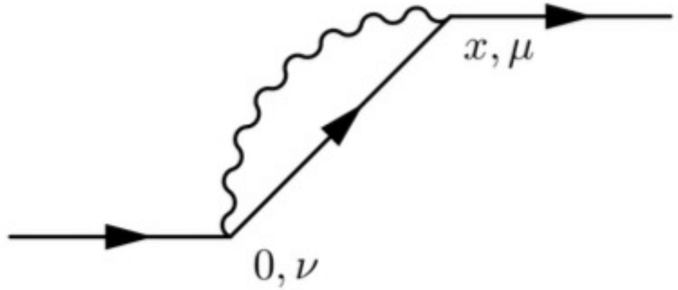
$$D(t) \equiv m_\pi^2 \frac{\partial \tilde{H}(t, \vec{p})}{\partial |\vec{p}|^2}\bigg|_{|\vec{p}|^2=0} = -\frac{m_\pi^2}{3!} \int d^3\vec{x}\, |\vec{x}|^2 H(x)$$

  - Consider the finite-volume effects

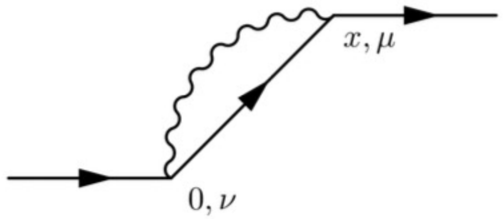
$$\frac{D^{(L)}(t)}{\tilde{H}^{(L)}(t, \vec{0})} = \sum_{n=0}^{\infty} \beta_n^{(L)}(t)\, c_n$$

Design the specific weight function to incorporate the FV effects

QED self energies



Long range photon on the lattice



$m_\gamma = 0 \Rightarrow$ long-range propagator enclosed in the lattice box

$\Rightarrow$ power-law finite-size effects

Various methods proposed to treat photon on the lattice

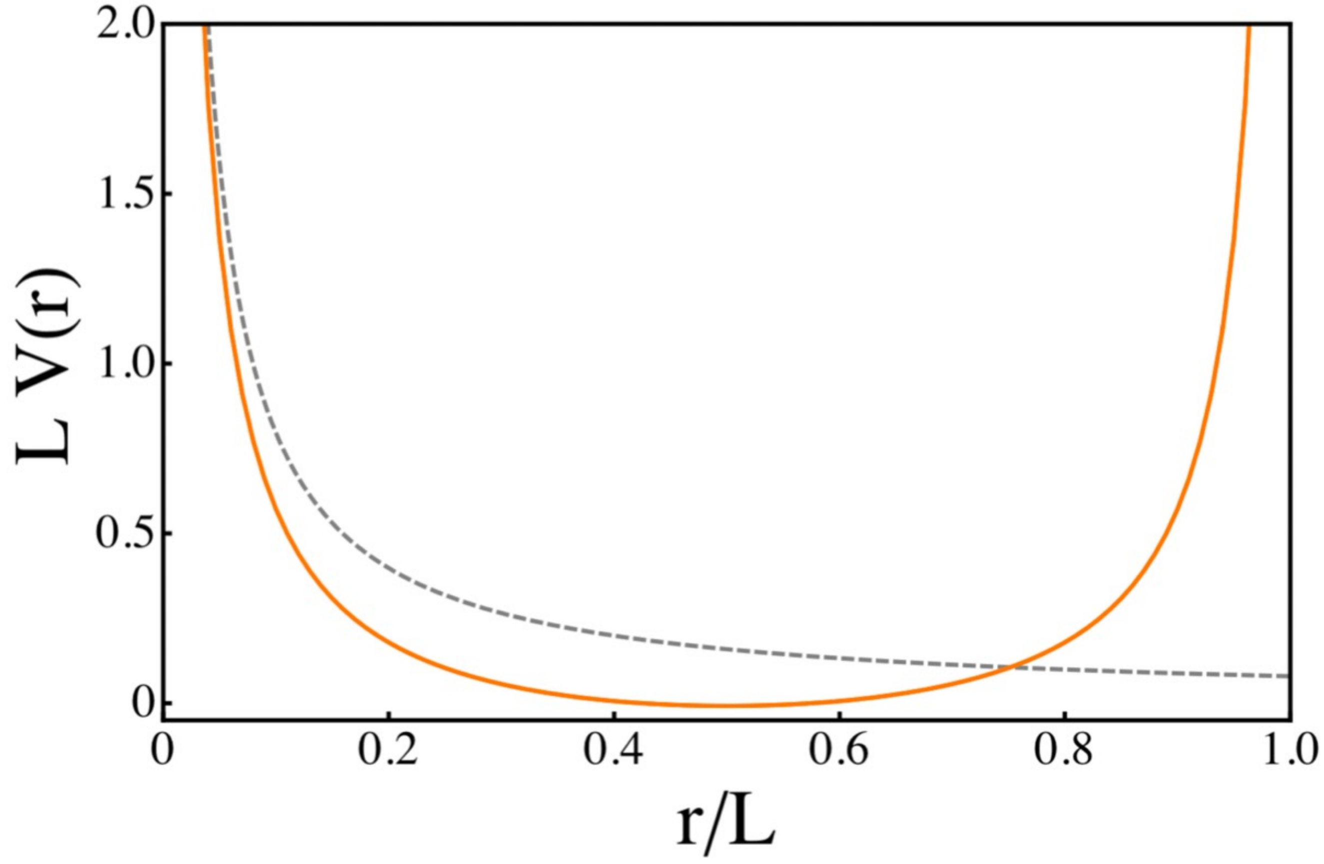
- $\mathrm{QED_L}$ and $\mathrm{QED_{TL}}$ [Hayakawa & Uno, 2008; S. Borsany et. al., 2015]
- Massive photon [M. Endres et. al., 2016]
- C^* boundary condition [B. Lucini et. al., 2016]

Change photon propagators to make it suitable for lattice

Remove zero mode - QED_L

Infinite volume propagator ⇒ finite-volume propagator

$$S_\infty(x) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2} = \frac{1}{4\pi^2x^2} \quad \Rightarrow \quad S_L(x) = \frac{1}{VT}\sum_p{}' \frac{e^{ipx}}{p^2}, \quad p = \frac{2\pi}{L}n \neq 0$$



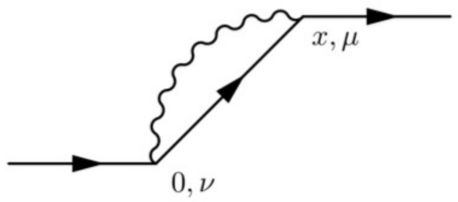
[Davoudi, Savage, PRD90 (2014) 054503]

Power-law ($1/L^n$) finite volume effect as lattice volume L increase

Starting from QED$_\infty$

XF, L. Jin, PRD 100 (2019) 094509

QED self energy



- We start with infinite volume [QED$_\infty$ method, used in HVP & HLbL]

$$\mathcal{I} = \frac{1}{2} \int d^4x \, \mathcal{H}_{\mu,\nu}(x) S^\gamma_{\mu,\nu}(x)$$

where $\mathcal{H}_{\mu,\nu}(x)$ is the hadronic function

$$\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t,\vec{x}) = \langle P | T[J_\mu(t,\vec{x}) J_\nu(0)] | P \rangle$$

$S^\gamma_{\mu,\nu}(x)$ is the photon propagator in the infinite volume

$$S^\gamma_{\mu,\nu}(x) = \frac{\delta_{\mu\nu}}{4\pi^2 x^2}$$

- Propose to replace $\mathcal{H}_{\mu,\nu}(x)$ by $\mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(x)$

However, this still leads to power-law FV effects

Analysis of hadronic function

- We have proposed to replace $\mathcal{H}_{\mu,\nu}(x)$ by $\mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(x)$
  - $\mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(x)$ mainly differs from $\mathcal{H}_{\mu,\nu}(x)$ at $x \sim L$
- The hadronic part $\mathcal{H}_{\mu,\nu}(x)$ is given by

$$\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t,\vec{x}) = \langle P | T[J_\mu(t,\vec{x}) J_\nu(0)] | P \rangle$$

  - $J_\mu(t,\vec{x}) J_\nu(0) \quad \to \quad e^{-M\sqrt{t^2+\vec{x}}} \quad \Rightarrow \quad$ exp. suppressed
  - $\langle P | J_\mu(t,\vec{x}) \quad \to \quad e^{Mt} \quad \Rightarrow \quad$ exp. enhanced

For small $|t|$, we have exponentially suppressed FV effects:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M|\vec{x}|} \quad \Rightarrow \quad \text{Exponentially suppressed FV effects}$$

For large $|t|$, we shall have:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M\frac{\vec{x}^2}{2t}} \sim O(1) \quad \Rightarrow \quad \text{Pow-law suppressed FV effects}$$

Infinite volume reconstruction method

Realizing at large $t > t_s$ we have ground state dominance:

$$\langle P|J_\mu(t,\vec{x})J_\nu(0)|P\rangle \sim \int \frac{d^3\vec{k}}{(2\pi)^3}\langle P|J_\mu(0)|P(\vec{k})\rangle\langle P(\vec{k})|J_\nu(0)|P\rangle e^{-E_{\vec{k}}t+Mt}e^{-i\vec{k}\cdot\vec{x}}$$

- Reconstruct $\mathcal{H}_{\mu,\nu}(t,\vec{x})$ at large t using $\mathcal{H}_{\mu,\nu}(t_s,\vec{x})$ at modest t_s

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}') \approx \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x})\int \frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}e^{-(E_{\vec{k}}-M)(t-t_s)}e^{-i\vec{p}\cdot\vec{x}'}$$

Replace

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \quad \Leftarrow \quad \mathcal{H}_{\mu,\nu}(t_s,\vec{x}) \quad \Leftarrow \quad \mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(t_s,\vec{x})$$

The replacement only amounts for exponentially suppressed FV effects

Master formula

To sum up, we split the integral $\mathcal{I}$ into two parts

$$
\begin{aligned}
\mathcal{I} &= \mathcal{I}^{(s)} + \mathcal{I}^{(l)} \\
\mathcal{I}^{(s)} &= \frac{1}{2}\int_{-t_s}^{t_s}\int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(x)S^{\gamma}_{\mu,\nu}(x) \\
\mathcal{I}^{(l)} &= \int_{t_s}^{\infty}\int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(x)S^{\gamma}_{\mu,\nu}(x) \\
&= \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x})L_{\mu,\nu}(t_s,\vec{x})
\end{aligned}
$$

where $L_{\mu,\nu}(t_s,\vec{x})$ is known

$$
L_{\mu,\nu}(t_s,\vec{x}) = \frac{\delta_{\mu\nu}}{2\pi^2}\int_0^{\infty} dp\,\frac{\sin(p|\vec{x}|)}{2(p+E_p-M)|\vec{x}|}e^{-pt_s}
$$

At $t \le t_s$,

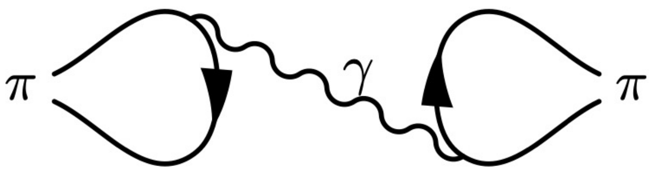
$$
\mathcal{H}_{\mu,\nu}(t,\vec{x}) \quad \Leftarrow \quad \mathcal{H}^{\mathrm{lat}}_{\mu,\nu}(t,\vec{x})
$$

Ground-state dominance can be verified by the t_s dependence

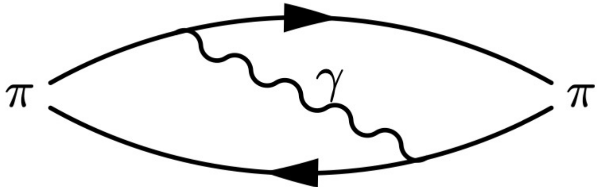
Example: Pion mass splitting

XF, L. Jin, M. Riberdy, PRL 128 (2022) 052003

$m_{\pi^+} - m_{\pi^0}$:



type 1



type 2

Isospin breaking effects: EM (α_e) + strong ($\frac{m_u - m_d}{\Lambda_{\mathrm{QCD}}}$) contributions

- Strong IB breaking appears at $O\left(\left(\frac{m_u - m_d}{\Lambda_{\mathrm{QCD}}}\right)^2\right)$ $\Rightarrow$ dominated by EM effect

- Previous calculation by RM123, 2013

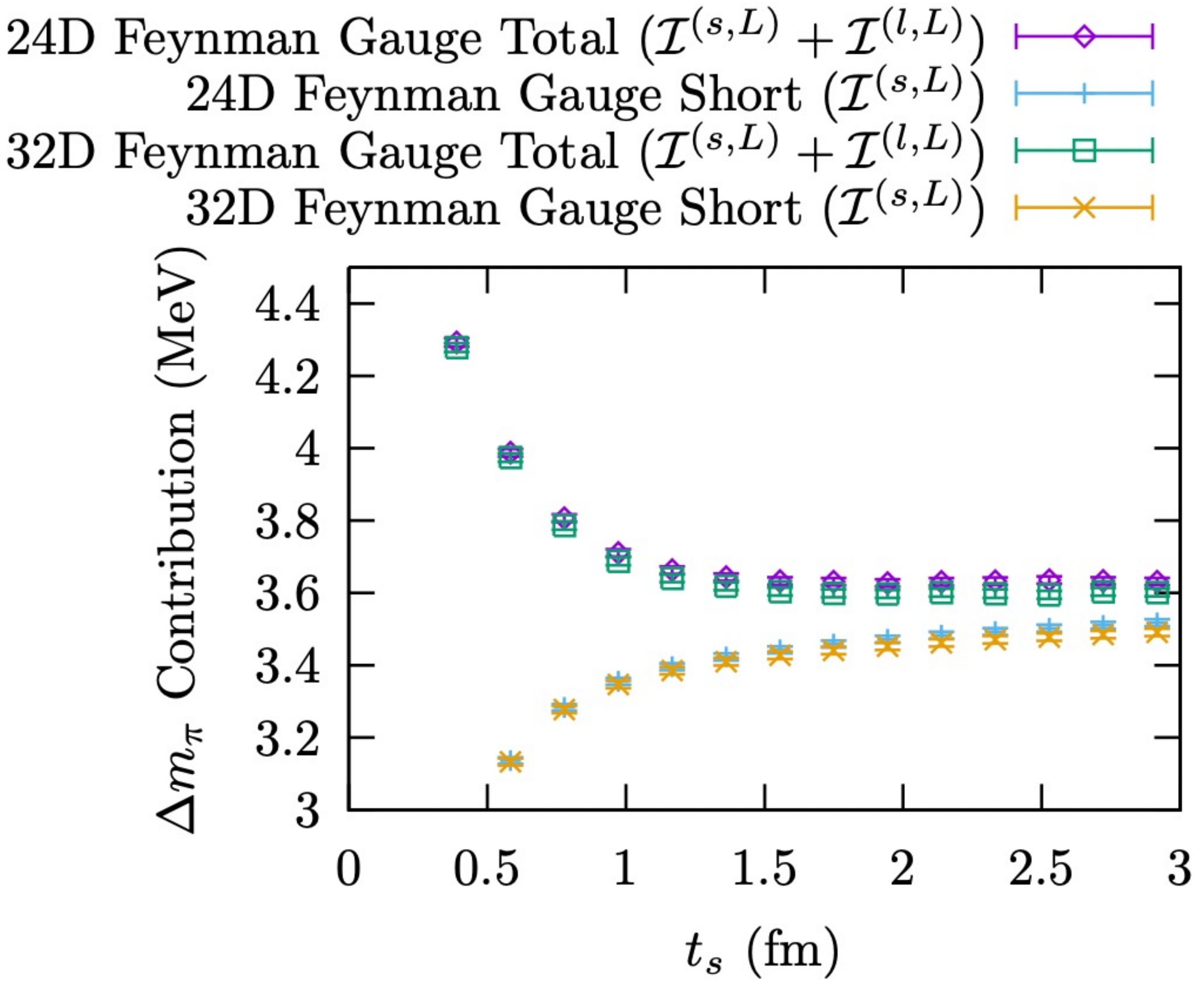
$$M^2_{\pi^+} - M^2_{\pi^0} = 1.44(13)_{\mathrm{stat}}(16)_{\mathrm{chiral}} \times 10^3 \ \mathrm{MeV}^2$$

including type 2 diagram only

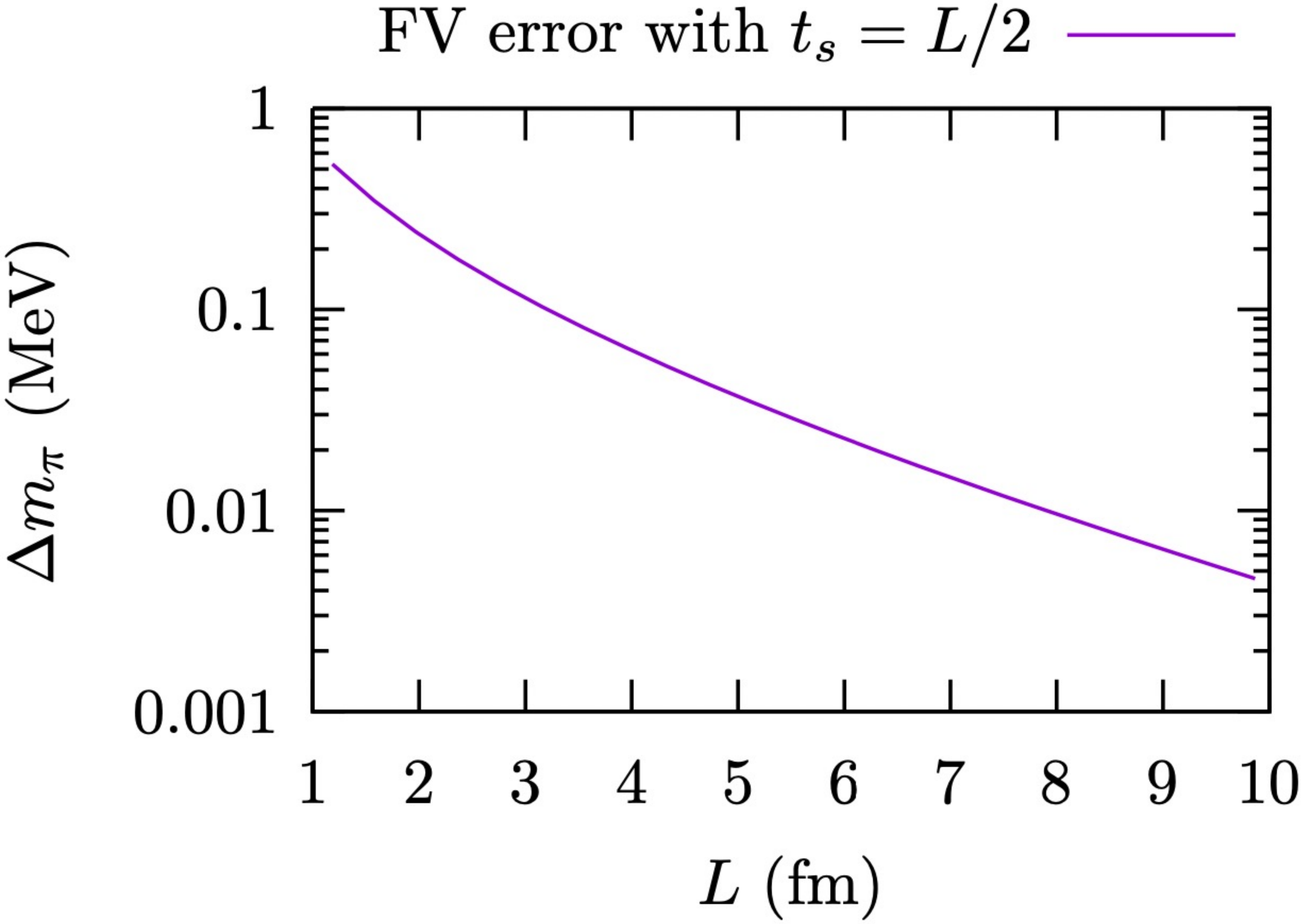
Using infinite-volume reconstruction

- List of gauge ensembles used, DWF @ phys. Pion mass

	Volume	a^{-1} (GeV)	L (fm)	M_π (MeV)
48I	$48^3 \times 96$	1.730(4)	5.5	135
64I	$64^3 \times 128$	2.359(7)	5.4	135
24D	$24^3 \times 64$	1.0158(40)	4.7	142
32D	$32^3 \times 64$	1.0158(40)	6.2	142
32Dfine	$32^3 \times 64$	1.378(7)	4.6	144
24DH	$24^3 \times 64$	1.0158(40)	4.7	341

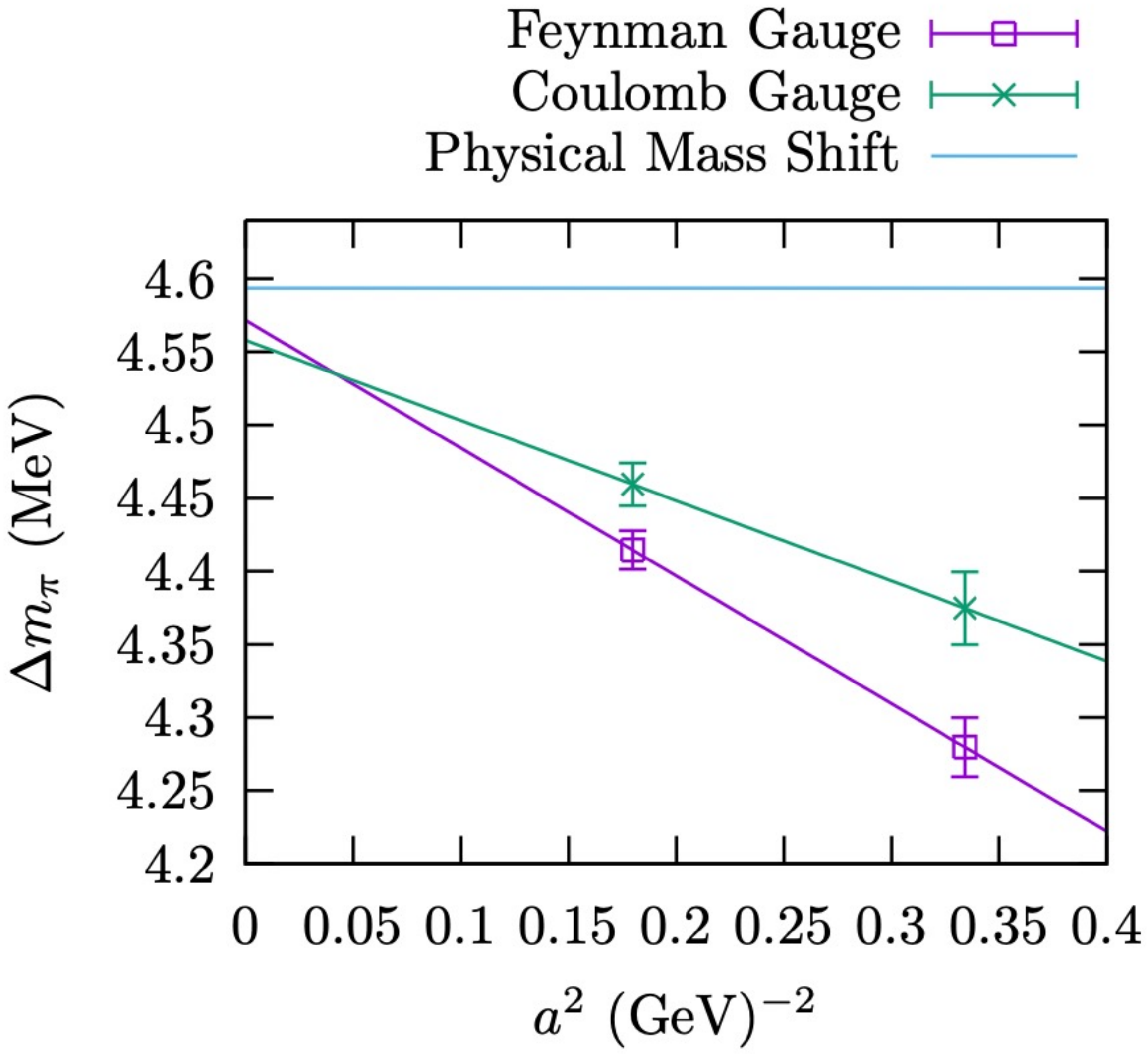


FV effects from scalar QED



FV error exponentially suppressed

Pion mass splitting

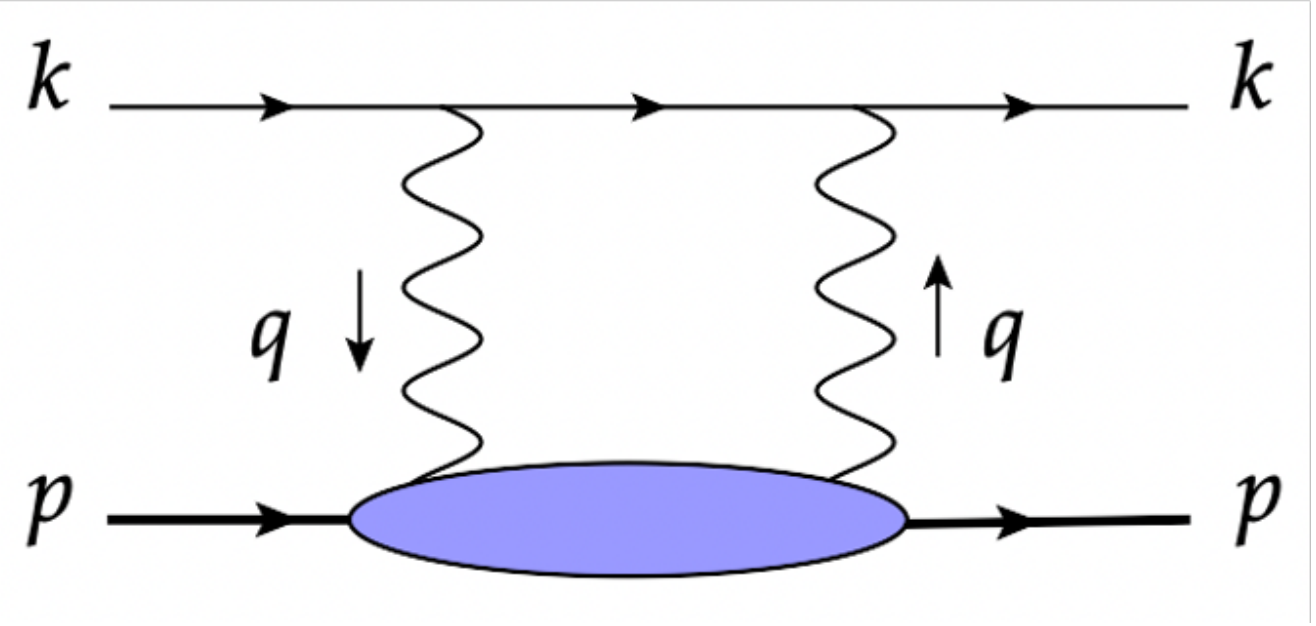


XF, L. Jin, M. Riberdy, PRL 128 (2022) 052003

Reference	$m_{\pi^\pm} - m_{\pi^0}$ (MeV)
RM123 2013 [5]	$5.33(48)_{\rm stat}(59)_{\rm sys}$ [a]
R. Horsley et al. 2016 [7]	$4.60(20)_{\rm stat}$
RM123 2017 [9]	$4.21(23)_{\rm stat}(13)_{\rm sys}$
This work	$4.534(42)_{\rm stat}(43)_{\rm sys}$

- Pion mass splitting with a percent-level uncertainty, which is about 5-10 times smaller than previous lattice QCD calculations

Two-photon exchange contribution to Lamb shift

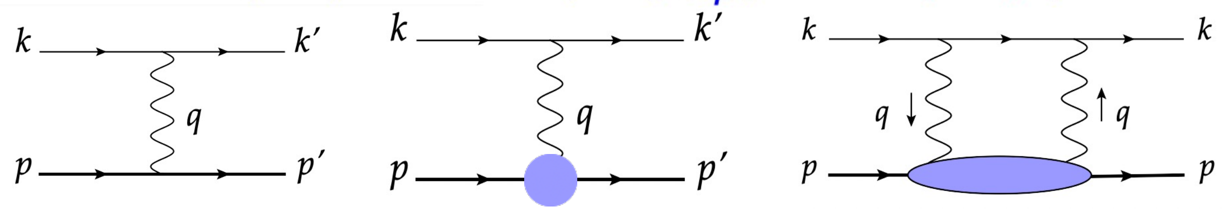


Various contributions to μH Lamb shift

Experiment		Theory				
$E_{2P}-E_{2S}$	$=$	ΔE_{QED}	$+$	$\Delta E_{\text{proton size}}$	$+$	ΔE_{TPE}

【Science 339 (2013) 417】

$$202370.6(2.3) = 206033.6(1.5) - 5227.5(1.0)\langle r_p^2\rangle + 33.2(2.0)\ \mu\mathrm{eV}$$



- Exp. vs Theory ➡ $r_p = 0.84087(39)$ fm ➡ Proton size puzzle
- Largest theoretical uncertainty from two-photon exchange (TPE)
- Uncertainty for structure independent contribution is further reduced

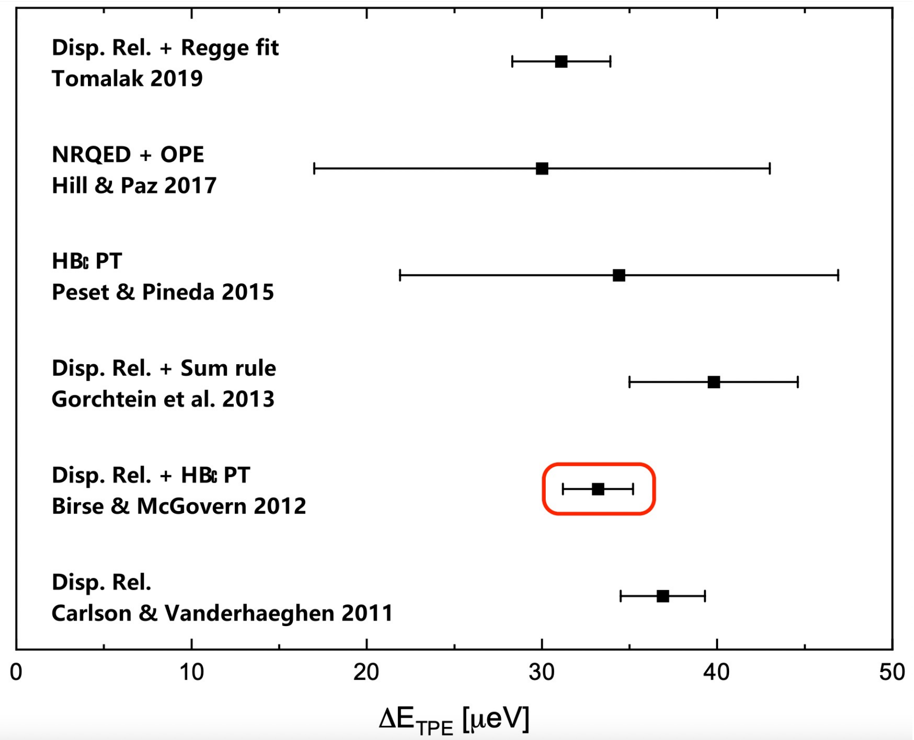
  $\Delta E_{\mathrm{QED}} = 206034.7(0.3)\quad \mu\mathrm{eV}$ 【Ann.Rev.Nucl.Part.Sci.72 (2022) 389】

  Upgrade of CREMA@PSI can reduce Exp. error by a factor of 5

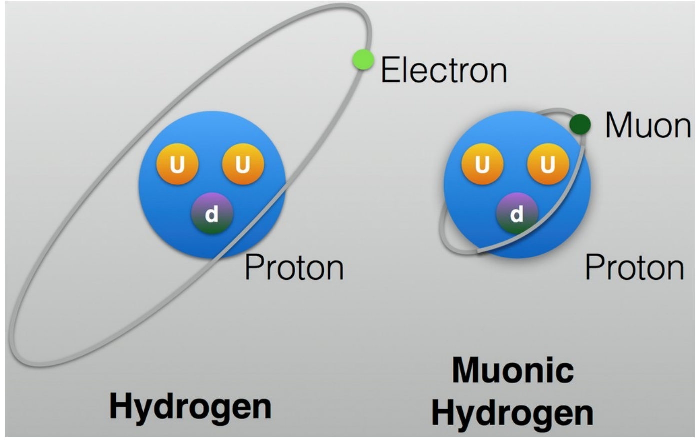
  Leaving TPE the important source for uncertainty!

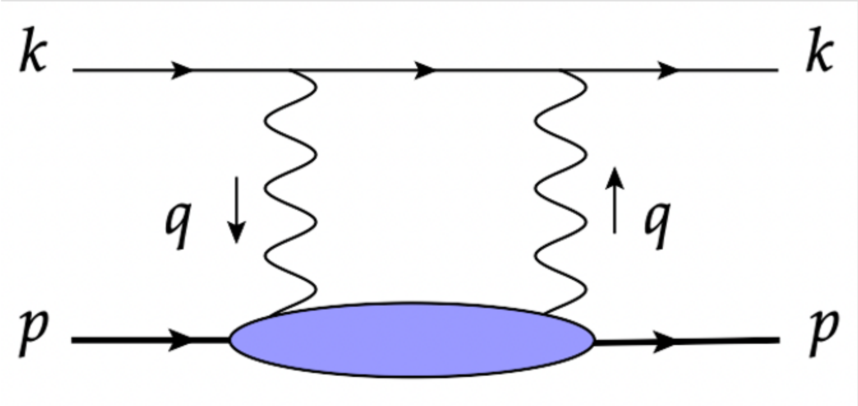
- Theoretical prediction for TPE relies on data + models and ranges from 20 to 50 μeV

  Our target: obtain TPE from first principles ➡ Lattice QCD



Challenges from TPE (1): IR divergence



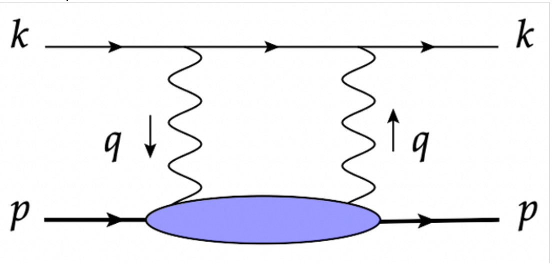


- Binding energy of μH serves as a natural IR cutoff
  - ➡ Bound-state QED
  - ➡ Proton treated as point-like particle + charge radius correction

No divergence, but rich structure information lost

- QCD+QED: complete information of proton structure
  - ➡ Loop integral sensitive to hadronic scale → highly NP
  - ➡ Bound lepton → free lepton → IR divergence

Solution: subtract the divergence



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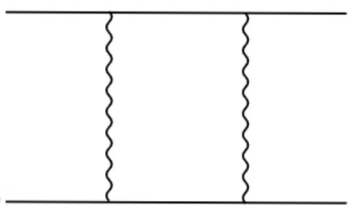
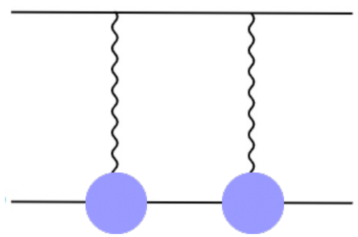


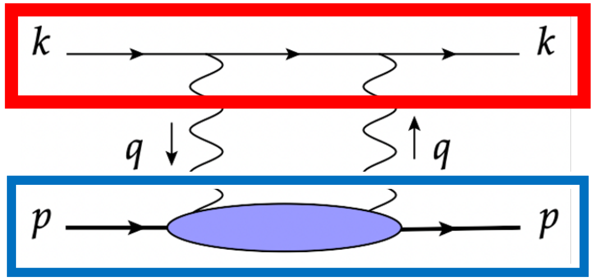
Point-like particle

-



Charge radius

Challenges from TPE (1): IR divergence



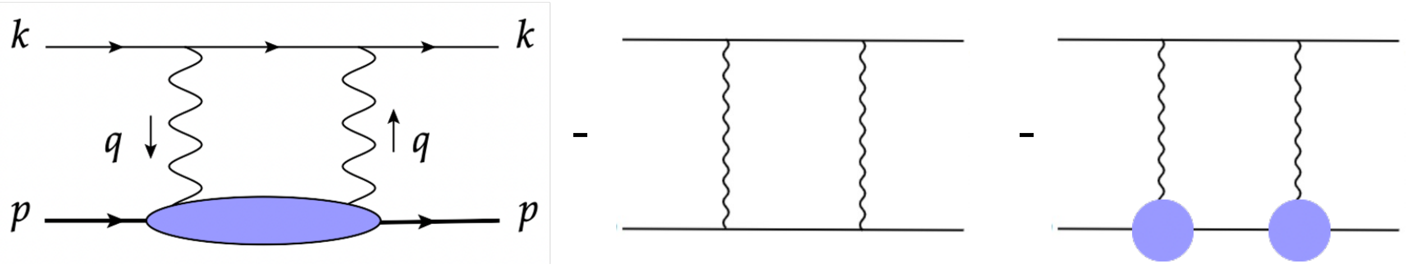
Leptonic part: $L_{\mu\nu}(q)$ $\rightarrow$ Analytically known

Hadronic part: $H_{\mu\nu}(q)$ $\rightarrow$ Provided by LQCD (statistical errors)

Loop integral

$$\Delta E^{\text{IR-}\infty} = \int \frac{d^4q}{(2\pi)^4} L_{\mu\nu}(q) H_{\mu\nu}(q) = \int d^4x \, L_{\mu\nu}(x) H_{\mu\nu}(x)$$

IR subtraction



divergence - divergence - divergence = finite

Key technical problem

Three diagrams contain diff. stat. errors $\Rightarrow$ How to maintain the error cancellation?

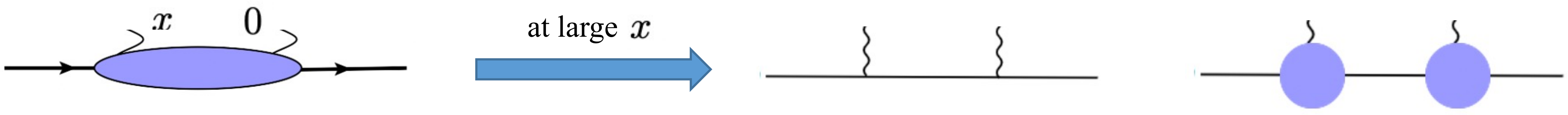
If signal cancels and error does not, then signal is completely hidden by error

Challenges from TPE (1): IR divergence

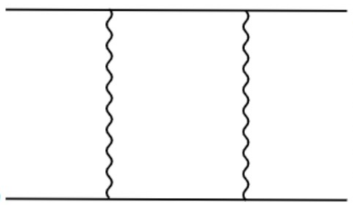
To solve IR divergence: infinite-volume reconstruction method 【X. Feng, L. Jin, PRD 100 (2019) 094509】

Basic idea: low-energy structure information is contained in the long-distance part of hadronic function

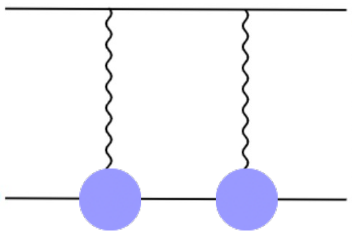
Use $H_{\mu\nu}(x)$ to reconstruct the quantities such as charge radius



- Find the appropriate weight functions $L_{\mu\nu}^{\text{pt-like}}$ and $L_{\mu\nu}^{\text{radius}}$ for the subtraction terms, yielding

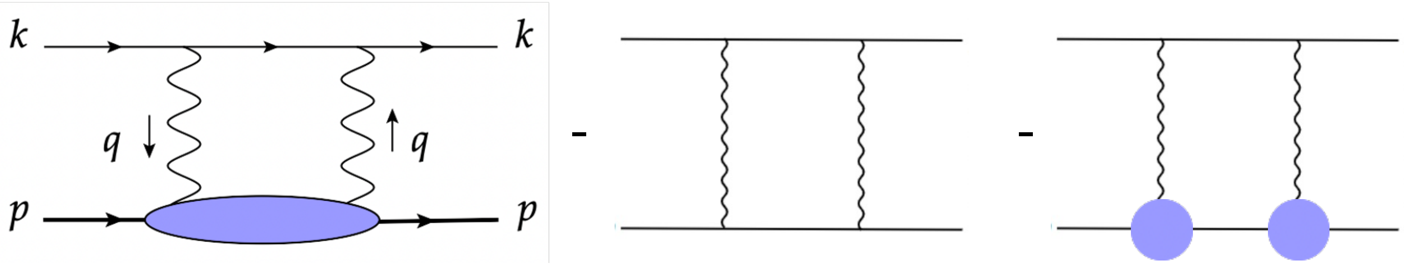


$$= \int d^4x\, L_{\mu\nu}^{\text{pt-like}}(x) H_{\mu\nu}(x)$$



$$= \int d^4x\, L_{\mu\nu}^{\text{radius}}(x) H_{\mu\nu}(x)$$

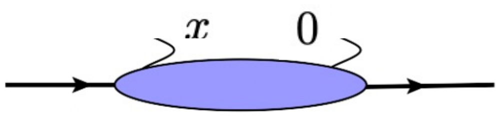
- Cancellation of IR divergence is rigorously fulfilled via the subtraction of weight functions



$$L_{\mu\nu}^{\text{sub}} = L_{\mu\nu} - L_{\mu\nu}^{\text{pt-like}} - L_{\mu\nu}^{\text{radius}}$$

$$\Delta E^{\text{IR-finite}} = \int d^4x\, L_{\mu\nu}^{sub}(x) H_{\mu\nu}(x)$$ Error decreases coherently as signal. IR divergence is solved cleanly!

Challenges from TPE (2): Signal-to-noise problem

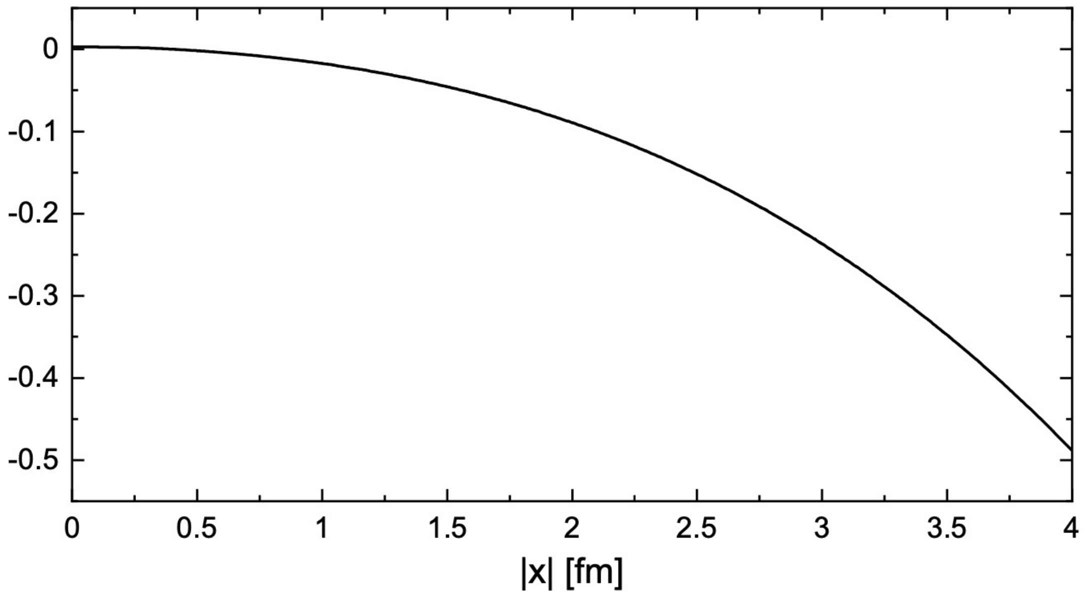


$H_{\mu\nu}(x)$

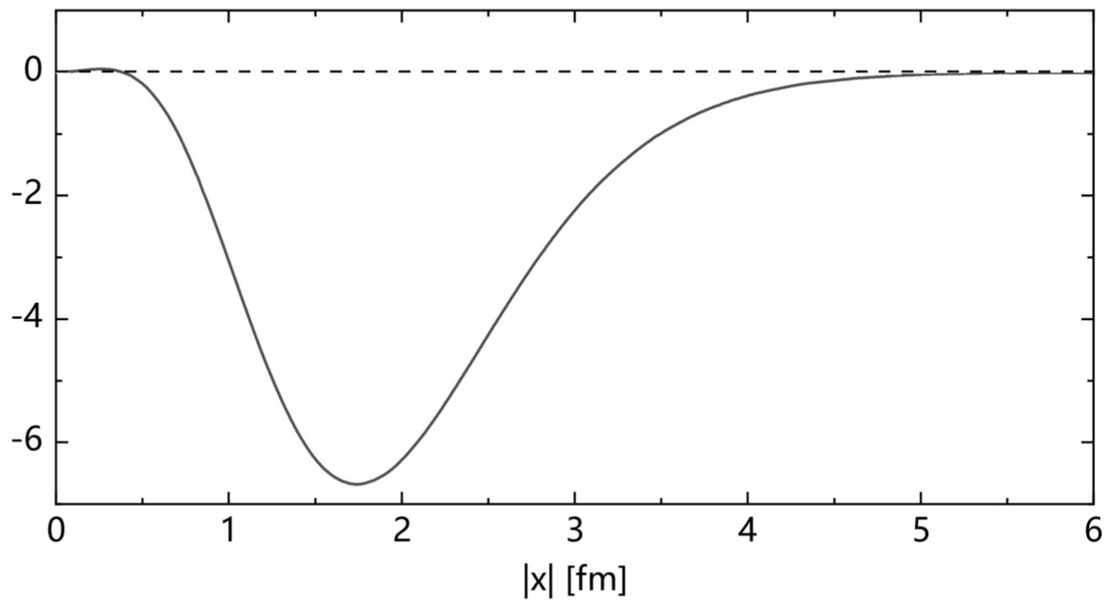
Property of lattice data:

As x increases, proton matrix element $H_{\mu\nu}(x)$ decreases as $e^{-M_p|x|}$

However, error decreases as $e^{-\frac{3}{2}M_\pi|x|}$



Weight function $L_{\mu\nu}^{sub}(x)$ increases fast, as x increases



Model estimate: Combine leptonic and hadronic part

Conclusion: take x as large as 5 fm to guarantee no information lost

Require a 10 fm lattice for simulation

Decrease of S/N ratio seems an inevitable problem

Challenges from TPE (2): Signal-to-noise problem

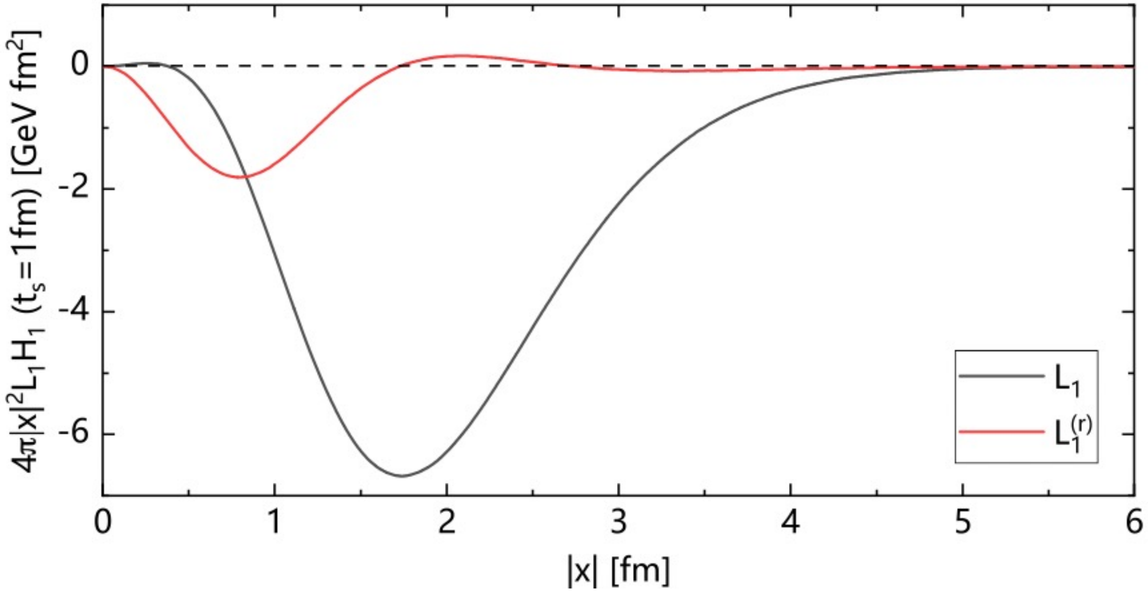
To solve S/N problem: optimized subtraction scheme 【Y. Fu, X. Feng, L. Jin, C. Lu, PRL 128 (2022) 172002】

Trick: A = (A – B) + B

Define the reduced weight function

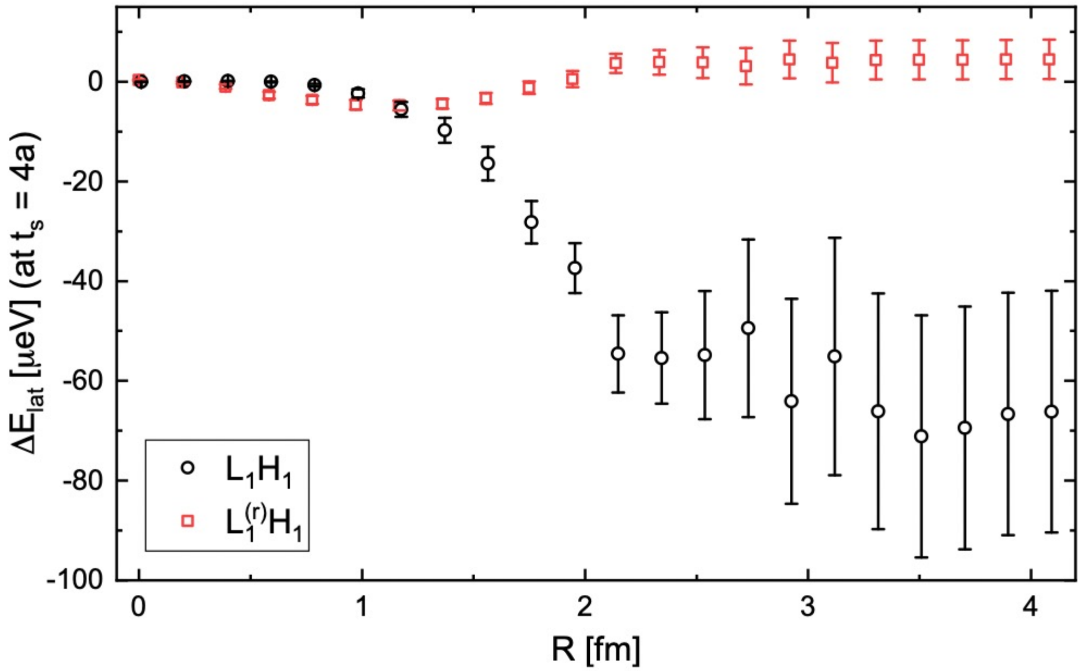
$$L^{(r)}(x) = L^{\text{sub}}(x) - c_0\, L^{\text{pt-like}}(x) - c_r\, L^{\text{radius}}(x)$$

- Choose c_0, c_r to minimize $L^{(r)}(x)$ in the region of 1-3 fm
- Using $L^{(r)}(x)$, (A-B) part is illustrated by the red curve



- Total contribution is $\Delta E = \Delta E^{(r)} + c_0 + c_r \cdot \langle r_p^2 \rangle$

Use optimized subtraction scheme in realistic calculation



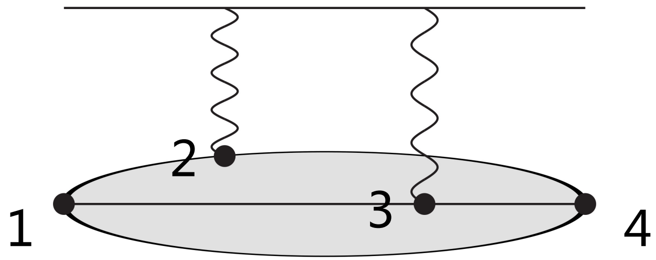
Integral within the range R:

using $L^{(r)}(x)$, error reduced by a factor of 6!

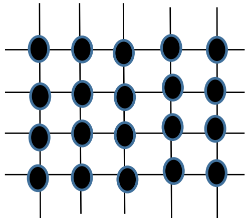
S/N problem is solved

Challenges from TPE (3): Computation of 4-point function

- TPE - hadronic part from a typical 4-point function

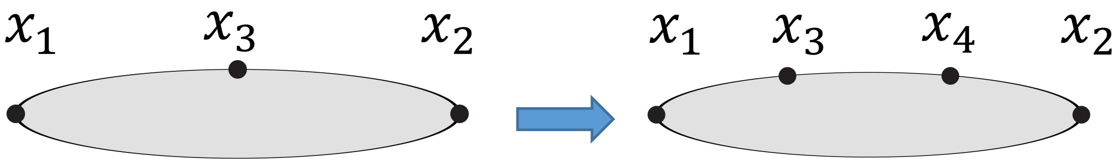


- Perform the volume summation for each point



$$\sum_{x \in L^3}$$

- From 3-point to 4-point function



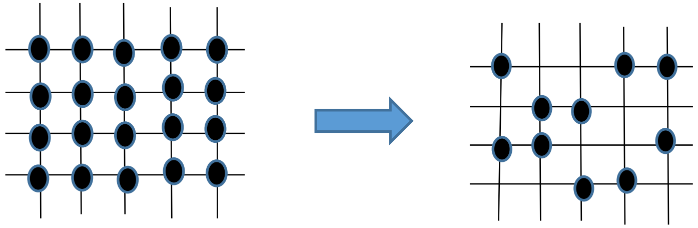
3-point: L^6 summation 4-point: L^9 summation

Increasing each point, computational cost increases by 10^4-10^5 times!
Cannot be solved by increasing resources …

Solution： Field sparsening method

【Y. Li, S. Xia, X. Feng, L. Jin, C. Liu, PRD 103 (2021) 014514】

【W. Detmold, D. Murphy, et. al. PRD 104 (2021) 034502】



- Less summation points may lead to lower precision
- It is not the case because of high correlation in lattice data

10^2-10^3 times less points yields similar precision

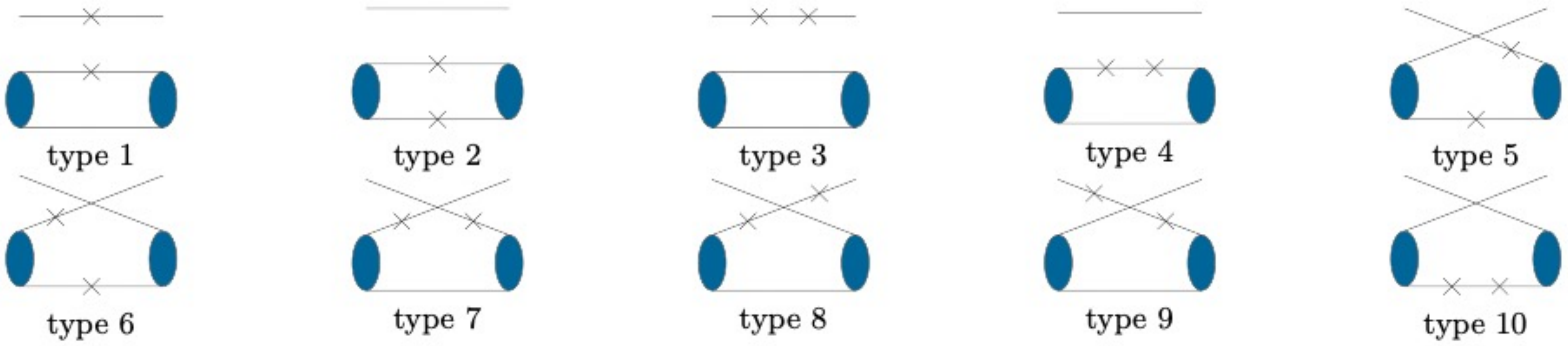
- Used for diff. physical system to confirm the universality

Utilize field sparsening method

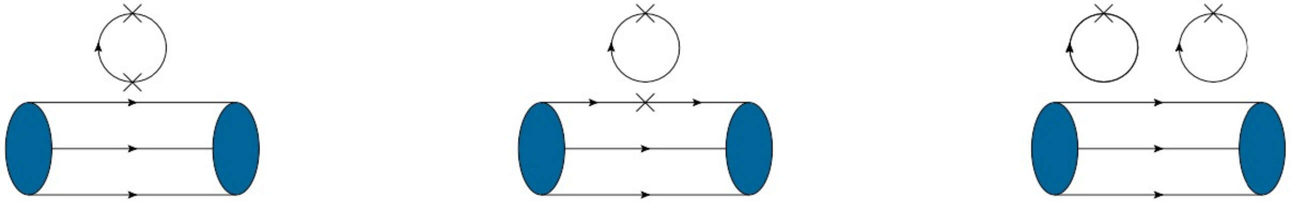
- Reduce the computational cost by a factor of 10^2-10^3 with almost no loss of precision!

Challenges from TPE (3): Computation of 4-point function

- Complicated quark field contraction for nucleon 4-point function – 10 types of connected diagrams

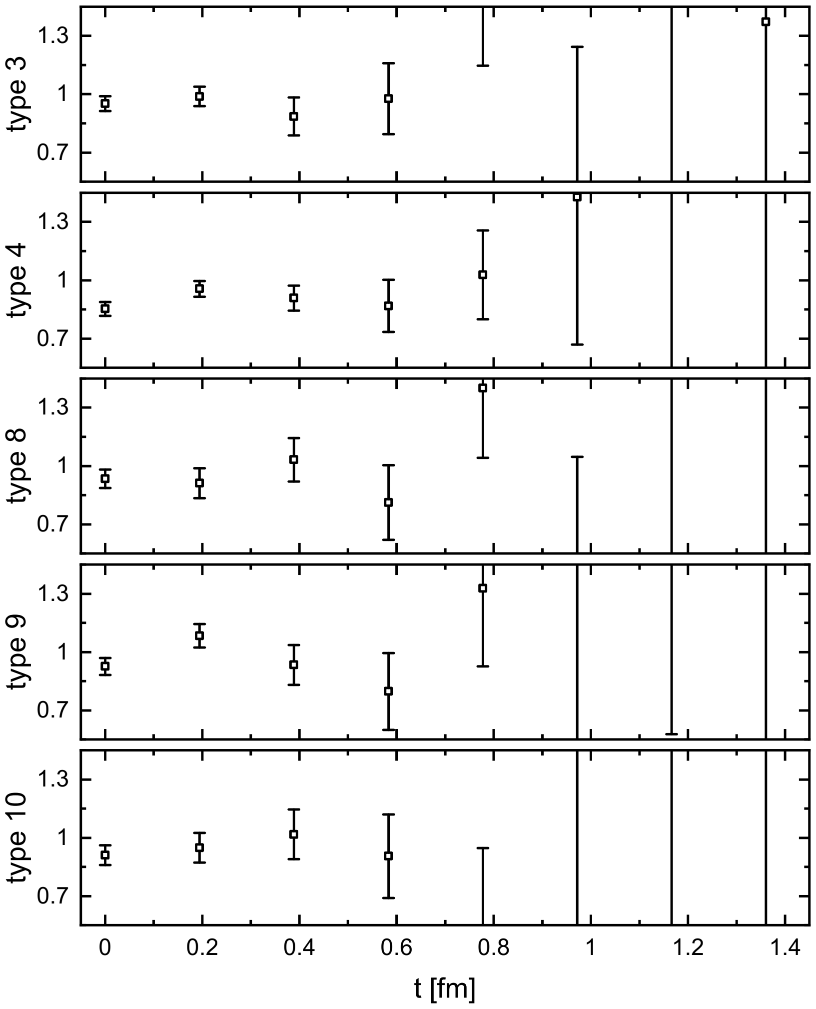


- There are also disconnected diagrams – notorious for high cost and bad S/N ratio

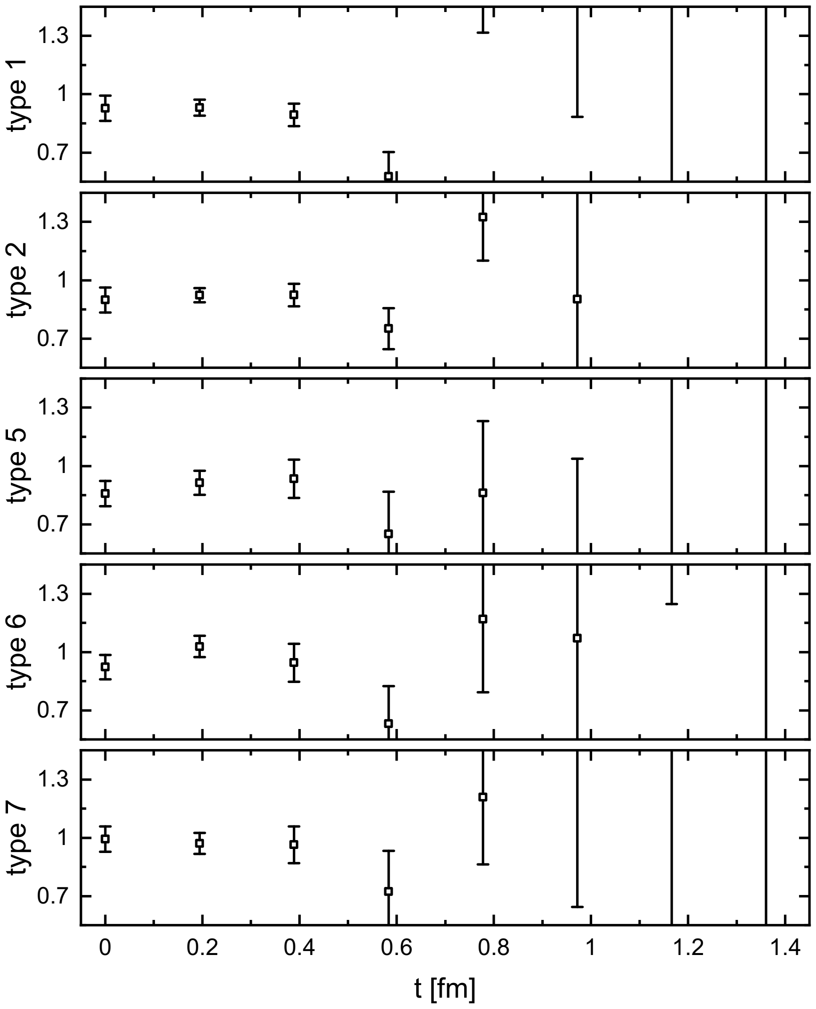


Our calculation contains both connected and the main disconnected diagrams

Challenges from TPE (3): Computation of 4-point function



Two currents inserted in one quark line



Two currents inserted in two quark lines

Using the conditions such as charge conservation to verify the contraction code

Lattice results

- Gauge ensemble used – nearly physical pion mass

Ensemble	m_π [MeV]	L/a	T/a	a [fm]	$N_{\rm conf}$
24D	142	24	64	0.1943(8)	131

$$\Delta E_{\rm lat} = \begin{cases} 27.6(4.5)\ \mu\text{eV}, & \text{connected part,} \\ 2.1(2.1)\ \mu\text{eV}, & \text{disconnected part,} \\ 29.7(4.9)\ \mu\text{eV}, & \text{total contribution.} \end{cases}$$

- The total TPE contribution is given by

$$\Delta E_{\rm TPE} = 0.77 + 93.72 \cdot \langle r_p^2 \rangle - \Delta E_{\rm lat}$$

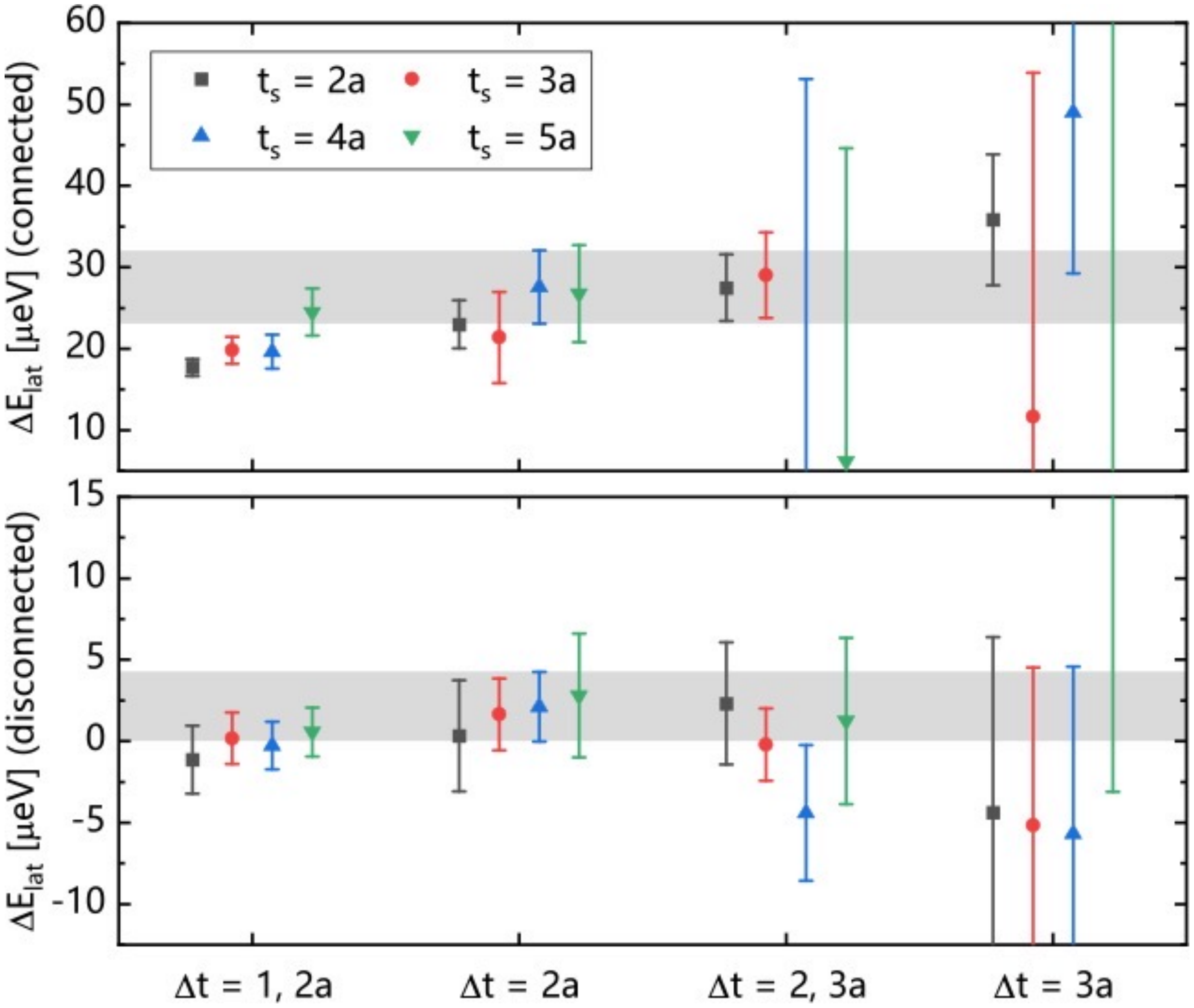
- Matching $\Delta E_{\rm TPE}$ with Exp. measurement, one gets

$$\sqrt{\langle r_p^2 \rangle} = 0.84136(65)\ \text{fm}$$

consistent with $0.84087(39)$ fm quoted by μH Exp

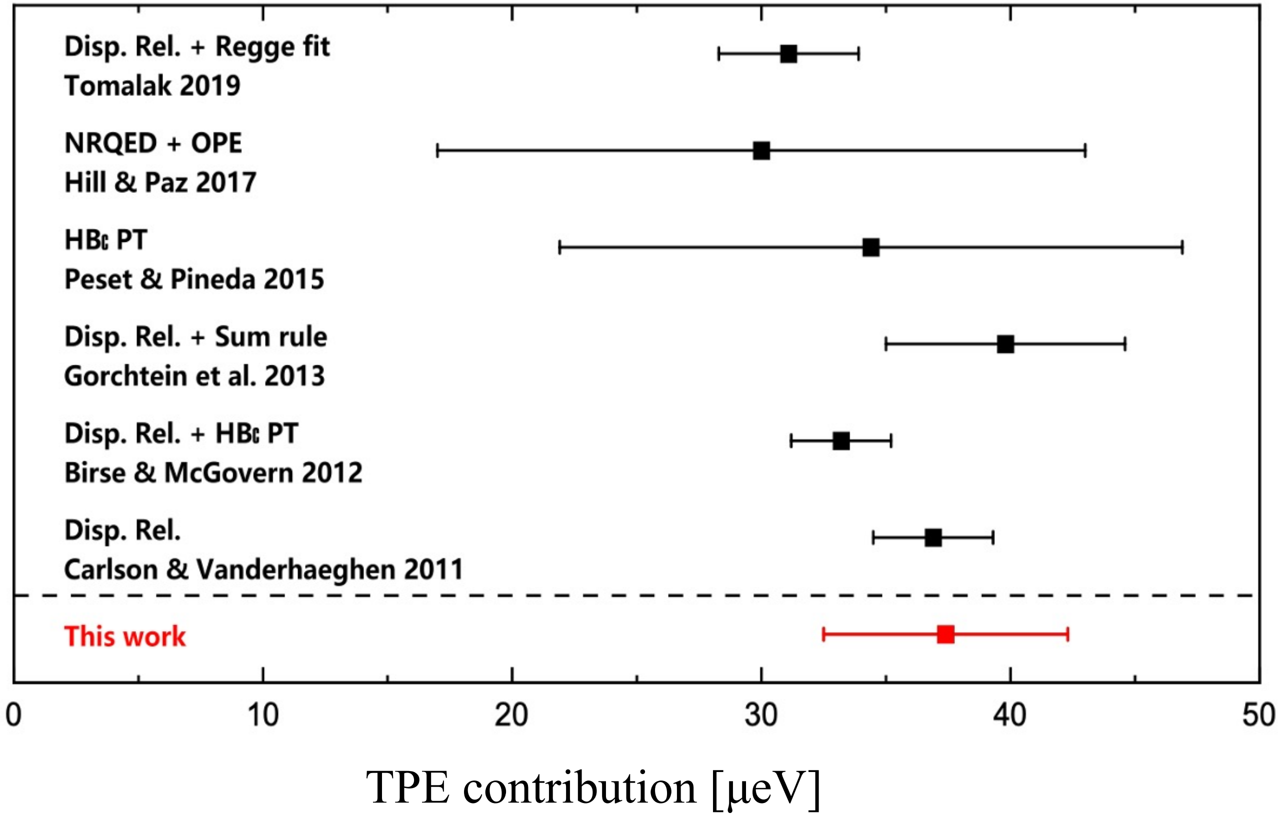
- Using μH value of charge radius as input, one gets

$$\Delta E_{\rm TPE} = 37.4(4.9)\ \mu\text{eV}$$



Lattice results

➢ Compared with other theoretical work



Y. Fu, XF, L. Jin, C. Lu, PRL 128 (2022) 17, 172002

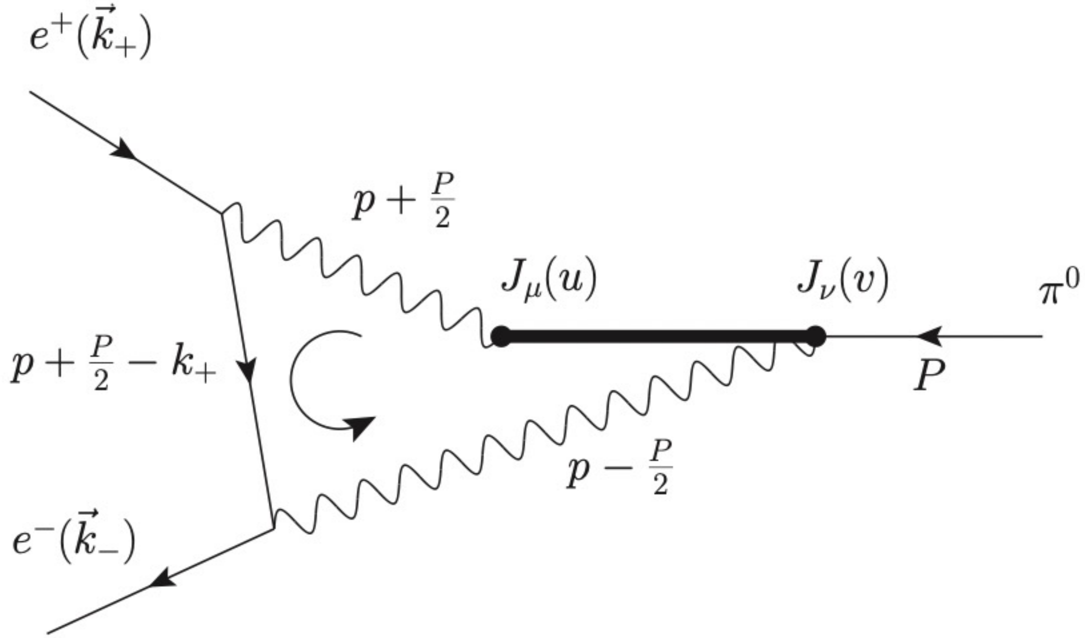
- First lattice result @ m_π=142 MeV

$$\Delta E_{\mathrm{TPE}} = 37.4(4.9) \quad \mu\mathrm{eV}$$

- Need to increase statistics and control systematic effects

Outlook: to help answer the question – what is the exact size of proton

$\pi^0 \to e^+e^-$ decays



First look at $\pi^0 \to \gamma\gamma$

- Step 1 - Calculate hadronic matrix element in position space

$$\mathcal{H}_{\mu\nu}(x) = \langle 0 | T[J_\mu(x) J_\nu(0)] | \pi^0(q) \rangle$$

- Step 2 - Choose on-shell momentum for the two photons

$$p = (im_\pi/2, \vec{p}), \quad p' = (im_\pi/2, -\vec{p}), \quad q = (im_\pi, \vec{0}), \quad |\vec{p}| = m_\pi/2.$$

Perform Fourier transform

$$\mathcal{F}_{\mu\nu}(q, p, p') = \int d^4x \, e^{-ipx} \mathcal{H}_{\mu\nu}(x)$$

We have

$$\mathcal{F}_{\mu\nu}(q, p, p') = \varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta F_{\pi\gamma\gamma}(m_\pi^2, 0, 0)$$

- Step 3 - Obtain a Lorentz scalar amplitude

$$\begin{aligned} \mathcal{I} &= \varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta \int d^4x \, e^{-ipx} \mathcal{H}_{\mu\nu}(x) \\ &= m_\pi \int d^4x \, e^{-ipx} \underbrace{\varepsilon_{\mu\nu\alpha 0} \frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha}}_{\text{Lorentz scalar}} \end{aligned}$$

First look at $\pi^0 \to \gamma\gamma$

- Step 4 - Average over the spatial direction for $\vec{p}$

$$\begin{aligned}
\mathcal{I} &= m_\pi \int dt\, e^{m_\pi t/2} \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\, \varepsilon_{\mu\nu\alpha 0} \frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} \\
&= m_\pi \int dt\, e^{m_\pi t/2} \int d^3\vec{x}\, j_0\left(\frac{m_\pi |\vec{x}|}{2}\right) \varepsilon_{\mu\nu\alpha 0} \frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} \\
&= \int dt\, e^{m_\pi t/2} \int d^3\vec{x}\, \frac{m_\pi^2}{2|\vec{x}|} j_1\left(\frac{m_\pi |\vec{x}|}{2}\right) \varepsilon_{\mu\nu\alpha 0} x_\alpha \mathcal{H}_{\mu\nu}(x)
\end{aligned}$$

- Step 5 - Master formula

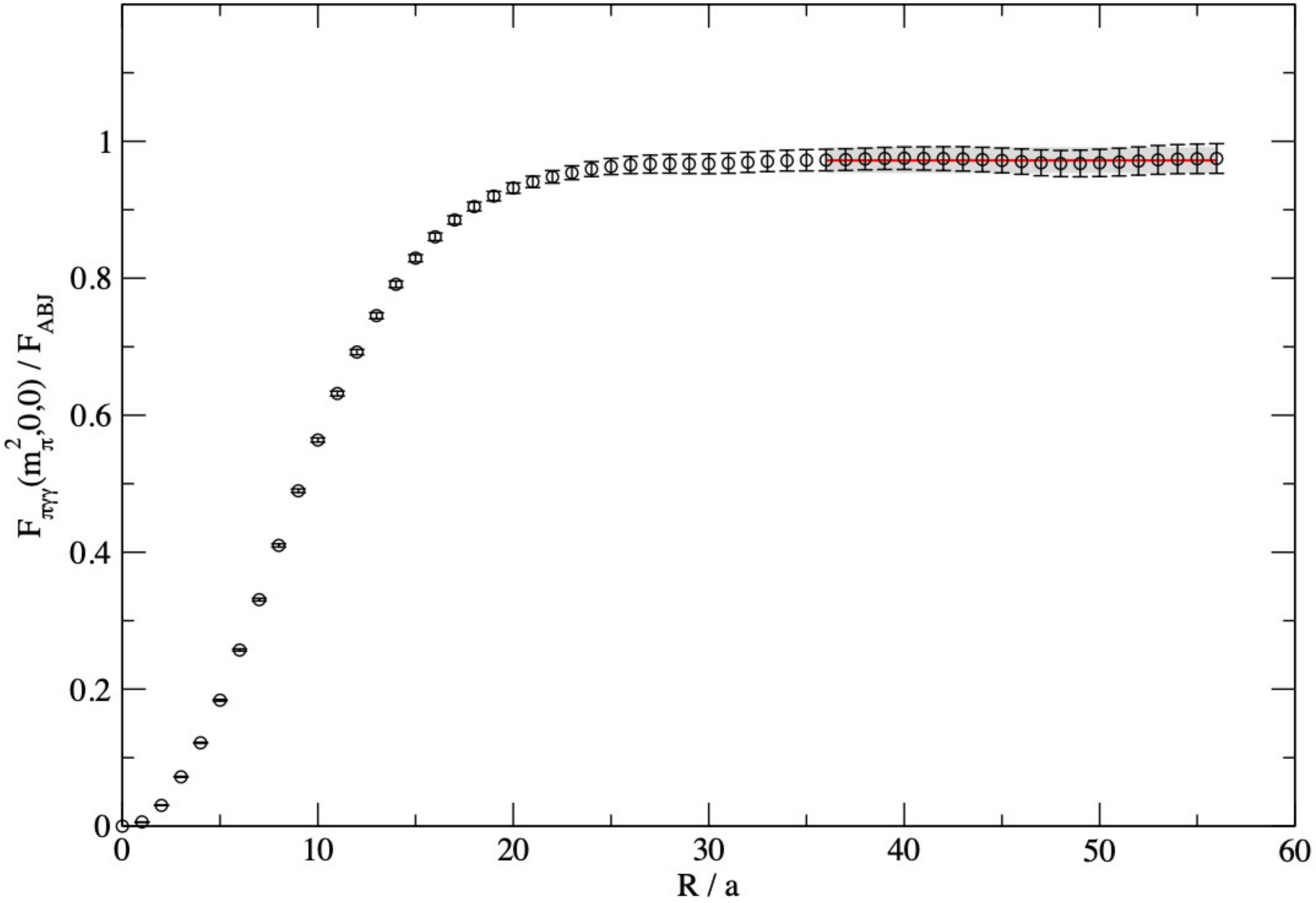
$$F_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = \frac{\mathcal{I}}{[\varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta][\varepsilon_{\mu\nu\rho\sigma} p_\rho q_\sigma]} = -\frac{2}{m_\pi^4}\mathcal{I} = \int d^4x\, \omega(x) H(x)$$

Weight function $\omega(x)$ is known analytically

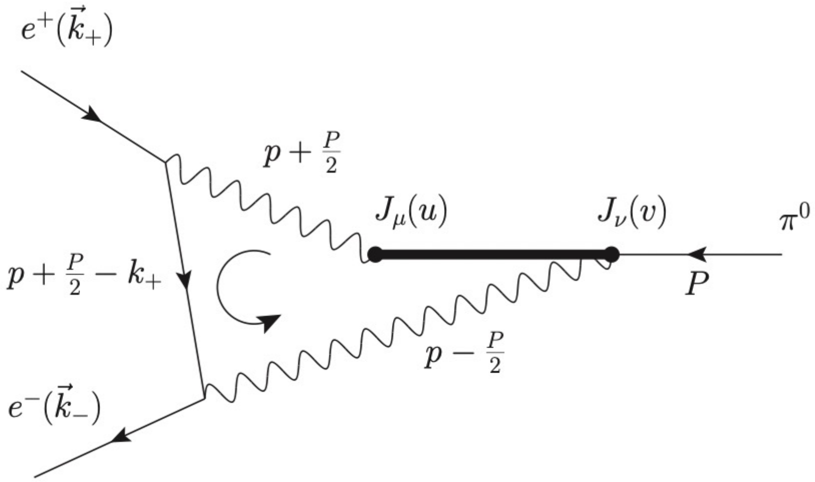
Key quantity required from lattice QCD is $H(x) = \varepsilon_{\mu\nu\alpha 0} x_\alpha \mathcal{H}_{\mu\nu}(x)$

Results for $\pi^0 \to \gamma\gamma$

Perform the integral in the region of $\sqrt{t^2 + \vec{x}^2} < R$



Move to $\pi^0 \rightarrow e^+e^-$



【N. Christ, X. Feng, L. Jin, Y. Zhao, PRL 130 (2023) 191901】

- Decay amplitude

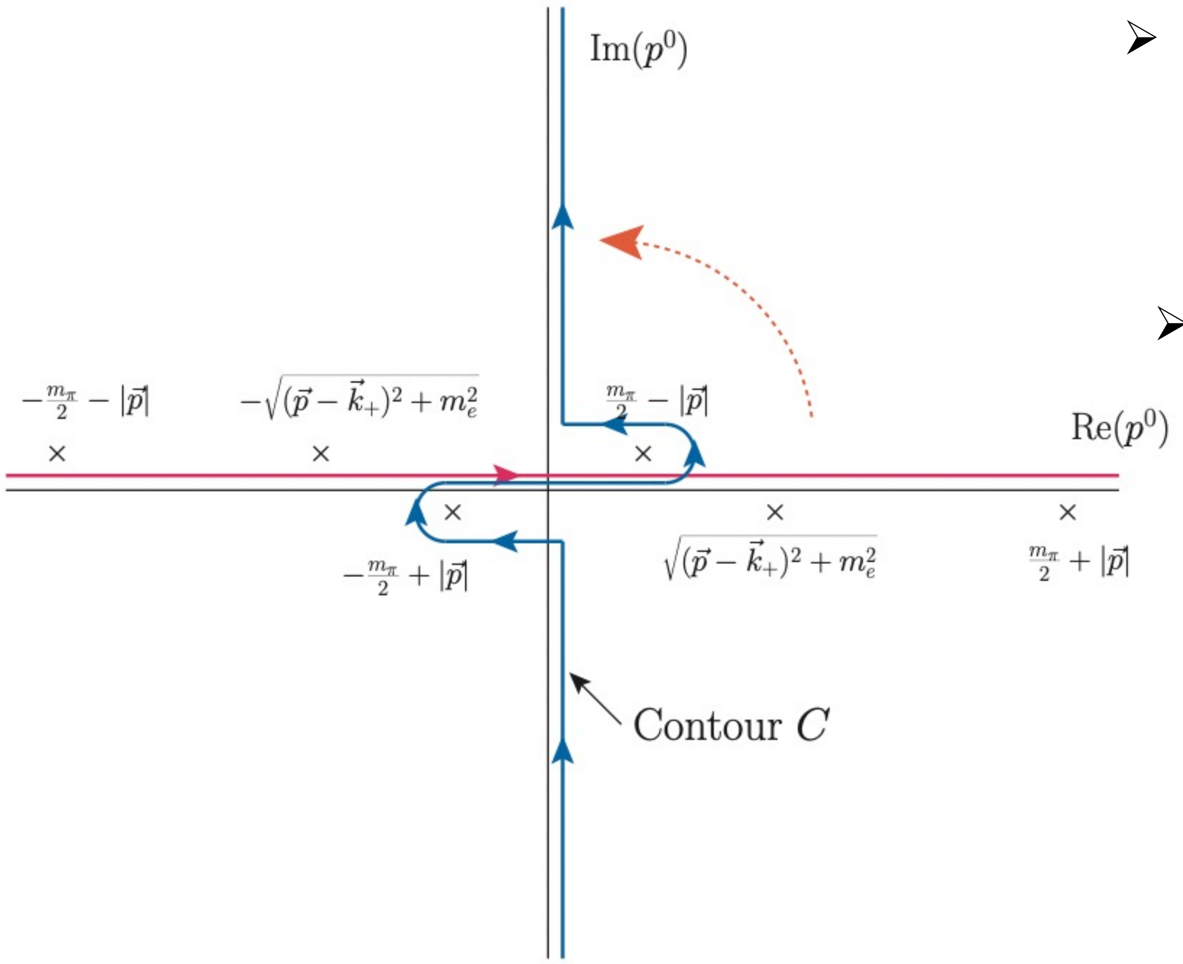
$$\mathcal{A} = e^4 \int \mathrm{d}^4 w \, \langle 0 | T\{J_\mu(\frac{w}{2}) J_\nu(-\frac{w}{2})\} | \pi^0 \rangle$$

Hadronic function from lattice QCD

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot w} \left[\frac{g_{\mu\mu'}}{(p+\frac{P}{2})^2 - i\epsilon} \right] \left[\frac{g_{\nu\nu'}}{(p-\frac{P}{2})^2 - i\epsilon} \right] \overline{u}(k_-, h) \gamma_{\mu'} \left[\frac{\gamma\cdot(p+\frac{P}{2}-k_+) + m_e}{(p+\frac{P}{2}-k_+)^2 + m_e^2 - i\epsilon} \right] \gamma_{\nu'} v(k_+, h)$$

Analytically known weight function

Singularity of weight function



- Express the weight function in position space with coordinates w

$$\mathcal{A} = \int \mathrm{d}^4 w \; L_{\mu\nu}(w) H_{\mu\nu}(w)$$

- Weight function contains both imaginary and real part

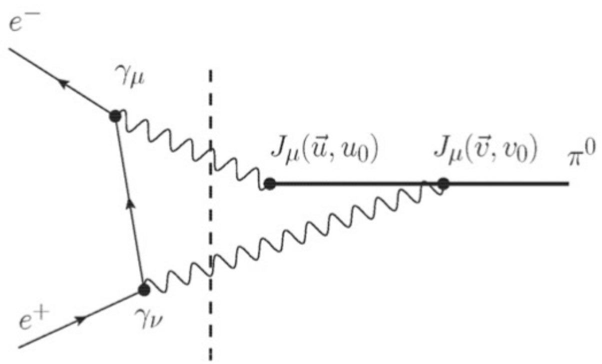
e.g.

$$L^{im}(w) = \frac{2\pi m_e \alpha^2}{m_\pi^2 \beta} \ln \frac{1-\beta}{1+\beta} \frac{M_\pi}{2} |\vec{w}| j_1\left(\frac{m_\pi}{2}|\vec{w}|\right)$$

with $\beta = \sqrt{1 - 4m_e^2/m_\pi^2}$

$\gamma\gamma \to e^+e^-$ part

$\pi^0 \to \gamma\gamma$ part

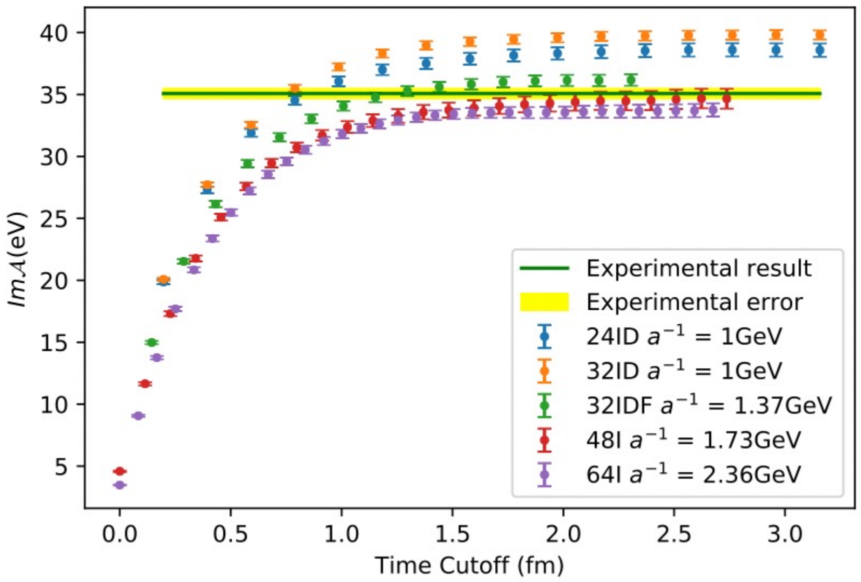


Results

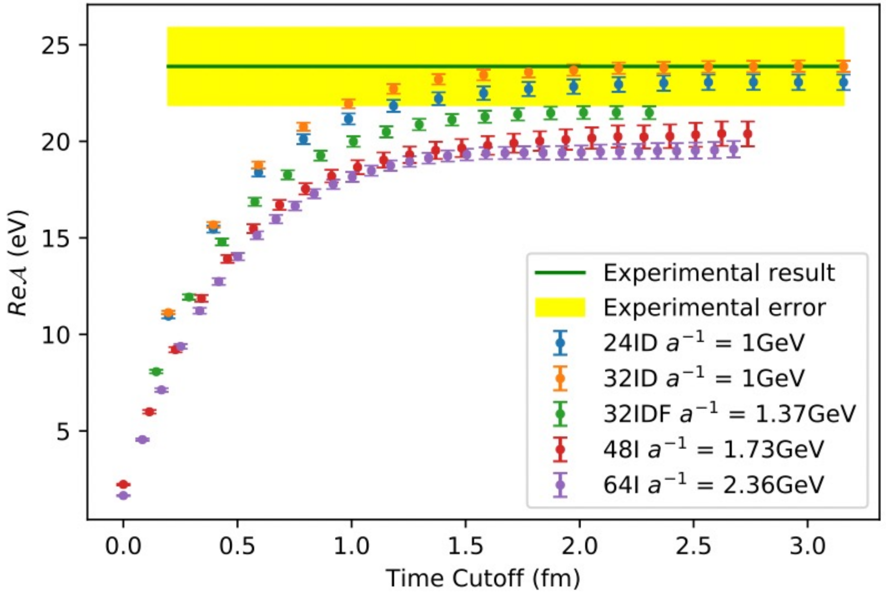
➢ List of gauge ensembles used, DWF @ phys. Pion mass

	24ID	32ID	32IDF	48I	64I
a^{-1} (GeV)	1.015	1.015	1.37	1.73	2.36
M_π (MeV)	140	140	143	135	135
Configuration separation	10	10	10	10	20
Configurations	47	47	61	32	49
point sources	1024	2048	1024	1024	1024
Δt	10	10	14	16	22

【N. Christ, X. Feng, L. Jin, Y. Zhao, PRL 130 (2023) 191901】



(a) Imaginary part



(b) Real part

Results

➢ List of gauge ensembles used, DWF @ phys. Pion mass

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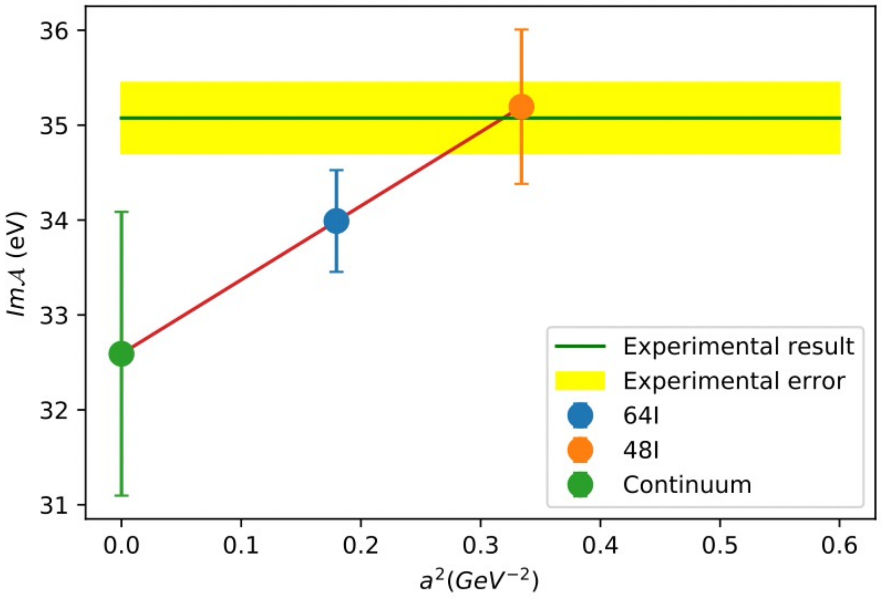
Configuration separation	10	10	10	10	20
Configurations	47	47	61	32	49
point sources	1024	2048	1024	1024	1024
Δt	10	10	14	16	22

➢ Lattice result is 1.8 σ below experimental value, where radiative corrections have been removed

【N. Christ, X. Feng, L. Jin, Y. Zhao, PRL 130 (2023) 191901】



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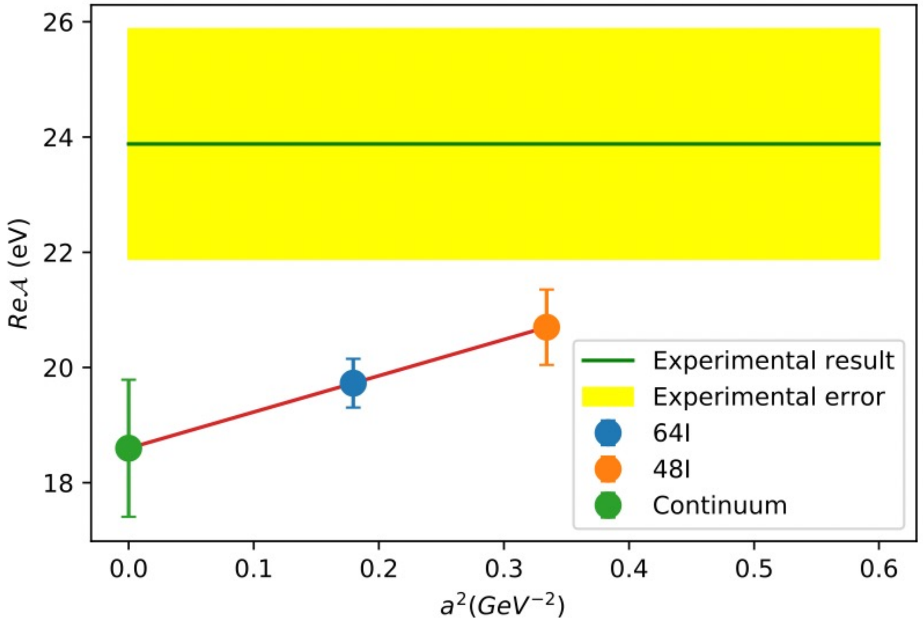
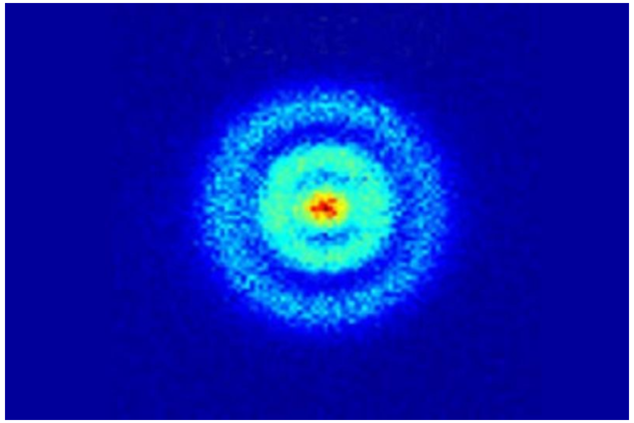
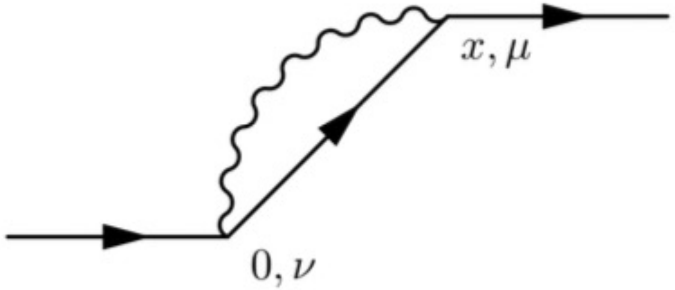


(b) Real part

Conclusion

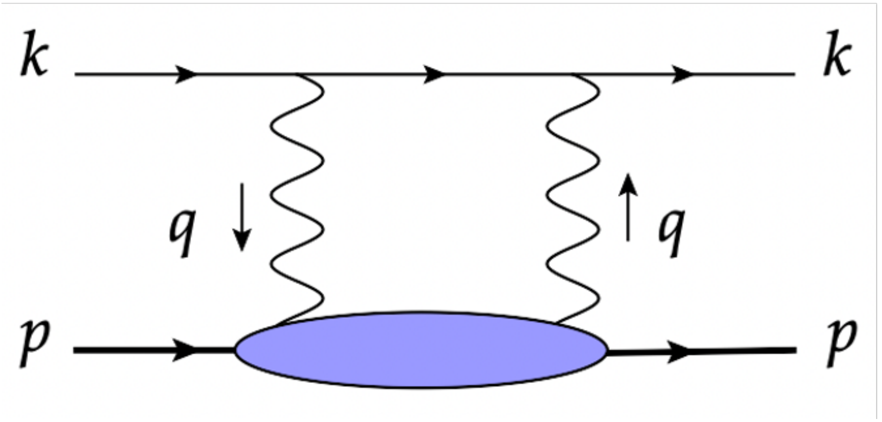


Hadron charge radius

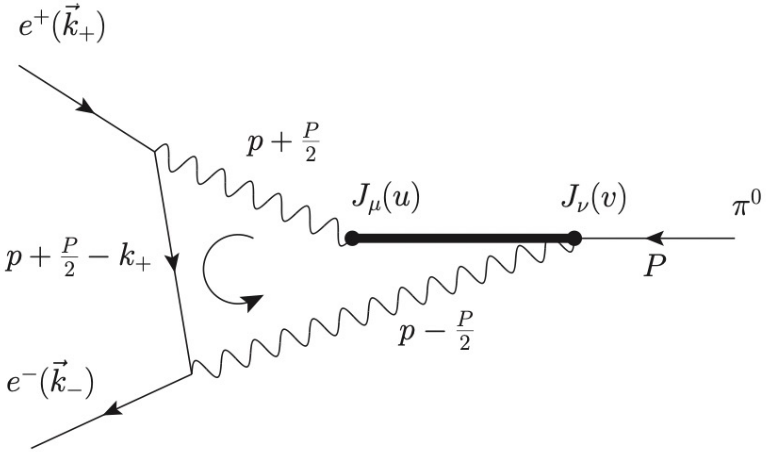


QED self energy

$$O = \sum_{\vec{x}} \omega(\vec{x}, t) H^{(L)}(\vec{x}, t)$$



Two photon exchange



$\pi^0 \to e^+e^-$ decays