Planar Two-Loop $t\bar{t}H$ integrals with a Light Quark Loop

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With F. Febres Cordero, G. Figueiredo, M. Kraus, L. Reina

based on [arXiv:2312.08131]

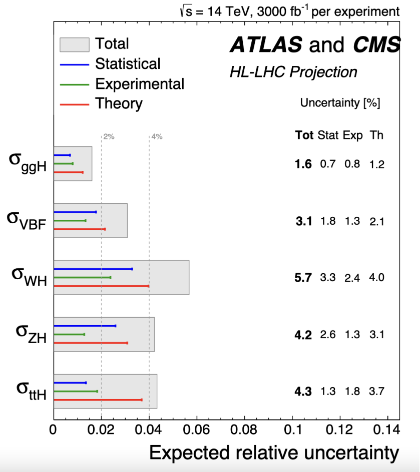
Motivation

- $t\bar{t}H$ production $\Rightarrow$ precisely measure top Yukawa.
- Projected precision of 2% at end of Hi-Lumi phase.
- NNLO real radiation contribution available.

  [Catani et al '21, Catani et al '22]

  [Chiara's talk]
- Two-loop virtuals missing.

  [This talk, Vitaly's talk]



Status of NNLO Five-Point Massive Double Virtual

$pp \to t\bar{t}j$

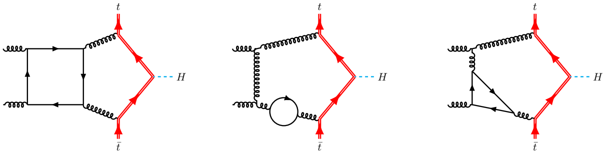
- One-loop amplitudes @ $\mathcal{O}(\epsilon^2)$. [Badger, Bechetti, Chaubey, Marzucca, Sarandrea]
- Family of planar two-loop integrals [Badger, Bechetti, Chaubey, Marzucca, Sarandrea] [Matteo's talk]

$pp \to t\bar{t}H$

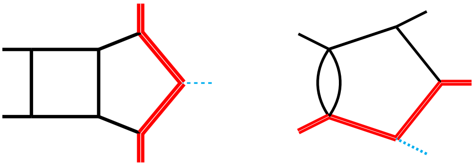
- One-loop $gg \to ttH$ @ $\mathcal{O}(\epsilon^2)$ [Buccioni, Kreer, Liu, Tancredi '23]
- IR divergences of 2-loop amplitudes [Chen, Ma, Wang, Yang, Ye '22].
- Two-loop amplitudes in boosted limit, soft Higgs limit: [Wang, Xia, Yang, Ye, '24; Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Savoini '22]
- Two-loop $q\bar{q} \to t\bar{t}H$ amplitudes, N_f piece [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson '24] [Vitaly's talk]
- Two-loop planar master integrals for light-quark loop. [Febres Cordero, Figueiredo, Kraus, BP, Reina '23] [This talk]

Light Fermion Loops in Leading Color $pp \to t\bar{t}H$

- All integrals contained in $gg \to t\bar{t}H$. Three top topologies.



- Two families of two-loop Feynman integrals.



Scattering Kinematics

$$\text{Channel: } q(p_4)q(p_5) \to t(p_1)H(p_2)\bar{t}(p_3)$$

- Five-point, two massless legs. On-shell tops, generic Higgs.

$$p_1^2 = p_3^2 = m_t^2, \quad p_2^2 = q^2, \quad p_4^2 = p_5^2 = 0.$$

- 7 independent mandelstam invariants.

$$\vec{s} = \{q^2, m_t^2, v_{12}, v_{23}, v_{34}, v_{45}, v_{15}\}, \qquad v_{ij} = 2p_i \cdot p_j.$$

- Physical region:

$$m_t^2 > 0, \quad q^2 > 0, \quad v_{12} \geq 2m_t q, \quad v_{23} \geq 2m_t q,$$
$$v_{34} \leq 0, \quad v_{15} \leq 0, \quad v_{45} \geq (2m_t + q)^2$$

$$G(p_i, p_j, p_k) \geq 0, \quad G(p_1, p_2, p_3, p_4) \leq 0, \quad G(q_1, \ldots, q_{n-1}) = \det(q_i \cdot q_j).$$

Calculational Strategy

1. Work in differential equations framework.

$$\mathrm{d}J = \overline{\mathbf{M}}\, J.$$

[Kotikov '91; Remiddi '97; Gehrmann, Remiddi '01]

2. Find basis with ϵ-factorized form.

$$J = UI \quad \rightarrow \quad \mathrm{d}I = \epsilon \mathbf{M}\, I.$$

[Henn '13]

3. Organize differential matrix into basis of forms.

$$\mathbf{M}_{kl} = \epsilon M_{klm} \boldsymbol{\omega}_{\boldsymbol{m}}$$

4. Numerical solutions via series expansion. (Preliminary).

[Moriello '19, Hidding '20, Liu, Ma '22]

Pure Basis Construction

Many automated tools, but difficult to apply with many scales

[Prausa '17, Gituliar, Magerya '17, Dlapa, Henn, Yan '20, Lee '21], ...

Our approach:

1. From experience, choose initial basis satisfying ϵ-linear DE:

$$\overline{\mathbf{M}}(\epsilon, \vec{s}) = \overline{\mathbf{M}}^{(0)}(\vec{s}) + \epsilon \overline{\mathbf{M}}^{(1)}(\vec{s}).$$

2. Change to ϵ-factorizing basis $J = UI$, where U satisfies

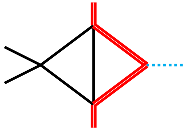
$$\mathrm{d}U_{ij} = \overline{\mathbf{M}}^{(\mathbf{0})}_{\mathbf{ix}} U_{xj}.$$

- Solve DE blockwise with Magnus expansion

[Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi '14]

Magnus Machinations: *kite*$_7$

- Solving $\mathrm{d}U = \overline{\mathbf{M}}^0 U$ blockwise non-trivial for *kite*$_7$ topology.



$$\mathrm{d}U = \frac{1}{4}\begin{pmatrix} 0 & \mathrm{d}\log(\underbrace{A}_{\text{algebraic}}) \\ \mathrm{d}\log(\underbrace{A}_{\text{algebraic}}) & 0 \end{pmatrix} U$$

- Solution to DE (and integral basis) involves nested root,

$$\sqrt{N_+} = \sqrt{q^2\left(N_b + \sqrt{N_b^2 - N_c}\right)},$$

$$N_b = q^2[(v_{14}+v_{15})^2 + (v_{34}+v_{35})^2] - 2m_t^2(v_{24}+v_{25})^2,$$
$$N_c = q^2(q^2-4m^2)(v_{12}-v_{23})^2(v_{24}+v_{25}+2v_{45})^2.$$

- Novel analytic structure! [See also Matteo's Talk]

Reconstructing ϵ-factorized DEs

- We express DE in basis given by subset of DE entries:

$$\mathrm{d}I_k = \epsilon \mathbf{M_{kl}} I_l, \qquad \mathbf{M_{kl}} = \underbrace{M_{klm}}_{\mathbb{Q}} \underbrace{\boldsymbol{\omega_m}[\vec{s}]}_{\text{algebraic}}$$

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

- M_{klm} are rational numbers: compute via finite-field samples.

[von Manteuffel, Schabinger '14; Peraro '17] [Abreu, BP, Zeng, '17]

- $\boldsymbol{\omega_m}$ are algebraic, controlled by "Galois transformations".

$$\sqrt{x} \to -\sqrt{x}$$

$$\alpha : \sqrt{N_+} \to \sqrt{q^2(q^2-4m_t^2)} \frac{q^2(v_{12}-v_{23})(2q^2+v_{12}+v_{23}-2v_{45})}{\sqrt{N_+}}.$$

Problem reduced to computation of basis of $\boldsymbol{\omega}_m$.

Analytic Reconstruction of the ω_m

- Differential forms have well understood structure:

$$\boldsymbol{\omega_m} = \frac{1}{\prod_i (\underbrace{W_i^e}_{\text{polynomial}})^{q_{im}}} \underbrace{\boldsymbol{\Omega_m}}_{\text{algebraic form}},$$

- Observation: All W_i^e arise in polynomial $\boldsymbol{\omega_m}$.

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

- For polynomial $\boldsymbol{\Omega_m}$: use denominators for $\mathrm{d}\log$ form Ansatz.

$$\boldsymbol{\omega_m} = \sum_i c_{mi} \mathrm{d}\log(W_i).$$

- NB: Completely avoid complicated $\boldsymbol{\Omega_m}$ reconstruction!

Algebraic Letters Ansätze [Febres Cordero, Figueiredo, Kraus, BP, Reina '23]

For $\sqrt{N_+}$ independent assume $\mathrm{d}\log$-form, exploit root structure.

$$\mathrm{d}\log\left(\frac{w+\sqrt{R}}{w-\sqrt{R}}\right) = \frac{1}{\sqrt{R}}\frac{2R(\mathrm{d}w)-w(\mathrm{d}R)}{w^2-R}, \qquad w^2-R = \prod_i W_i^{e}.$$

- Write rational Ansatz for w

$$w = \frac{w_N}{w_D}, \qquad w_X = \sum_{\vec{\gamma}} w_{x,\vec{\gamma}} \prod_{i=1}^{7} s_i^{\gamma_i}.$$

- Constrain Ansatz by requiring denominator structure

$$w_N^2 - R\, w_D^2 \quad \mathrm{mod} \quad \prod_i W_i^{e} = 0.$$

see also [Heller, von Manteuffel, Schabinger '19]

- Solving for $w_{x,\vec{\gamma}}$ with Gröbner basis techniques.

Alphabet Summary

- 152 letters*. 9 are "irrelevant": arise in integrals @ $\mathcal{O}(\epsilon)$.

  *Much larger than single family for $pp \to Wjj$

- 122 relevant letters do not depend on $\sqrt{N_+}$.

Mass dimension	1	2	3	4	5	6	Σ
# Polynomial Letters	19	10	8	5	0	1	43
# Algebraic, single odd	14	12	15	0	0	1	42
# Algebraic, double odd	5	21	10	1	0	1	37

- Remaining 21 relevant letters depend on $\sqrt{N_+}$.

Letters Involving $\sqrt{N_\pm}$

- Many $\mathrm{d}\log$-forms, e.g.

$$\mathrm{d}\log\left(\frac{q^2[v_{45}+s_{13}-q^2]-\sqrt{N_+}}{q^2[v_{45}+s_{13}-q^2]+\sqrt{N_+}}\right),\quad \mathrm{d}\log\left(\frac{q^2(v_{12}-v_{23})(v_{45}+s_{13}-q^2)+\sqrt{r_1}\sqrt{N_+}}{q^2(v_{12}-v_{23})(v_{45}+s_{13}-q^2)-\sqrt{r_1}\sqrt{N_+}}\right).$$

- We were unable to find a $\mathrm{d}\log$ form for 4 cases, generated by

$$\omega^E = \frac{\Omega^E}{m_t^2(q^2-v_{23})\sqrt{G(p_2,p_3)}\sqrt{N_+}\sqrt{N_b^2-N_c W_{32}}}.$$

- Unclear if possible. Would like to have decision procedure.

Numerical Solutions

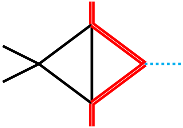
- Use AMFlow to determine 100-digit physical region boundary.

  [Liu, Ma '22]

- Can solve our DE using series expansion techniques.

  [Moriello '19]

- Public implementations dislike $\sqrt{N_+}$. Use auxiliary kite basis.



$$\left[\epsilon^3 (q^2)^2 \left(\frac{1}{\rho_3} + \frac{1}{\rho_2}\right), \epsilon^3 (q^2)^2 \left(\frac{1}{\rho_3} - \frac{1}{\rho_2}\right)\right]$$

- Proof of concept DiffExp implementation in ancillaries.

  [Hidding '20]

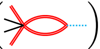
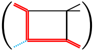
Summary and Outlook

- We compute a family of integrals for $pp \to t\bar{t}H$ @ NNLO.
- We provide a differential equation in ϵ-factorized form.
- We find a novel analytic structure: nested square root $\sqrt{N_+}$.
- Looking forward:
  - Form of DE lends itself to one-fold integral solutions.
    à la [Chicherin, Sotnikov '20]
  - Other pentabox families for $pp \to t\bar{t}H$ are elliptic.

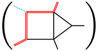
Algebraic Structure of ϵ-Factorized DE

- DE is algebraic. Involves roots of Gram/Cayley determinants.

$$\sqrt{G(p_1,p_2)}, \sqrt{G(p_1,p_{23})}, \quad (1,3) \leftrightarrow (4,5), \quad \sqrt{G(p_2,p_{34})}, \sqrt{G(p_1,p_2,p_3,p_4)},$$

$\sqrt{\mathcal{C}\Big($  $\Big)}$, $\sqrt{\mathcal{C}\Big($  $\Big)}$

- Other roots are two-loop like:

$\sqrt{\mathrm{LS}\Big($  $\Big)}$, $\quad (1,3) \leftrightarrow (4,5), \quad$ $\sqrt{\mathrm{LS}\Big($  $\Big)}$, $\sqrt{N_\pm}, \sqrt{N_b^2 - N_c}$