Massive Integrals

Outlook 000

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

Thomas Stone

In collaboration with Stephen Jones (IPPP) & Anton Olsson (KIT)

(and the rest of the pySecDec collaboration: G. Heinrich, M. Kerner, V. Magerya, J. Schlenk)

16th April 2024





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Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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Calculating Loop Integrals

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Calculating Loop Integrals

 Many loop integrals appearing in state-of-the-art amplitude calculations are analytically intractable

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Calculating Loop Integrals

- Many loop integrals appearing in state-of-the-art amplitude calculations are analytically intractable
- Numerical methods developed to tackle these integrals (Monte Carlo techniques, differential equation methods etc.)

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Calculating Loop Integrals

- Many loop integrals appearing in state-of-the-art amplitude calculations are analytically intractable
- Numerical methods developed to tackle these integrals (Monte Carlo techniques, differential equation methods etc.)
- Exploring singularity structure of Feynman integrals can help us understand how to integrate in the Minkowski regime



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Minkowski Regime

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Massive Integrals

Minkowski Regime

• Often have trouble numerically calculating in the Minkowski regime due to poles on the contour of integration

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

Massive Integrals

Minkowski Regime

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 \Rightarrow Contour Deformation

Massive Integrals

Minkowski Regime

• Often have trouble numerically calculating in the Minkowski regime due to poles on the contour of integration

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• Causal prescription provided by Feynman $i\delta$

[Schwinger; Feynman, Landau; Eden, Landshoff, Olive, Polkinghorne; Hannesdottir, Mizera; ...]

Massive Integrals

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Minkowski Regime

• Often have trouble numerically calculating in the Minkowski regime due to poles on the contour of integration

 \Rightarrow Contour Deformation



- Causal prescription provided by Feynman $i\delta$ [Schwinger; Feynman, Landau; Eden, Landshoff, Olive, Polkinghorne; Hannesdottir, Mizera; ...]
- Methods being explored to remove need for contour deformation in momentum space [Buchta, Chachamis, Draggiotis, Rodrigo; Anastasiou, Haindl, Sterman, Yang, Zeng; Aguilera-Verdugo,

Hernandez-Pinto, Sborlini, Torres Bobadilla; Capatti, Hirschi, Kermanschah, Pelloni, Ruijl; ...]

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- Can we do the same in Feynman parameter space?

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Recap: Feynman Parameterisation

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Recap: Feynman Parameterisation

Momentum Space Integral

$$I = \int_{-\infty}^{+\infty} \left(\prod_{l=1}^{L} \frac{\mathrm{d}^{D} k_{l}}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{N} \frac{1}{P_{j}^{\nu_{j}} \left(\{k\}, \{p\}, m_{j}^{2} \right)}$$

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Feynman-Parameterised Integral

$$I = \frac{(-1)^{\nu} \Gamma(\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_{\mathbb{R}_{\geq 0}^{N}} \left(\prod_{j=1}^{N} \mathrm{d}x_j x_j^{\nu_j - 1} \right) \frac{\mathcal{U}(\mathbf{x})^{\nu - (L+1)D/2}}{(\mathcal{F}(\mathbf{x}, \mathbf{s}) - i\delta)^{\nu - LD/2}} \delta\left(1 - \sum_{j=1}^{N} x_j \right)$$

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Recap: Feynman Parameterisation

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 $\mathcal{U} \And \mathcal{F}$ constructable directly from Feynman diagrams with $\mathcal{U} \ge 0$ and \mathcal{F} depending on both parameters **and** kinematics

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Contour Deformation in Parameter Space

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Contour Deformation in Parameter Space

• The $i\delta$ prescription in momentum space induces $\mathcal{F} - i\delta$ in Feynman parameter space

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Contour Deformation in Parameter Space

- The $i\delta$ prescription in momentum space induces $\mathcal{F}-i\delta$ in Feynman parameter space
- This is needed when \mathcal{F} is 0 within the boundary of integration (i.e. within $\mathbb{R}^N_{>0}$)

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Contour Deformation in Parameter Space

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Implementation in Parameter Space

$$\mathcal{F}(\vec{z}) = \mathcal{F}(\vec{x}) - i \sum_{j} \tau_{j} \frac{\partial \mathcal{F}(\vec{x})}{\partial x_{j}} \qquad \tau_{j} = \lambda_{j} x_{j} \left(1 - x_{j}\right) \frac{\partial \mathcal{F}(\vec{x})}{\partial x_{j}}$$

[Soper; Binoth, Guillet, Heinrich, Pilon, Schubert; Nagy; Anastasiou, Beerli, Daleo; Borowka, Carter; ...]

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• Many techniques, including neural networks [Winterhalder, Magerya, Villa, Jones, Kerner, Butter, Heinrich, Plehn; ...], have been used to optimise this choice but overall can still slow down integration massively

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- Many techniques, including neural networks [Winterhalder, Magerya, Villa, Jones, Kerner, Butter, Heinrich, Plehn; ...], have been used to optimise this choice but overall can still slow down integration massively
- There are even cases where this procedure fails entirely! [see Stephen Jones' talk]

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Contour Deformation in Parameter Space

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Contour Deformation in Parameter Space

"Can we avoid this sometimes?"

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Contour Deformation in Parameter Space

"Can we avoid this sometimes?"

Yes!

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Contour Deformation in Parameter Space

"Can we avoid this sometimes?"

Yes!

"Always?"

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Contour Deformation in Parameter Space

"Can we avoid this sometimes?"

Yes!

"Always?"

Maybe...?

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The Idea

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The Idea

Construct transformations of the Feynman parameters to map zeroes of the ${\cal F}$ polynomial to the boundary of integration

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The Idea

Construct transformations of the Feynman parameters to map zeroes of the ${\cal F}$ polynomial to the boundary of integration



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The Idea

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The Idea

 These transformations look like: a) positive parameter rescalings: e.g. x_j → αx_j or x_j → x_ix_j

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Massive Integrals

The Idea

 These transformations look like: a) positive parameter rescalings: e.g. x_i → αx_i or x_i → x_ix_j

Cheng-Wu Theorem [Cheng, Wu] [see e.g. Smirnov; Weinzierl; ...]

$$\forall S \subseteq \{1, ..., N\} \land S \neq \emptyset :$$

$$\delta \left(1 - \sum_{j=1}^{N} x_j\right) \rightarrow \delta \left(1 - \sum_{j \in S} x_j\right) \text{ leaves } I \text{ invariant}$$

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 and b) the introduction of hierarchies between Feynman parameters which generate split integrals to cover the entire original parameter space: e.g. x_j → x_i + x_j

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Massive Integrals

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Heaviside Identity

Under the integral sign: $\theta(x_a - x_b) + \theta(x_b - x_a) = 1$

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The Idea

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The Idea

• These transformations result in us only integrating multiple manifestly non-negative integrands

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The Idea

- These transformations result in us only integrating multiple manifestly non-negative integrands
- For the transformations which make *F* non-positive, extract an overall minus sign and bring it out of the integral along with the *iδ* to generate the physically-correct imaginary part

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Overall Construction

$$I = \sum_{n_{+}=1}^{N_{+}} I_{n_{+}}^{+} + (-1 - i\delta)^{-(\nu - LD/2)} \sum_{n_{-}=1}^{N_{-}} I_{n_{-}}^{-}$$

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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- We stitch everything together as follows:

Overall Construction

$$I = \sum_{n_{+}=1}^{N_{+}} I_{n_{+}}^{+} + (-1 - i\delta)^{-(\nu - LD/2)} \sum_{n_{-}=1}^{N_{-}} I_{n_{-}}^{-}$$

 Can be much faster numerically to only calculate the manifestly non-negative integrals {*I*⁺_{n+}, *I*⁻_{n-}}!

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Massive Integrals

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The Idea

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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The Idea

Lets consider the one-loop massless box with an offshell leg to make this concrete:

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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The Idea

Lets consider the one-loop massless box with an offshell leg to make this concrete:



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Introduction & Motivation

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1-Loop Off-Shell Box

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1-Loop Off-Shell Box



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1-Loop Off-Shell Box



• *U* =

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• *U* =**x**₀

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1-Loop Off-Shell Box



• $\mathcal{U} = x_0 + x_1$

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1-Loop Off-Shell Box



• $\mathcal{U} = x_0 + x_1 + x_2$

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1-Loop Off-Shell Box



• $U = x_0 + x_1 + x_2 + x_3$

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U =x₀+x₁+x₂+x₃ *F* =

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1-Loop Off-Shell Box



• $\mathcal{U} = x_0 + x_1 + x_2 + x_3$

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•
$$\mathcal{F} = -sx_0x_2$$

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• $\mathcal{U} = x_0 + x_1 + x_2 + x_3$

•
$$\mathcal{F} = -sx_0x_2 - tx_1x_3$$

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U = x₀+x₁+x₂+x₃ *F* = −sx₀x₂−tx₁x₃−p₁²x₀x₁

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U = x₀+x₁+x₂+x₃ *F* = −sx₀x₂−tx₁x₃−p₁²x₀x₁

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Let's consider the regime: s > 0, $p_1^2 > 0$ & $t < 0 \implies$ zeroes of \mathcal{F} within the integration volume for $\{x_0, x_1, x_2, x_3\} \in \mathbb{R}^4_{\geq 0}$

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•
$$\mathcal{F} = -sx_0x_2 + |t|x_1x_3 - p_1^2x_0x_1$$

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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1-Loop Off-Shell Box

•
$$\mathcal{F} = -sx_0x_2 + |t|x_1x_3 - p_1^2x_0x_1$$

• First, rescale
$$x_0 \& x_3$$
: $x_0 \to \frac{x_0 x_1}{s}$, $x_3 \to \frac{x_2 x_3}{|t|}$

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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1-Loop Off-Shell Box

•
$$\mathcal{F} = -sx_0x_2 + |t|x_1x_3 - p_1^2x_0x_1$$

• First, rescale
$$x_0$$
 & x_3 : $x_0 \rightarrow \frac{x_0 x_1}{s}$, $x_3 \rightarrow \frac{x_2 x_3}{|t|}$

•
$$\mathcal{F} \to x_1 \left(x_2 \left(x_3 - x_0 \right) - \frac{p_1^2}{s} x_0 x_1 \right)$$

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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1-Loop Off-Shell Box

- $\mathcal{F} = -sx_0x_2 + |t|x_1x_3 p_1^2x_0x_1$
- First, rescale x_0 & x_3 : $x_0 \rightarrow \frac{x_0x_1}{s}$, $x_3 \rightarrow \frac{x_2x_3}{|t|}$
- $\mathcal{F} \to x_1 \left(x_2 \left(x_3 x_0 \right) \frac{p_1^2}{s} x_0 x_1 \right)$
- Introduce the hierarchy $x_0 > x_3$ by *shifting* x_0 : $x_0 \rightarrow x_0 + x_3$

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1-Loop Off-Shell Box

- $\mathcal{F} = -sx_0x_2 + |t|x_1x_3 p_1^2x_0x_1$
- First, rescale x_0 & x_3 : $x_0 \rightarrow \frac{x_0 x_1}{s}$, $x_3 \rightarrow \frac{x_2 x_3}{|t|}$
- $\mathcal{F} \to x_1 \left(x_2 \left(x_3 x_0 \right) \frac{p_1^2}{s} x_0 x_1 \right)$
- Introduce the hierarchy $x_0 > x_3$ by *shifting* x_0 : $x_0 \rightarrow x_0 + x_3$
- $\mathcal{F} \to -\frac{1}{s} \left(x_1 \left(s x_0 x_2 + p_1^2 x_1 \left(x_0 + x_3 \right) \right) \right)$

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1-Loop Off-Shell Box

- $\mathcal{F} = -sx_0x_2 + |t|x_1x_3 p_1^2x_0x_1$
- First, rescale x_0 & x_3 : $x_0 \rightarrow \frac{x_0 x_1}{s}$, $x_3 \rightarrow \frac{x_2 x_3}{|t|}$
- $\mathcal{F} \to x_1 \left(x_2 \left(x_3 x_0 \right) \frac{p_1^2}{s} x_0 x_1 \right)$
- Introduce the hierarchy $x_0 > x_3$ by *shifting* x_0 : $x_0 \rightarrow x_0 + x_3$
- $\mathcal{F} \to -\frac{1}{s} \left(x_1 \left(s x_0 x_2 + p_1^2 x_1 \left(x_0 + x_3 \right) \right) \right) =: -\mathcal{F}_1^-$

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1-Loop Off-Shell Box

- $\mathcal{F} = -sx_0x_2 + |t|x_1x_3 p_1^2x_0x_1$
- First, rescale x_0 & x_3 : $x_0 \rightarrow \frac{x_0 x_1}{s}$, $x_3 \rightarrow \frac{x_2 x_3}{|t|}$
- $\mathcal{F} \to x_1 \left(x_2 \left(x_3 x_0 \right) \frac{p_1^2}{s} x_0 x_1 \right)$
- Introduce the hierarchy $x_0 > x_3$ by *shifting* $x_0: x_0 \rightarrow x_0 + x_3$
- $\mathcal{F} \to -\frac{1}{s} \left(x_1 \left(s x_0 x_2 + p_1^2 x_1 \left(x_0 + x_3 \right) \right) \right) =: -\mathcal{F}_1^-$
- \mathcal{F}_1^- non-negative as required!

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1-Loop Off-Shell Box

- $\mathcal{F} = -sx_0x_2 + |t|x_1x_3 p_1^2x_0x_1$
- First, rescale x_0 & x_3 : $x_0 \rightarrow \frac{x_0 x_1}{s}$, $x_3 \rightarrow \frac{x_2 x_3}{|t|}$
- $\mathcal{F} \to x_1 \left(x_2 \left(x_3 x_0 \right) \frac{p_1^2}{s} x_0 x_1 \right)$
- Introduce the hierarchy $x_0 > x_3$ by *shifting* x_0 : $x_0 \rightarrow x_0 + x_3$
- $\mathcal{F} \to -\frac{1}{s} \left(x_1 \left(s x_0 x_2 + p_1^2 x_1 \left(x_0 + x_3 \right) \right) \right) =: -\mathcal{F}_1^-$
- \mathcal{F}_1^- non-negative as required!
- To cover the whole original space, the Heaviside identity tells us we need to consider the converse case (x₃ > x₀) so *shift* x₃: x₃ → x₀ + x₃

1-Loop Off-Shell Box

- $\mathcal{F} = -sx_0x_2 + |t|x_1x_3 p_1^2x_0x_1$
- First, rescale x_0 & x_3 : $x_0 \rightarrow \frac{x_0 x_1}{s}$, $x_3 \rightarrow \frac{x_2 x_3}{|t|}$
- $\mathcal{F} \to x_1 \left(x_2 \left(x_3 x_0 \right) \frac{p_1^2}{s} x_0 x_1 \right)$
- Introduce the hierarchy $x_0 > x_3$ by *shifting* x_0 : $x_0 \rightarrow x_0 + x_3$

•
$$\mathcal{F} \to -\frac{1}{s} \left(x_1 \left(s x_0 x_2 + p_1^2 x_1 \left(x_0 + x_3 \right) \right) \right) =: -\mathcal{F}_1^-$$

- \mathcal{F}_1^- non-negative as required!
- To cover the whole original space, the Heaviside identity tells us we need to consider the converse case (x₃ > x₀) so *shift* x₃:
 x₃ → x₀ + x₃

•
$$\mathcal{F} \to x_1 \left(-\frac{p_1^2}{s} x_0 x_1 + x_2 x_3 \right)$$

1-Loop Off-Shell Box

- $\mathcal{F} = -sx_0x_2 + |t|x_1x_3 p_1^2x_0x_1$
- First, rescale x_0 & x_3 : $x_0 \rightarrow \frac{x_0 x_1}{s}$, $x_3 \rightarrow \frac{x_2 x_3}{|t|}$
- $\mathcal{F} \to x_1 \left(x_2 \left(x_3 x_0 \right) \frac{p_1^2}{s} x_0 x_1 \right)$
- Introduce the hierarchy $x_0 > x_3$ by *shifting* x_0 : $x_0 \rightarrow x_0 + x_3$

•
$$\mathcal{F} \to -\frac{1}{s} \left(x_1 \left(s x_0 x_2 + p_1^2 x_1 \left(x_0 + x_3 \right) \right) \right) =: -\mathcal{F}_1^-$$

To cover the whole original space, the Heaviside identity tells us we need to consider the converse case (x₃ > x₀) so *shift* x₃: x₃ → x₀ + x₃

•
$$\mathcal{F} \to x_1 \left(-\frac{p_1^2}{s} x_0 x_1 + x_2 x_3 \right)$$

• This is **not** of uniform sign \Rightarrow needs further work!

Introduction & Motivation

Massless Integrals

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1-Loop Off-Shell Box

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1-Loop Off-Shell Box

• Rescale again:
$$x_2 \rightarrow \frac{p_1^2 x_0 x_2}{s}, x_1 \rightarrow x_1 x_3$$

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1-Loop Off-Shell Box

• *Rescale* again:
$$x_2 \rightarrow \frac{p_1^2 x_0 x_2}{s}, x_1 \rightarrow x_1 x_3$$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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1-Loop Off-Shell Box

• Rescale again: $x_2 \rightarrow \frac{p_1^2 x_0 x_2}{s}, x_1 \rightarrow x_1 x_3$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

• Suggests introducing new hierarchy! First $x_2 > x_1 \Rightarrow shift x_2$:

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1-Loop Off-Shell Box

• Rescale again: $x_2
ightarrow rac{p_1^2 x_0 x_2}{s}, x_1
ightarrow x_1 x_3$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

• Suggests introducing new hierarchy! First $x_2 > x_1 \Rightarrow shift x_2$:

•
$$\mathcal{F} \rightarrow \frac{p_1^2}{s} x_0 x_1 x_2 x_3^2$$

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1-Loop Off-Shell Box

• Rescale again: $x_2
ightarrow rac{p_1^2 x_0 x_2}{s}, x_1
ightarrow x_1 x_3$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

• Suggests introducing new hierarchy! First $x_2 > x_1 \Rightarrow shift x_2$:

•
$$\mathcal{F} \to rac{p_1^2}{s} x_0 x_1 x_2 x_3^2 =: \mathcal{F}_1^+$$

1-Loop Off-Shell Box

• Rescale again: $x_2
ightarrow rac{p_1^2 x_0 x_2}{s}, x_1
ightarrow x_1 x_3$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

• Suggests introducing new hierarchy! First $x_2 > x_1 \Rightarrow shift x_2$:

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_2 x_3^2 =: \mathcal{F}_1^+$$

• Now
$$x_1 > x_2 \Rightarrow shift x_1$$
:

1-Loop Off-Shell Box

• Rescale again: $x_2 \rightarrow \frac{p_1^2 x_0 x_2}{s}, x_1 \rightarrow x_1 x_3$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

• Suggests introducing new hierarchy! First $x_2 > x_1 \Rightarrow shift x_2$:

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_2 x_3^2 =: \mathcal{F}_1^+$$

• Now $x_1 > x_2 \Rightarrow$ *shift* x_1 :

•
$$\mathcal{F} \to -\frac{p_1^2}{s} x_0 x_1 (x_1 + x_2) x_3^2$$

1-Loop Off-Shell Box

• Rescale again: $x_2 \rightarrow \frac{p_1^2 x_0 x_2}{s}, x_1 \rightarrow x_1 x_3$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

• Suggests introducing new hierarchy! First $x_2 > x_1 \Rightarrow shift x_2$:

•
$$\mathcal{F} \to rac{p_1^2}{s} x_0 x_1 x_2 x_3^2 =: \mathcal{F}_1^+$$

• Now $x_1 > x_2 \Rightarrow$ *shift* x_1 :

•
$$\mathcal{F} \to -\frac{p_1^2}{s} x_0 x_1 (x_1 + x_2) x_3^2 =: -\mathcal{F}_2^-$$

1-Loop Off-Shell Box

• Rescale again: $x_2 \rightarrow \frac{p_1^2 x_0 x_2}{s}, x_1 \rightarrow x_1 x_3$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

• Suggests introducing new hierarchy! First $x_2 > x_1 \Rightarrow shift x_2$:

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_2 x_3^2 =: \mathcal{F}_1^+$$

- Now $x_1 > x_2 \Rightarrow$ *shift* x_1 :
- $\mathcal{F} \to -\frac{p_1^2}{s} x_0 x_1 (x_1 + x_2) x_3^2 =: -\mathcal{F}_2^-$
- Generate U₁⁺, U₁⁻ & U₂⁻ by applying the corresponding transformations to U

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1-Loop Off-Shell Box

• Rescale again: $x_2
ightarrow rac{p_1^2 x_0 x_2}{s}, x_1
ightarrow x_1 x_3$

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_3^2 (x_2 - x_1)$$

• Suggests introducing new hierarchy! First $x_2 > x_1 \Rightarrow shift x_2$:

•
$$\mathcal{F} \to \frac{p_1^2}{s} x_0 x_1 x_2 x_3^2 =: \mathcal{F}_1^+$$

• Now $x_1 > x_2 \Rightarrow shift x_1$:

•
$$\mathcal{F} \to -\frac{p_1^2}{s} x_0 x_1 (x_1 + x_2) x_3^2 =: -\mathcal{F}_2^-$$

- Generate \mathcal{U}_1^+ , \mathcal{U}_1^- & \mathcal{U}_2^- by applying the corresponding transformations to \mathcal{U}
- Generate the absolute values of the corresponding Jacobian determinants: \mathcal{J}_1^+ , \mathcal{J}_1^- & \mathcal{J}_2^-

Massive Integrals

1-Loop Off-Shell Box: Putting the Jigsaw Pieces Together

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1-Loop Off-Shell Box: Putting the Jigsaw Pieces Together

• We converted our initial single integral *I* into a sum over 3 easier integrals

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1-Loop Off-Shell Box: Putting the Jigsaw Pieces Together

• We converted our initial single integral *I* into a sum over 3 easier integrals

1-Loop Off-Shell Box

$$I = I_1^+ + (-1 - i\delta)^{-2-\varepsilon} \left(I_1^- + I_2^- \right)$$

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1-Loop Off-Shell Box: Putting the Jigsaw Pieces Together

• We converted our initial single integral *I* into a sum over 3 easier integrals

1-Loop Off-Shell Box

$$I = I_1^+ + (-1 - i\delta)^{-2-\varepsilon} \left(I_1^- + I_2^-\right)$$

• Each of the manifestly non-negative integrands has the following structure:

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1-Loop Off-Shell Box: Putting the Jigsaw Pieces Together

• We converted our initial single integral *I* into a sum over 3 easier integrals

1-Loop Off-Shell Box

$$I = I_1^+ + (-1 - i\delta)^{-2-\varepsilon} \left(I_1^- + I_2^- \right)$$

• Each of the manifestly non-negative integrands has the following structure:

$$\mathcal{J}_{n_{\pm}}^{\pm} \left(\mathcal{U}_{n_{\pm}}^{\pm} \right)^{2\varepsilon} \left(\mathcal{F}_{n_{\pm}}^{\pm} \right)^{-2-\varepsilon}$$

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1-Loop Off-Shell Box: Putting the Jigsaw Pieces Together

• We converted our initial single integral *I* into a sum over 3 easier integrals

1-Loop Off-Shell Box

$$I = I_1^+ + (-1 - i\delta)^{-2-\varepsilon} \left(I_1^- + I_2^- \right)$$

• Each of the manifestly non-negative integrands has the following structure:

$$\mathcal{J}_{n_{\pm}}^{\pm}\left(\mathcal{U}_{n_{\pm}}^{\pm}
ight)^{2arepsilon}\left(\mathcal{F}_{n_{\pm}}^{\pm}
ight)^{-2-arepsilon}$$

 \bullet Verified numerical result against the known analytic result \checkmark

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2-Loop Non-Planar Box

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2-Loop Non-Planar Box



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2-Loop Non-Planar Box



• $\mathcal{U} = x_0 x_1 + x_0 x_2 + x_0 x_3 + x_0 x_4 + x_1 x_2 + x_1 x_3 + x_1 x_5 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_4 x_5$

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2-Loop Non-Planar Box



• $\mathcal{U} = x_0x_1 + x_0x_2 + x_0x_3 + x_0x_4 + x_1x_2 + x_1x_3 + x_1x_5 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5$

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• $\mathcal{F} = -sx_1x_2x_5 - tx_0x_1x_3 - ux_0x_2x_4$

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2-Loop Non-Planar Box



Momentum conservation implies $s + t + u = 0 \Rightarrow u = -(s + t)$

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2-Loop Non-Planar Box



Momentum conservation implies $s + t + u = 0 \Rightarrow u = -(s + t)$ Hence, \mathcal{F} can be 0 within $\{x_i\} \in \mathbb{R}^6_{\geq 0}$ even with s > 0, t > 0

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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2-Loop Non-Planar Box



Momentum conservation implies $s + t + u = 0 \Rightarrow u = -(s + t)$ Hence, \mathcal{F} can be 0 within $\{x_i\} \in \mathbb{R}^6_{\geq 0}$ even with s > 0, t > 0Not possible to define a Euclidean region at all!

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2-Loop Non-Planar Box



Momentum conservation implies $s + t + u = 0 \Rightarrow u = -(s + t)$ Hence, \mathcal{F} can be 0 within $\{x_i\} \in \mathbb{R}^6_{\geq 0}$ even with s > 0, t > 0Not possible to define a Euclidean region at all! Nevertheless, the method works

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2-Loop Non-Planar Box

•
$$\mathcal{F} = -sx_1x_2x_5 - tx_0x_1x_3 + (s+t)x_0x_2x_4$$

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2-Loop Non-Planar Box

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• $x_1 \rightarrow x_1 x_4$ then $x_2 \rightarrow x_1 x_2$ then $x_0 \rightarrow x_0 x_1$

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2-Loop Non-Planar Box

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$$\mathcal{F} \to x_1^2 x_4 \left(s x_2 \left(x_0 - x_5 \right) + |t| x_0 \left(x_3 - x_2 \right) \right)$$

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2-Loop Non-Planar Box

- $\mathcal{F} = -sx_1x_2x_5 tx_0x_1x_3 + (s+t)x_0x_2x_4$
- $x_1 \rightarrow x_1 x_4$ then $x_2 \rightarrow x_1 x_2$ then $x_0 \rightarrow x_0 x_1$
- $\mathcal{F} \to x_1^2 x_4 \left(s x_2 \left(x_0 x_5 \right) + |t| x_0 \left(x_3 x_2 \right) \right)$
- Let's look at one set of splittings to see how the method works in the kinematic region s>-t=1

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2-Loop Non-Planar Box

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2-Loop Non-Planar Box

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2-Loop Non-Planar Box

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•
$$\mathcal{F} \to -x_1^2 x_2 x_4 x_5 (s-1+sx_3) =: -\mathcal{F}_1^-$$

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2-Loop Non-Planar Box: Putting the Pieces Together

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2-Loop Non-Planar Box: Putting the Pieces Together

• To cover all the space in this kinematic regime, we find we need 6 integrals



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2-Loop Non-Planar Box: Putting the Pieces Together

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2-Loop Non-Planar Box

$$I = (I_1^+ + I_2^+ + I_3^+) + (-1 - i\delta)^{-2-2\varepsilon} (I_1^- + I_2^- + I_3^-)$$

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2-Loop Non-Planar Box: Putting the Pieces Together

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2-Loop Non-Planar Box

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• Verified numerical result against the known analytic result \checkmark $_{\rm [Tausk 99]}$

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2-Loop Non-Planar Box: Putting the Pieces Together

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2-Loop Non-Planar Box: Putting the Pieces Together

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- For the kinematic regime 0 < s < -t = 1, we find a different set of split integrals (still 6 in total)

$\delta\text{-}\mathsf{Function}$ Issue

Some rescalings can modify our $\delta\text{-function}$ in an impractical way for our numerical integration setup - let's avoid that!

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2-Loop Non-Planar Box: pySecDec Timing Comparison

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3-Loop Non-Planar Box



Diagram by Yao Ma

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3-Loop Non-Planar Box

• By considering all permutations of hierarchies for the Feynman parameters and considering symmetry, we find 6 integrals

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3-Loop Non-Planar Box

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- For the kinematic regime s > -t > 0, we find 2 of these integrals would usually require contour deformation

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

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•
$$\mathcal{F}^{a} = x_{1}x_{3}x_{5}x_{7}\left[-sx_{0}x_{2}+|t|(x_{0}+x_{4})(x_{2}+x_{4})\right]$$

• $\mathcal{F}^{b} = x_{1}x_{3}x_{5}x_{7}\left[sx_{6}\left(x_{0}+x_{2}+x_{6}\right)-\left|t\right|\left(x_{0}+x_{6}\right)\left(x_{2}+x_{6}\right)\right]$

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3-Loop Non-Planar Box

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• Through a more involved combination of splits and rescalings, we can express each of these integrals in terms of 4 others

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3-Loop Non-Planar Box

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$$\mathcal{F}^{a} = x_{1}x_{3}x_{5}x_{7}\left[-sx_{0}x_{2}+|t|(x_{0}+x_{4})(x_{2}+x_{4})\right]$$

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$$\mathcal{F}^{b} = x_{1}x_{3}x_{5}x_{7} [sx_{6}(x_{0} + x_{2} + x_{6}) - |t|(x_{0} + x_{6})(x_{2} + x_{6})]$$

- Through a more involved combination of splits and rescalings, we can express each of these integrals in terms of 4 others
- This gives us 12 integrals to compute, none of which require contour deformation!

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3-Loop Non-Planar Box: Putting the Pieces Together

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3-Loop Non-Planar Box: Putting the Pieces Together

3-Loop Non-Planar Box

$$I = \sum_{n_{+}=1}^{8} I_{n_{+}}^{+} + (-1 - i\delta)^{-2-3\varepsilon} \sum_{n_{-}=1}^{4} I_{n_{-}}^{-}$$

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[Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21]

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3-Loop Non-Planar Box: Putting the Pieces Together

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3-Loop Non-Planar Box: Putting the Pieces Together

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- Struggled to go beyond leading pole $(\frac{1}{\varepsilon^4})$ with contour deformation
- Avoiding contour deformation allowed us to go to higher orders (tried up to $\frac{1}{c^2}$)

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3-Loop Non-Planar Box: pySecDec Precision Comparison

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3-Loop Non-Planar Box: pySecDec Precision Comparison

Let's look at the leading pole for $I_a \& I_b$

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3-Loop Non-Planar Box: pySecDec Precision Comparison

Let's look at the leading pole for $I_a \& I_b$

 I_a & I_b (s = 1, t = -1/5) after $\mathcal{O}(\mathsf{mins})$ with pySecDec

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3-Loop Non-Planar Box: pySecDec Precision Comparison

Let's look at the leading pole for $I_a \& I_b$

$I_a \& I_b \ (s = 1, t = -1/5)$ after $\mathcal{O}(\text{mins})$ with pySecDec

$$\begin{split} I_a^{\rm CD} &= \varepsilon^{-4} \left[(18.5195704502 - 15.707988011i) \pm (5.897 * 10^{-5} + 5.897 * 10^{-5}i) \right] + \ldots \\ I_a^{\rm NOCD} &= \varepsilon^{-4} \left[(18.51948920208488 - 15.70796326794897i) \pm (4.032 * 10^{-11} + 4.592 * 10^{-11}i) \right] + \ldots \end{split}$$

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3-Loop Non-Planar Box: pySecDec Precision Comparison

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What about the full result?

Let's look at the leading pole for $I_a \& I_b$

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I(s = 1, t = -1/5) after $\mathcal{O}(\mathsf{mins})$ with pySecDec

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Let's look at the leading pole for $I_a \& I_b$

$I_a \& I_b \ (s = 1, t = -1/5)$ after $\mathcal{O}(\text{mins})$ with pySecDec

$$\begin{split} & I_a^{\text{CD}} = \varepsilon^{-4} \left[(18.5195704502 - 15.707988011i) \pm (5.897 * 10^{-5} + 5.897 * 10^{-5}i) \right] + \dots \\ & I_a^{\text{NOCD}} = \varepsilon^{-4} \left[(18.51948920208488 - 15.70796326794897i) \pm (4.032 * 10^{-11} + 4.592 * 10^{-11}i) \right] + \dots \\ & I_b^{\text{CD}} = \varepsilon^{-4} \left[(12.7432949988 - 23.561968275i) \pm (1.605 * 10^{-5} + 1.415 * 10^{-5}i) \right] + \dots \\ & I_b^{\text{NOCD}} = \varepsilon^{-4} \left[(12.74326469721394 - 23.5619449018131i) \pm (4.125 * 10^{-11} + 6.919 * 10^{-11}i) \right] + \dots \end{split}$$

What about the full result?

I(s = 1, t = -1/5) after $\mathcal{O}(mins)$ with pySecDec

 $I^{\text{CD}} = \varepsilon^{-4} [8.34055 - 52.3608i] + \mathcal{O} (\varepsilon^{-3})$

Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

Thomas Stone

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Let's look at the leading pole for $I_a \& I_b$

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What about the full result?

I (s = 1, t = -1/5) after \mathcal{O} (mins) with pySecDec

$$\begin{split} I^{\text{CD}} &= \varepsilon^{-4} \left[8.340 \textbf{55} - 52.36 \textbf{08} i \right] + \mathcal{O} \left(\varepsilon^{-3} \right) \\ I^{\text{NOCD}} &= \varepsilon^{-4} \left[8.3400403920 \textbf{28} - 52.35987755983 \textbf{47} i \right] + \mathcal{O} \left(\varepsilon^{-3} \right) \end{split}$$

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Let's look at the leading pole for $I_a \& I_b$

$I_a \& I_b \ (s = 1, t = -1/5)$ after $\mathcal{O}(\text{mins})$ with pySecDec

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What about the full result?

$$I(s = 1, t = -1/5)$$
 after $\mathcal{O}(\mathsf{mins})$ with pySecDec

$$I^{\text{CD}} = \varepsilon^{-4} [8.34055 - 52.3608i] + \mathcal{O} (\varepsilon^{-3})$$
$$I^{\text{NOCD}} = \varepsilon^{-4} [8.340040392028 - 52.3598775598347i] + \mathcal{O} (\varepsilon^{-3})$$

 $I_{\text{analytic}} = \varepsilon^{-4} \left[8.34004039223768 - 52.35987755984493 i \right] + \mathcal{O} \left(\varepsilon^{-3} \right)$

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3-Loop Non-Planar Box: pySecDec Timing Comparison

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3-Loop Non-Planar Box: pySecDec Timing Comparison



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3-Loop Non-Planar Box: pySecDec Timing

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3-Loop Non-Planar Box: pySecDec Timing



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Massive Integrals

Massive Integrals

 Integrals with massive internal propagators appear in a variety of phenomenologically-relevant amplitudes

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Massive Integrals

- Integrals with massive internal propagators appear in a variety of phenomenologically-relevant amplitudes
- Often no known analytic solution \Rightarrow numerical methods essential

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Massive Integrals

- Integrals with massive internal propagators appear in a variety of phenomenologically-relevant amplitudes
- Often no known analytic solution \Rightarrow numerical methods essential
- Does the method extends to this class of integrals?

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Massive Integrals

- Integrals with massive internal propagators appear in a variety of phenomenologically-relevant amplitudes
- Often no known analytic solution \Rightarrow numerical methods essential
- Does the method extends to this class of integrals?

${\mathcal F}$ for Massive Integrals

$$\mathcal{F}(\mathbf{x}, \mathbf{s}) = \mathcal{F}_0(\mathbf{x}, \mathbf{s})$$

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Massive Integrals

- Integrals with massive internal propagators appear in a variety of phenomenologically-relevant amplitudes
- Often no known analytic solution \Rightarrow numerical methods essential
- Does the method extends to this class of integrals?

${\mathcal F}$ for Massive Integrals

$$\mathcal{F}(\mathbf{x}, \mathbf{s}) = \mathcal{F}_{\mathbf{0}}(\mathbf{x}, \mathbf{s}) + \mathcal{U}(\mathbf{x}) \sum_{j=1}^{N} m_{j}^{2} x_{j}$$

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Massive Integrals

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Massive Integrals

- Integrals with massive internal propagators appear in a variety of phenomenologically-relevant amplitudes
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${\mathcal F}$ for Massive Integrals

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• x_j can now appear quadratically in \mathcal{F} due to new term

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Massive Integrals

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Massive Integrals

- Integrals with massive internal propagators appear in a variety of phenomenologically-relevant amplitudes
- Often no known analytic solution \Rightarrow numerical methods essential
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${\mathcal F}$ for Massive Integrals

$$\mathcal{F}(\mathbf{x}, \mathbf{s}) = \mathcal{F}_{\mathbf{0}}(\mathbf{x}, \mathbf{s}) + \mathcal{U}(\mathbf{x}) \sum_{j=1}^{N} m_{j}^{2} x_{j}$$

- x_j can now appear quadratically in \mathcal{F} due to new term
- Viable transformations difficult even for trivial integrals

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Massive Integrals

- Integrals with massive internal propagators appear in a variety of phenomenologically-relevant amplitudes
- Often no known analytic solution \Rightarrow numerical methods essential
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${\mathcal F}$ for Massive Integrals

$$\mathcal{F}(\mathbf{x}, \mathbf{s}) = \mathcal{F}_{\mathbf{0}}(\mathbf{x}, \mathbf{s}) + \mathcal{U}(\mathbf{x}) \sum_{j=1}^{N} m_{j}^{2} x_{j}$$

- x_j can now appear quadratically in \mathcal{F} due to new term
- Viable transformations difficult even for trivial integrals
- Can we use geometry to guide us in the right direction?

Massive Integrals

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• $\mathcal{F} = -p^2 x_1 x_2$

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• $\mathcal{F} = -p^2 x_1 x_2 + (x_1 + x_2) \left(\frac{m_1^2 x_1 + m_2^2 x_2}{m_1^2 x_1 + m_2^2 x_2} \right)$

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• $\mathcal{F} = -p^2 x_1 x_2 + (x_1 + x_2) \left(\frac{m_1^2 x_1 + m_2^2 x_2}{m_1^2 - (m_1 + m_2)^2} \right)$ • Define $\beta^2 := \frac{p^2 - (m_1 + m_2)^2}{p^2 - (m_1 - m_2)^2} \in [0, 1)$

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- $\mathcal{F} = -p^2 x_1 x_2 + (x_1 + x_2) \left(\frac{m_1^2 x_1 + m_2^2 x_2}{m_1^2 x_1 + m_2^2 x_2} \right)$
- Define $\beta^2 := \frac{p^2 (m_1 + m_2)^2}{p^2 (m_1 m_2)^2} \in [0, 1)$
- Scale out dimension of \mathcal{F} via $x_i \rightarrow rac{x_i}{m_i}$

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• $\mathcal{F} = -p^2 x_1 x_2 + (x_1 + x_2) \left(\frac{m_1^2 x_1 + m_2^2 x_2}{m_1^2 - (m_1 - m_2)^2} \in [0, 1) \right)$ • Define $\beta^2 := \frac{p^2 - (m_1 + m_2)^2}{p^2 - (m_1 - m_2)^2} \in [0, 1)$ • Scale out dimension of \mathcal{F} via $x_i \to \frac{x_i}{m_i}$

$$\mathcal{F}
ightarrow \widetilde{\mathcal{F}} = x_1^2 + x_2^2 - 2rac{1+eta^2}{1-eta^2}x_1x_2$$

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 $\bullet\,$ Let's consider the variety of $\widetilde{\mathcal{F}}$

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- $\bullet\,$ Let's consider the variety of $\widetilde{\mathcal{F}}$
- 3 regions \Rightarrow 3 integrals

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- $\bullet\,$ Let's consider the variety of $\widetilde{\mathcal{F}}$
- 3 regions \Rightarrow 3 integrals

• 2 positive regions, 1 negative region

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- $\bullet\,$ Let's consider the variety of $\widetilde{\mathcal{F}}$
- 3 regions \Rightarrow 3 integrals
- 2 positive regions, 1 negative region

Massive Bubble

$$I = I_1^+ + I_2^+ + (-1 - i\delta)^{-\varepsilon} I_1^-$$

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Massive Integrals

Massive Bubble



- Let's consider the variety of $\widetilde{\mathcal{F}}$
- 3 regions \Rightarrow 3 integrals
- 2 positive regions, 1 negative region

Massive Bubble

$$I = I_1^+ + I_2^+ + (-1 - i\delta)^{-\varepsilon} I_1^-$$

• Construct transformations which directly send the variety to the integration boundary

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Massive Bubble

• The solutions to
$$\widetilde{\mathcal{F}} = 0$$
 are $x_2 = \frac{1+\beta}{1-\beta}x_1$ and $x_2 = \frac{1-\beta}{1+\beta}x_1$

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- The solutions to $\widetilde{\mathcal{F}}=0$ are $x_2=rac{1+eta}{1-eta}x_1$ and $x_2=rac{1-eta}{1+eta}x_1$
- For region I, we demand the y_2 -axis coincides with the x_2 -axis and the y_1 -axis coincides with the solution line $x_2 = \frac{1+\beta}{1-\beta}x_1$

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- The solutions to $\widetilde{\mathcal{F}} = 0$ are $x_2 = rac{1+eta}{1-eta} x_1$ and $x_2 = rac{1-eta}{1+eta} x_1$
- For region I, we demand the y_2 -axis coincides with the x_2 -axis and the y_1 -axis coincides with the solution line $x_2 = \frac{1+\beta}{1-\beta}x_1$
- Along with the constraint that points within region I get mapped to the positive quadrant in the new y_i variables, this uniquely defines the transformation:

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- The solutions to $\widetilde{\mathcal{F}}=0$ are $x_2=rac{1+eta}{1-eta}x_1$ and $x_2=rac{1-eta}{1+eta}x_1$
- For region I, we demand the y_2 -axis coincides with the x_2 -axis and the y_1 -axis coincides with the solution line $x_2 = \frac{1+\beta}{1-\beta}x_1$
- Along with the constraint that points within region I get mapped to the positive quadrant in the new y_i variables, this uniquely defines the transformation:

$$y_1 \stackrel{!}{=} x_1, y_2 \stackrel{!}{=} x_2 - \frac{1+\beta}{1-\beta}x_1 \Rightarrow x_1 \rightarrow y_1, x_2 \rightarrow y_2 + \frac{1+\beta}{1-\beta}y_1$$

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 $\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + rac{4eta}{1-eta^2} y_1
ight)$

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 $\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + rac{4eta}{1-eta^2} y_1
ight)$

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 $\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + rac{4eta}{1-eta^2} y_1
ight)$

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 $\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + rac{4\beta}{1-\beta^2} y_1
ight)$

 $\widetilde{\mathcal{F}}_1^- = rac{4\beta}{1-\beta^2} y_1 y_2$

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 $\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + rac{4\beta}{1-\beta^2} y_1
ight)$

 $\widetilde{\mathcal{F}}_1^- = rac{4\beta}{1-\beta^2} y_1 y_2$

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 $\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + rac{4\beta}{1-\beta^2} y_1
ight)$

 $\widetilde{\mathcal{F}}_1^- = rac{4\beta}{1-\beta^2} y_1 y_2$

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 $\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + \frac{4\beta}{1-\beta^2} y_1 \right)$

 $\widetilde{\mathcal{F}}_1^- = \frac{4\beta}{1-\beta^2} y_1 y_2$

 $\widetilde{\mathcal{F}}_2^+ = rac{y_1\left(4eta y_2 + (1+eta)^2 y_1
ight)}{1-eta^2}$

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 $\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + rac{4\beta}{1-\beta^2} y_1
ight)$

 $\widetilde{\mathcal{F}}_1^- = \frac{4\beta}{1-\beta^2} y_1 y_2$

 $\widetilde{\mathcal{F}}_2^+ = \frac{y_1\left(4\beta y_2 + (1+\beta)^2 y_1\right)}{1-\beta^2}$

Verified result numerically & analytically <

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$$\mathcal{F} = -p_3^2 x_0 x_2$$

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 $\mathcal{F} = -p_3^2 x_0 x_2 + (x_0 + x_1 + x_2) m^2 x_0$

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$$\mathcal{F} = -p_3^2 x_0 x_2 + (x_0 + x_1 + x_2) \, \mathbf{m}^2 x_0$$

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Can solve this using rescalings and shifts as before (verified \checkmark)

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$$\mathcal{F} = -p_3^2 x_0 x_2 + (x_0 + x_1 + x_2) \, \mathbf{m}^2 x_0$$

Can solve this using rescalings and shifts as before (verified \checkmark)

Massive Triangle $I = I_1^+ + I_2^+ + (-1 - i\delta)^{-1-arepsilon} I_1^-$

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• Visualising variety of \mathcal{F} suggests 2 regions

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• Visualising variety of \mathcal{F} suggests 2 regions $\stackrel{?}{\Rightarrow} I = I^+ + (-1 - i\delta)^{-1-\varepsilon} I^-$

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- Visualising variety of \mathcal{F} suggests 2 regions $\stackrel{?}{\Rightarrow} I = I^+ + (-1 - i\delta)^{-1-\varepsilon} I^-$
- Can geometric picture tell us the minimum number of integrals we need? (i.e. #Regions ? #Integrals)

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- Visualising variety of \mathcal{F} suggests 2 regions $\stackrel{?}{\Rightarrow} I = I^+ + (-1 - i\delta)^{-1-\varepsilon} I^-$
- Can geometric picture tell us the minimum number of integrals we need? (i.e. <u>#Regions</u> [?] <u>#Integrals</u>)

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• Subject of current work

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• Can we use this geometric picture to move away from "arbitrary" shifts & rescalings and towards a general algorithm?

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Outlook

Outlook

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Introduction	Motivation

Massive Integrals

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- Understanding 2-loop massive integrals could lead to **huge** time improvements
- Applying method to multi-loop massless integrals could allow for (currently impossible) numerical cross-checks of analytic/semi-analytic results
- Look towards an implementation in numerical loop integration packages Thank you for listening!

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