Evaluating Parametric Integrals in the Minkowski Regime without Contour Deformation

Thomas Stone

In collaboration with Stephen Jones (IPPP) & Anton Olsson (KIT)

(and the rest of the pySecDec collaboration:
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Outline

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Calculating Loop Integrals

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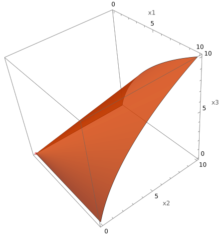
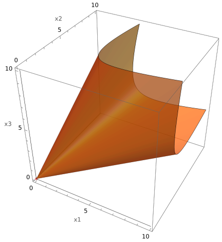
- Numerical methods developed to tackle these integrals (Monte Carlo techniques, differential equation methods etc.)
- Exploring singularity structure of Feynman integrals can help us understand how to integrate in the Minkowski regime





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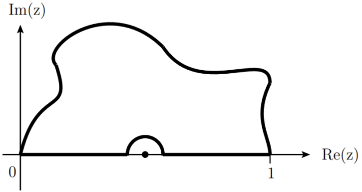

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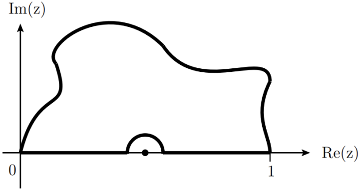

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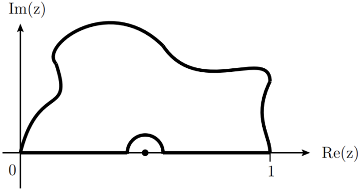

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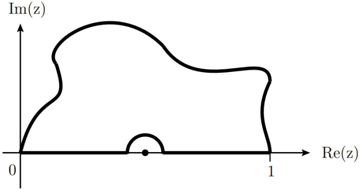

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- Can we do the same in Feynman parameter space?

Recap: Feynman Parameterisation

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Momentum Space Integral

$$I = \int_{-\infty}^{+\infty} \left(\prod_{l=1}^{L} \frac{\mathrm{d}^D k_l}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{N} \frac{1}{P_j^{\nu_j}\left(\{k\},\{p\},m_j^2\right)}$$

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Feynman-Parameterised Integral

$$I = \frac{(-1)^{\nu}\Gamma(\nu - LD/2)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_{\mathbb{R}^N_{\geq 0}} \left(\prod_{j=1}^{N} \mathrm{d}x_j x_j^{\nu_j - 1} \right) \frac{\mathcal{U}(\mathbf{x})^{\nu-(L+1)D/2}}{(\mathcal{F}(\mathbf{x},\mathbf{s}) - i\delta)^{\nu - LD/2}} \delta\left(1 - \sum_{j=1}^{N} x_j\right)$$

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$\mathcal{U}$ & $\mathcal{F}$ constructable directly from Feynman diagrams with $\mathcal{U} \geq 0$ and $\mathcal{F}$ depending on both parameters **and** kinematics

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Implementation in Parameter Space

$$\mathcal{F}(\vec{z}) = \mathcal{F}(\vec{x}) - i \sum_j \tau_j \frac{\partial \mathcal{F}(\vec{x})}{\partial x_j} \qquad \tau_j = \lambda_j x_j (1 - x_j) \frac{\partial \mathcal{F}(\vec{x})}{\partial x_j}$$

[Soper; Binoth, Guillet, Heinrich, Pilon, Schubert; Nagy; Anastasiou, Beerli, Daleo; Borowka, Carter; ...]

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- There are even cases where this procedure fails entirely! [see Stephen Jones' talk]

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"Can we avoid this sometimes?"

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Yes!

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Maybe...?

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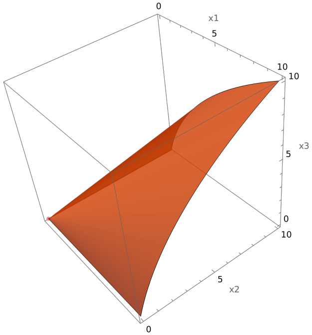

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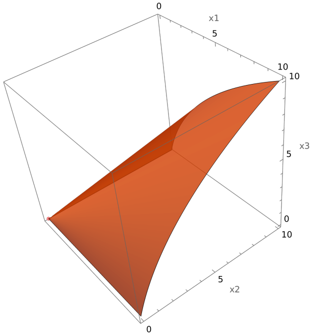

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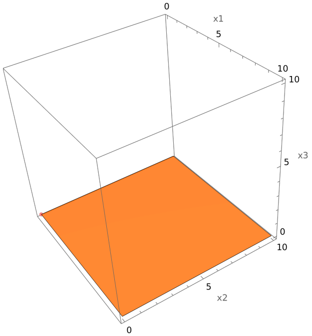

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Cheng-Wu Theorem [Cheng, Wu] [see e.g. Smirnov; Weinzierl; ...]

$$\forall S \subseteq \{1, ..., N\} \wedge S \neq \emptyset :$$
$$\delta\left(1 - \sum_{j=1}^{N} x_j\right) \to \delta\left(1 - \sum_{j \in S} x_j\right) \text{ leaves } I \text{ invariant}$$

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Heaviside Identity

Under the integral sign: $\theta(x_a - x_b) + \theta(x_b - x_a) = 1$

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Overall Construction

$$I = \sum_{n_+=1}^{N_+} I^+_{n_+} + (-1 - i\delta)^{-(\nu - LD/2)} \sum_{n_-=1}^{N_-} I^-_{n_-}$$

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Overall Construction

$$I = \sum_{n_+=1}^{N_+} I^+_{n_+} + (-1 - i\delta)^{-(\nu - LD/2)} \sum_{n_-=1}^{N_-} I^-_{n_-}$$

- Can be **much** faster numerically to only calculate the manifestly non-negative integrals $\{I^+_{n_+}, I^-_{n_-}\}$!

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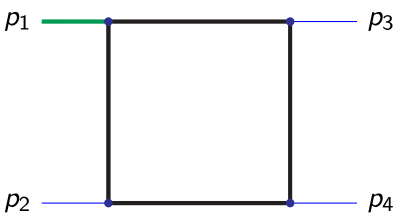

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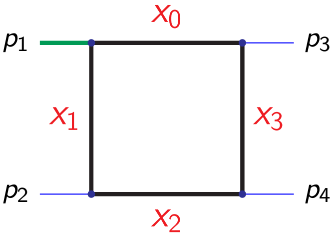

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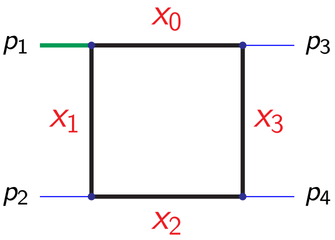

- $\mathcal{U} =$

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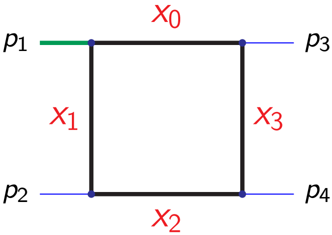

- $\mathcal{U} = x_0$

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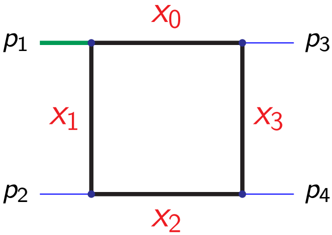

- $\mathcal{U} = x_0 + x_1$

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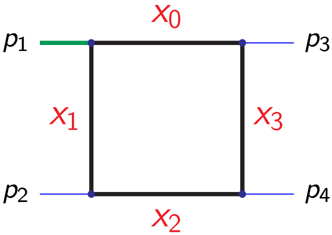

- $\mathcal{U} = x_0 + x_1 + x_2$

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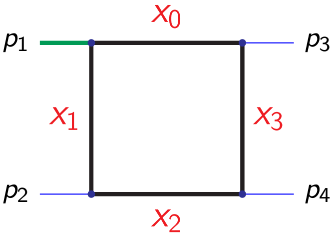

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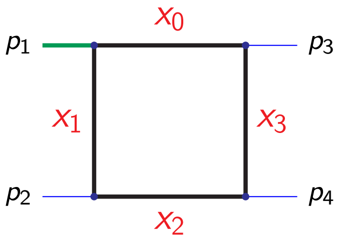

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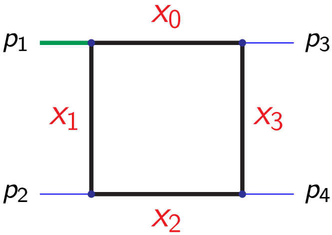

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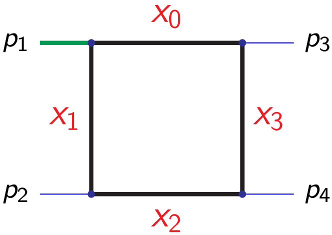

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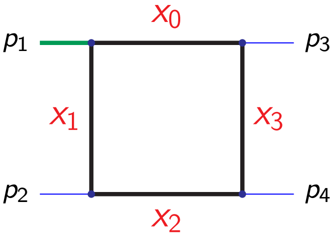

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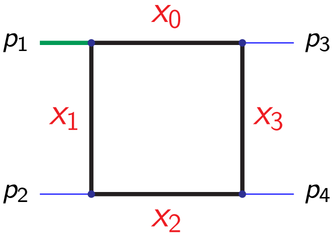

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Let's consider the regime: $s > 0$, $p_1^2 > 0$ & $t < 0 \quad \Rightarrow \quad$ zeroes of $\mathcal{F}$ *within* the integration volume for $\{x_0, x_1, x_2, x_3\} \in \mathbb{R}^4_{\geq 0}$

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- $\mathcal{F}_1^-$ non-negative as required!
- To cover the whole original space, the Heaviside identity tells us we need to consider the converse case $(x_3 > x_0)$ so *shift* x_3: $x_3 \to x_0 + x_3$

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- $\mathcal{F} = -sx_0x_2 + |t|x_1x_3 - p_1^2x_0x_1$
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- $\mathcal{F} \to x_1\left(x_2\left(x_3 - x_0\right) - \frac{p_1^2}{s}x_0x_1\right)$
- Introduce the hierarchy $x_0 > x_3$ by *shifting* x_0: $x_0 \to x_0 + x_3$
- $\mathcal{F} \to -\frac{1}{s}\left(x_1\left(sx_0x_2 + p_1^2x_1\left(x_0 + x_3\right)\right)\right) =: -\mathcal{F}_1^-$
- $\mathcal{F}_1^-$ non-negative as required!
- To cover the whole original space, the Heaviside identity tells us we need to consider the converse case $(x_3 > x_0)$ so *shift* x_3: $x_3 \to x_0 + x_3$
- $\mathcal{F} \to x_1\left(-\frac{p_1^2}{s}x_0x_1 + x_2x_3\right)$

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- This is **not** of uniform sign $\Rightarrow$ needs further work!

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- Generate the absolute values of the corresponding Jacobian determinants: $\mathcal{J}_1^+$, $\mathcal{J}_1^-$ & $\mathcal{J}_2^-$

1-Loop Off-Shell Box: Putting the Jigsaw Pieces Together

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- Verified numerical result against the known analytic result ✓

2-Loop Non-Planar Box

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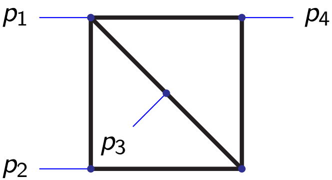

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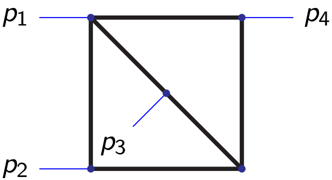

- $\mathcal{U} = x_0x_1 + x_0x_2 + x_0x_3 + x_0x_4 + x_1x_2 + x_1x_3 + x_1x_5 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5$

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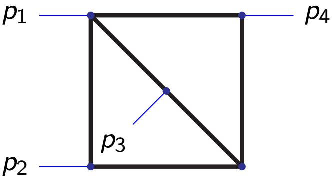

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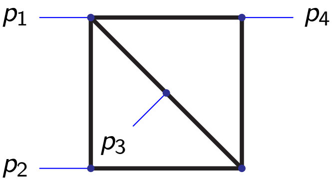

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Momentum conservation implies $s + t + u = 0 \Rightarrow u = -(s + t)$

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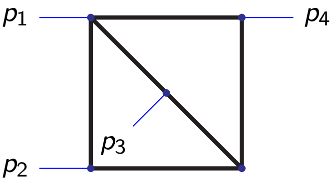

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Hence, $\mathcal{F}$ can be 0 *within* $\{x_i\} \in \mathbb{R}^6_{\geq 0}$ even with $s > 0, t > 0$

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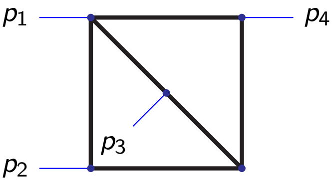

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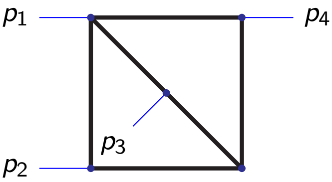

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Not possible to define a Euclidean region at all!
Nevertheless, the method works

2-Loop Non-Planar Box

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- $\mathcal{F} = -s x_1 x_2 x_5 - t x_0 x_1 x_3 + (s+t) x_0 x_2 x_4$

2-Loop Non-Planar Box

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[Tausk 99]

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δ-Function Issue

Some rescalings can modify our δ-function in an impractical way for our numerical integration setup - let's avoid that!

2-Loop Non-Planar Box: pySecDec Timing Comparison

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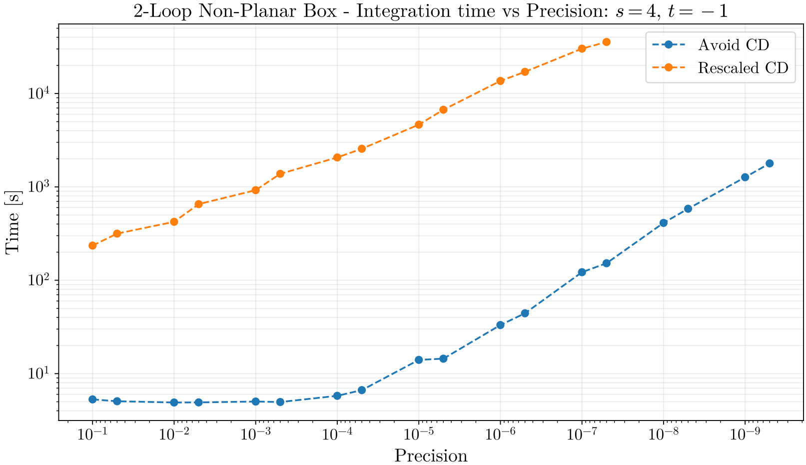

Evaluating up-to-and-including finite order with pySecDec
[Heinrich, Jones, Kerner, Magerya, Olsson, Schlenk]

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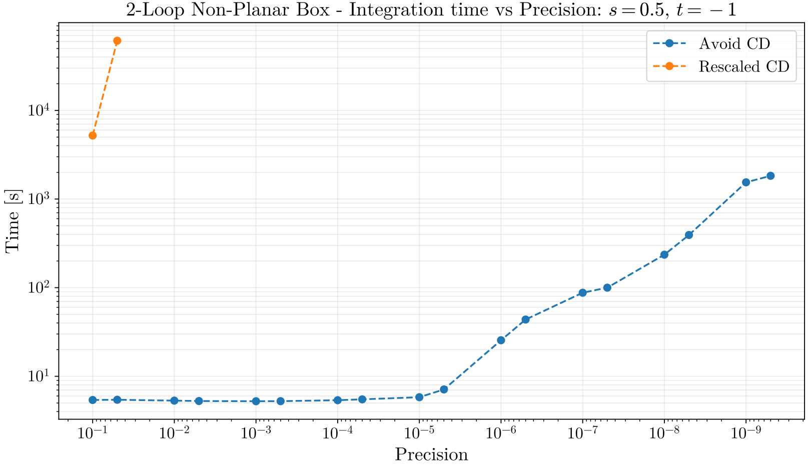

Evaluating up-to-and-including finite order with pySecDec

3-Loop Non-Planar Box

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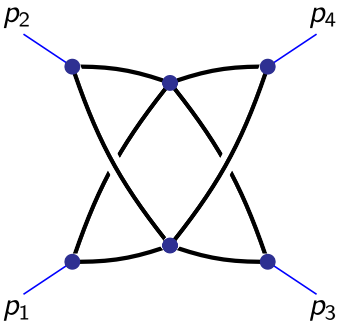

Diagram by Yao Ma

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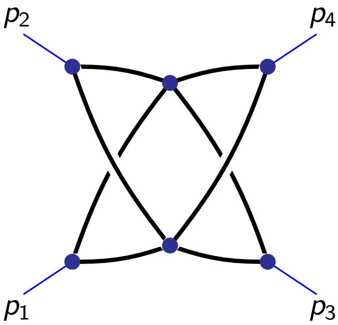

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$$\mathcal{F} = -\, s\,(x_1x_4 - x_0x_5)\,(x_3x_6 - x_2x_7) \\ -\, t\,(x_1x_2 - x_0x_3)\,(x_5x_6 - x_4x_7)$$

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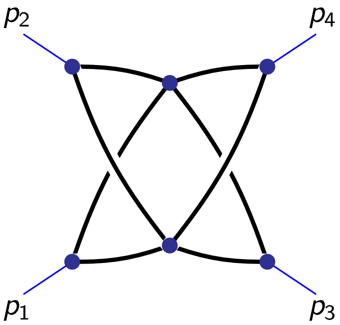

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[see Stephen Jones' talk]

$$\Downarrow$$

$$\begin{aligned}\mathcal{F} \to &- sx_1x_3x_5x_7\left(x_4 - x_0\right)\left(x_6 - x_2\right)\\ &- tx_1x_3x_5x_7\left(x_2 - x_6\right)\left(x_6 - x_4\right)\end{aligned}$$

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- This gives us 12 integrals to compute, none of which require contour deformation!

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- Struggled to go beyond leading pole ($\frac{1}{\varepsilon^4}$) *with* contour deformation
- Avoiding contour deformation allowed us to go to higher orders (tried up to $\frac{1}{\varepsilon^2}$)

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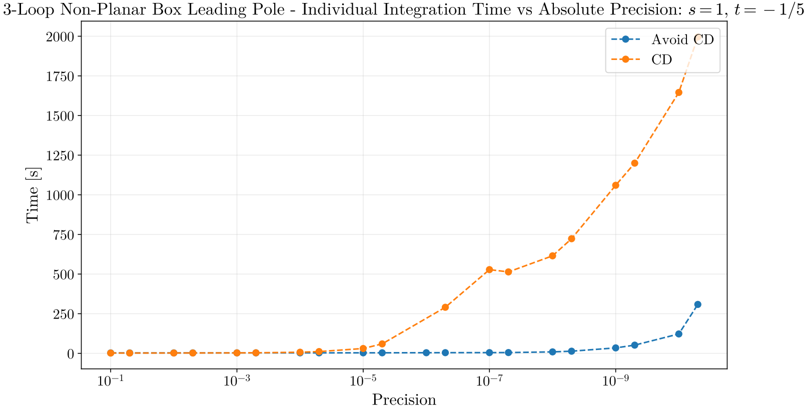

Evaluating leading pole ($\frac{1}{\varepsilon^4}$) with pySecDec

3-Loop Non-Planar Box: pySecDec Timing

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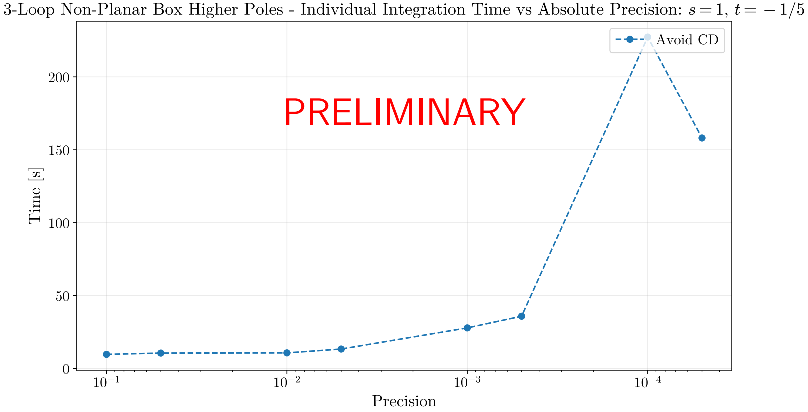

Evaluating up-to-and-including $\frac{1}{\varepsilon^2}$ with pySecDec

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- Can we use geometry to guide us in the right direction?

Massive Bubble

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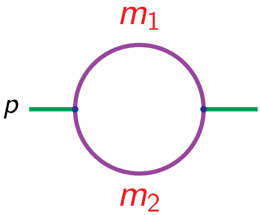

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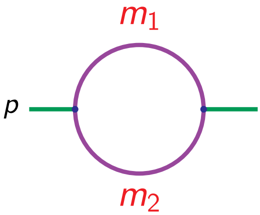

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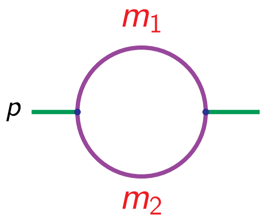

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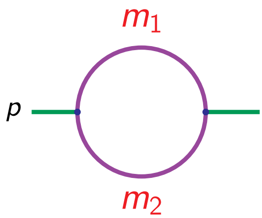

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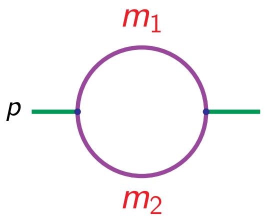

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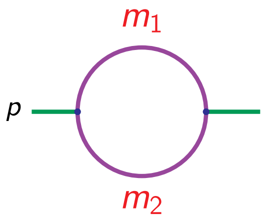

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$$\mathcal{F} \to \widetilde{\mathcal{F}} = x_1^2 + x_2^2 - 2\frac{1+\beta^2}{1-\beta^2} x_1 x_2$$

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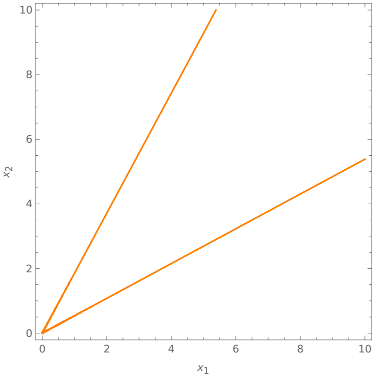

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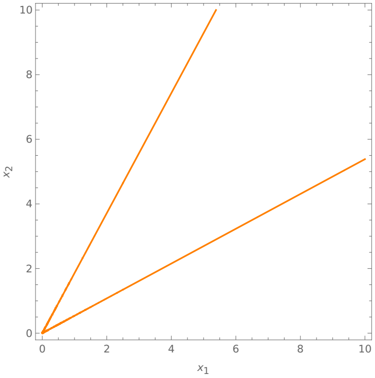

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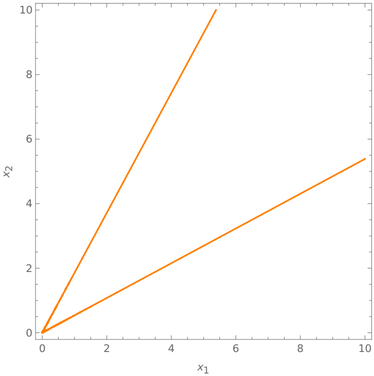

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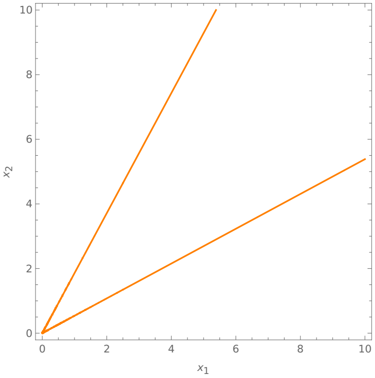
Massive Bubble



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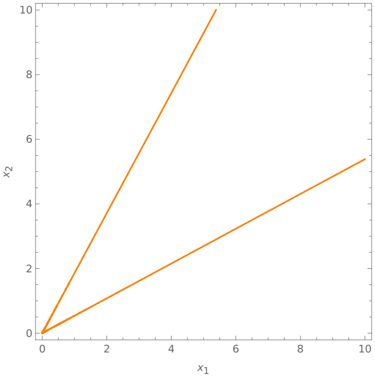

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- Construct transformations which directly send the variety to the integration boundary

Massive Bubble

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$$y_1 \overset{!}{=} x_1, y_2 \overset{!}{=} x_2 - \frac{1+\beta}{1-\beta} x_1 \Rightarrow x_1 \to y_1, x_2 \to y_2 + \frac{1+\beta}{1-\beta} y_1$$

Massive Bubble

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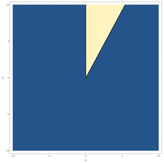

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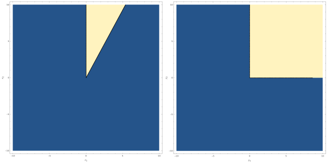

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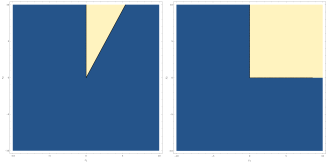
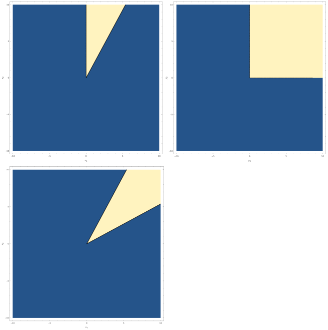


$$\widetilde{\mathcal{F}}_1^+ = y_2 \left(y_2 + \frac{4\beta}{1-\beta^2} y_1 \right)$$

Massive Bubble



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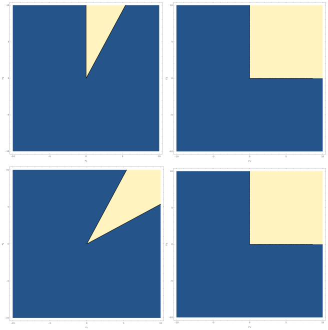

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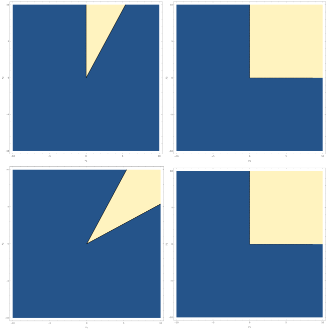

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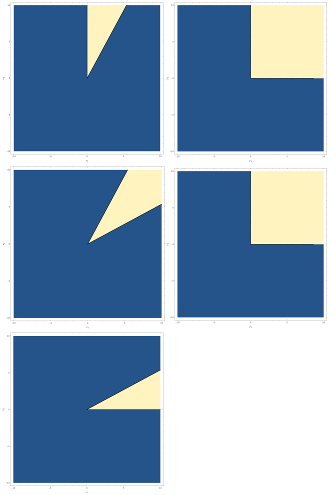

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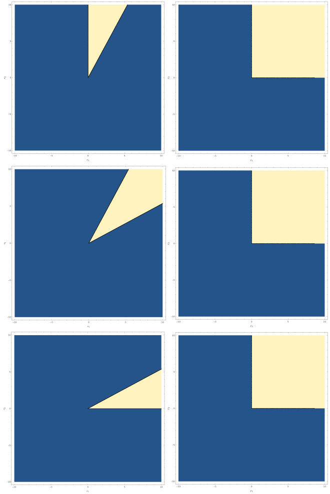

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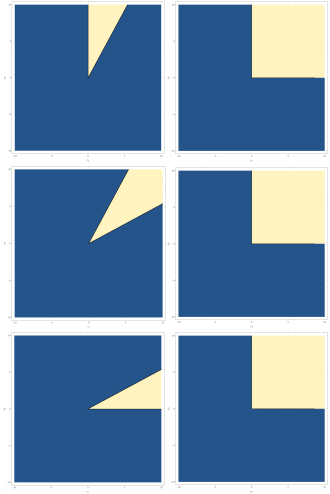

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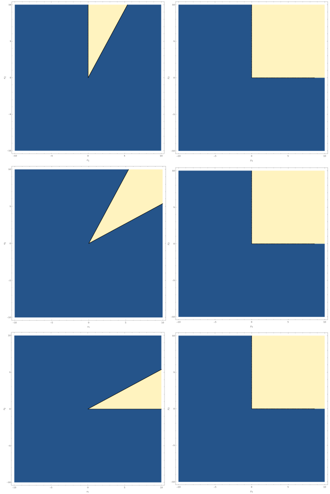

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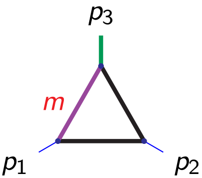
Verified result numerically **&** analytically ✓

Massive Triangle



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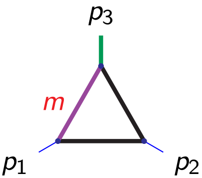

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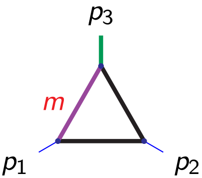

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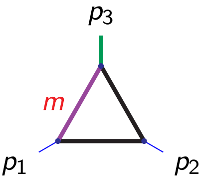

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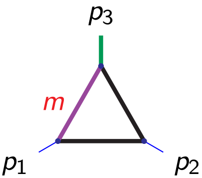

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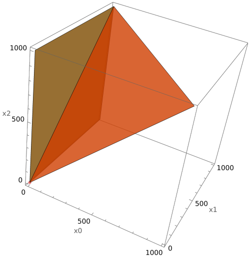

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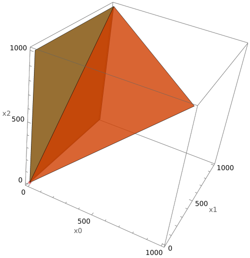

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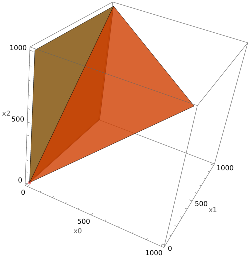

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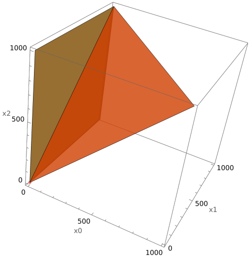

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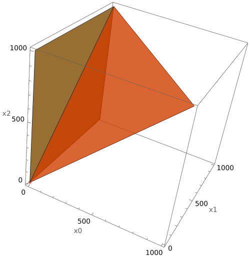

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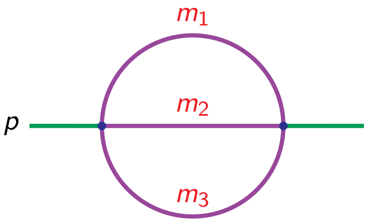

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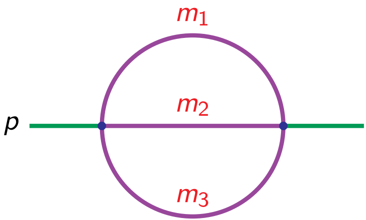

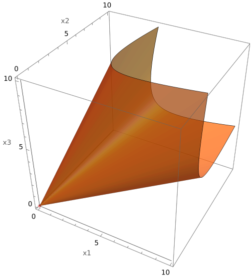

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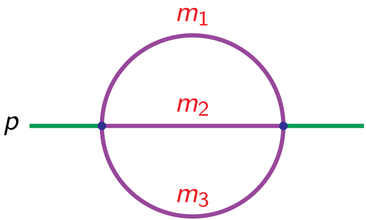

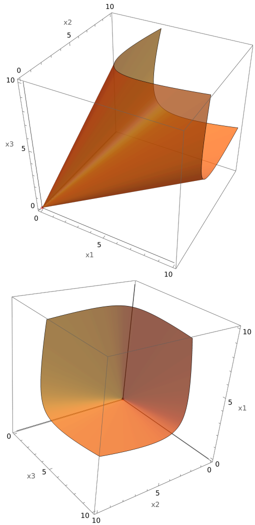

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Thank you for listening!