Evolution kernels of twist-two operators

A. N. Manashov

Hamburg University, Hamburg

Based on 2307.01763 Yao Ji, A.M., S. Moch

L & L, Wittenberg, 18 April 2024

- **Motivation**
- **Symmetries and Evolution kernels**
- **Kernels vs anomalous dimensions**
- **Conclusions**

Evolution equation for twist-two operators

$$\Big(\mu\partial_\mu + \beta(a)\partial_a + \mathrm{H}(a)\Big)\mathcal{O}(z_1,z_2) = 0,$$

$$\mathcal{O}(z_1,z_2) = \Big[\bar{q}(z_1 n)\gamma_+[z_1 n, z_2 n]q(z_2 n)\Big]_{\overline{\mathsf{MS}}},$$

- **DIS:** $\langle P|\mathcal{O}(z_1,z_2)|P\rangle$ **(DGLAP)** **Parton densities**
- **DAs** $\langle 0|\mathcal{O}(z_1,z_2)|P\rangle$ **(ERBL)** **Meson wave functions**
- **DVCS** $\langle P'|\mathcal{O}(z_1,z_2)|P\rangle$ **Generalized Parton Distributions**

$\mathrm{H}(a)-$**integral operator**

$$\mathrm{H}(a)\mathcal{O}(z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \omega(\alpha,\beta)\, \mathcal{O}(z_{12}^\alpha, z_{21}^\beta)$$

$z_{12}^\alpha = \bar\alpha z_1 + \alpha z_2$, $\bar\alpha \equiv 1-\alpha$.

Perturbative expansion:

$$\mathrm{H}(a) = a\,\mathrm{H}^{(1)} + a^2\,\mathrm{H}^{(2)} + a^3\,\mathrm{H}^{(3)} + \dots$$

$$\omega(\alpha,\beta) = a\,\omega^{(1)}(\alpha,\beta) + a^2\,\omega^{(2)}(\alpha,\beta) + a^3\,\omega^{(3)}(\alpha,\beta) + \dots$$

Anomalous dimensions ($\mathrm{H}(a) z_{12}^{N-1} = \gamma_N z_{12}^{N-1}$)

$$\gamma_N = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \omega(\alpha,\beta)(1-\alpha-\beta)^{N-1}$$

known at NNLO (Moch, Vermasseren, Vogt, 2004) + partial results at NNNLO

- **Makeenko, 1980** $\mathrm{H}^{(1)} = f(\gamma_N^{(1)})$. **(conformal symmetry).**
- **D. Müller, 1998** $\mathrm{H}^{(\ell)} = f(\gamma_N^{(\ell)}, \Delta^{(\ell-1)})$. **$\Delta-$ conformal anomaly.**
- **Belitsky, Müller, 1998-99** **all two loop kernels for all twist-2 operators in QCD.**
- **Braun, A.M., Moch, Strohmaier, 2015-2017** **QCD at the critical point, ($\beta(a_*) = 0$)**
  Three-loop kernels for flavor nonsinglet operators.

$$[\mathrm{S}_{\pm,0}(a), \mathrm{H}(a)] = 0\,.$$

$$\mathrm{S}_-(a) = -\partial_{z_1} - \partial_{z_2}$$

$$\mathrm{S}_0(a) = z_1\partial_{z_1} + z_2\partial_{z_2} + 2 + \bar\beta(a) + \frac{1}{2}\mathrm{H}(a)$$

$$\mathrm{S}_+(a) = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + (z_1+z_2)\left(2 + \bar\beta(a) + \frac{1}{2}\mathrm{H}(a)\right) + (z_1 - z_2)\Delta(a)$$

$\bar\beta(a) = -\beta(a)/2a = \beta_0 a + \beta_1 a^2 + \ldots$.

$$\mathrm{H}(a) = \widehat{\mathrm{H}}(a) + \Delta\mathrm{H}(a) \qquad\qquad [\mathrm{S}_{\pm,0}(0), \widehat{\mathrm{H}}(a)] = 0.$$

Canonically invariant kernels

$$\widehat{\mathrm{H}}\mathcal{O}(z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, \widehat{\omega}(\tau)\, \mathcal{O}(z_{12}^{\alpha}, z_{21}^{\beta}) \qquad\qquad \tau = \frac{\alpha\beta}{\bar\alpha\bar\beta}$$

Two problems to solve

$$\gamma_N \mapsto \widehat{\omega} \;\; ?? \qquad\qquad \widehat{\mathrm{H}} \mapsto \mathrm{H} = \mathrm{V}^{-1}\,\widehat{\mathrm{H}}\,\mathrm{V} \;\; ??$$

The operator V **has to intertwine the generators** $\mathrm{S}_\alpha(0)$ **with** $\mathrm{S}_\alpha(a) = \mathrm{S}_\alpha(\mathrm{H}, \Delta)$**.**

$$\mathrm{V}\,\mathrm{S}_\alpha(\mathrm{H}, \Delta) = \mathrm{S}_\alpha(0)\,\mathrm{V}$$

- **First step $\mapsto$ kill the anomaly:**

$$\mathrm{V}_1\,\mathrm{S}_+(\mathrm{H},\Delta) = \mathrm{S}_+(\mathbf{H},0)\,\mathrm{V}_1\,, \qquad\qquad \mathbf{H} = \mathrm{V}_1\,\mathrm{H}\,\mathrm{V}_1^{-1}$$

$\Big($ **Braun et al, 1703.09532 :** $\mathrm{V}_1 = e^{aX_1+a^2X_2+\dots}$ $\Big)$

- **Second step:** $\mathrm{S}_\alpha(\mathbf{H}) \mapsto \mathrm{S}_\alpha(0)$ **:**

Casimir operator: $\mathrm{J}^2 = \mathrm{S}_+\mathrm{S}_- + \mathrm{S}_0(\mathrm{S}_0-1)$ $\qquad$ $\Psi_N = z_{12}^{N-1}$ $\quad$ **(**$\Psi_{Nk} = S_+^k\Psi_N$**)**

$$\begin{aligned}\mathrm{J}^2(0)\Psi_N &= N(N+1)\Psi_N\\ \mathrm{J}^2(\mathbf{H})\Psi_N &= \left(N+\bar\beta+\frac{1}{2}\gamma_N\right)\left(N+1+\bar\beta+\frac{1}{2}\gamma_N\right)\Psi_N\end{aligned}$$

$$V_2\,S_\alpha(\mathbf{H}) = \mathrm{S}_\alpha(0)V_2, \qquad\qquad V_2 z_{12}^{N-1} \mapsto z_{12}^{N-1+\bar\beta+\gamma_N/2}$$

$$V_2 = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{L}^n \left(\bar\beta(a) + \frac{1}{2}\mathbf{H}\right)^n \quad \textbf{where} \quad \mathrm{L} = \ln z_{12}$$

- **If** $\Psi_N(z_1, z_2)$ **is an eigenfunction of** $\mathbf{H}$ **then**

$$V_2 \Psi_N(z_1, z_2) = z_{12}^{\bar\beta + \frac{1}{2}\gamma_N(a)} \Psi_N(z_1, z_2) = \widehat{\Psi}_{N+\bar\beta+\frac{1}{2}\gamma_N(a)}(z_1, z_2)$$

- V_2 **intertwines** $S_\alpha(\mathbf{H})$ **and** $\mathrm{S}_\alpha(0)$

$$V_2\, S_\alpha(\mathbf{H}) = \mathrm{S}_\alpha(0) V_2$$

- $\widehat{\mathrm{H}} = V_2\, \mathbf{H}\, V_2^{-1}$ **is canonically invariant operator,**

$$[\mathrm{S}_\alpha(\mathbf{H}), \mathbf{H}(a)] = 0 \Longrightarrow [\mathrm{S}_\alpha(0), \widehat{\mathrm{H}}(a)] = 0$$

$$V_2 = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{L}^n \left(\bar\beta(a) + \frac{1}{2}\mathbf{H}\right)^n \quad \textbf{where} \quad \mathrm{L} = \ln z_{12}$$

$$V_2^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathrm{L})^n \left(\bar\beta(a) + \frac{1}{2}\widehat{\mathrm{H}}(a)\right)^n$$

- **If $\Psi_N(z_1, z_2)$ is an eigenfunction of $\mathbf{H}$ then**

$$V_2 \Psi_N(z_1, z_2) = z_{12}^{\bar\beta + \frac{1}{2}\gamma_N(a)} \Psi_N(z_1, z_2) = \widehat{\Psi}_{N+\bar\beta+\frac{1}{2}\gamma_N(a)}(z_1, z_2)$$

- V_2 **intertwines** $S_\alpha(\mathbf{H})$ **and** $\mathrm{S}_\alpha(0)$

$$V_2\, S_\alpha(\mathbf{H}) = \mathrm{S}_\alpha(0) V_2$$

- $\widehat{\mathrm{H}} = V_2\, \mathbf{H}\, V_2^{-1}$ **is canonically invariant operator,**

$$[\mathrm{S}_\alpha(\mathbf{H}), \mathbf{H}(a)] = 0 \Longrightarrow [\mathrm{S}_\alpha(0), \widehat{\mathrm{H}}(a)] = 0$$

- **Eigenvalues**

$$\mathbf{H} z_{12}^{N-1} = \gamma(N) z_{12}^{N-1} \qquad \widehat{\mathrm{H}} z_{12}^{N-1} = \widehat{\gamma}(N) z_{12}^{N-1}$$

$$\gamma(N) = \widehat{\gamma}\left(N + \bar{\beta}(a) + \frac{1}{2}\gamma(N)\right)$$

$\widehat{\gamma}(N)-$ **parity respecting anomalous dimensions.**

Dokshitzer, Marchesini, Salam, 2006;
Basso, Korchemsky 2007,
Beccaria, Forini, 2009,
Alday, Bissi, Lukowski, 2015
....

$$\widehat{\gamma}(N) \simeq \widehat{\gamma}(-N-1) \qquad \text{for large } N$$

Kernels vs anomalous dimensions

$$\widehat{\mathrm{H}}\mathcal{O}(z_1, z_2) = \int d\alpha d\beta\, \omega(\tau)\, \mathcal{O}(z_{12}^{\alpha}, z_{21}^{\beta}) \qquad \widehat{\gamma}(N) = \int d\alpha d\beta\, \omega(\tau)\, (1-\alpha-\beta)^{N-1}$$

$\tau = \alpha\beta/\bar{\alpha}\bar{\beta}$.

Parity : **Large N vs Small τ**

$$\widehat{\gamma}\,(N) \underset{N\to\infty}{\simeq} f\,(N(N+1)) \qquad \Longleftrightarrow \qquad \omega(\tau) \underset{\tau\to 0}{\simeq} \sum_{mk} a_{mk}\tau^{k} \ln^{m} \tau$$

$\widehat{\gamma}(N)$ **contains only specific combinations of harmonic sums** $S_{a_1,\ldots,a_k}$**:**

$$S_{a_1,\ldots,a_k}(N) \mapsto \Omega_{a_1,\ldots,a_k}(N)$$

Dokshitzer, Marchesini, Beccaria, Forini, 2007-09

Dokshitzer, Marchesini, Beccaria, Forini, 2007-09

Parity:

$$\Omega_{a_1,\ldots,a_k}(N) \sim (-1)^P \Omega_{a_1,\ldots,a_k}(-N-1) \qquad (-1)^P = \prod_k p(a_k)$$

$p(a_k) = 1$ **if a_k positive odd or negative even**

$p(a_k) = -1$ **if a_k negative odd or positive even**

e.g. $\Omega_{1,-2},\ \Omega_{1,3,5},\ \Omega_{2,2}\ P = 1$ **and** $\Omega_{-2,2},\ \Omega_{3,-5}\ P = -1$

If $p(a_k) = 1$ for all k there is an effective method to calculate the corresponding kernel:

$$\Omega_{a_1,\ldots,a_k}(N) \mapsto \omega_{a_1,\ldots,a_k}(\tau)$$

Inverse transformation:

Braun, A.M, 2014

$$\omega(\tau) = \frac{1}{2\pi i} \int_C dN(2N+1)\,\widehat{\gamma}(N)\,P_N\left(\frac{1+\tau}{1-\tau}\right)$$

$\mathrm{Re}C > 0$, P_N **Legendre function:**

$$P_N(z) = P_{-N-1}(z)$$

Inverse transformation:

Braun, A.M, 2014

$$\omega(\tau) = \frac{1}{2\pi i}\int_C dN(2N+1)\,\widehat{\gamma}(N)\,P_N\left(\frac{1+\tau}{1-\tau}\right)$$

$\mathrm{Re}C > 0$, P_N **Legendre function:**

$$P_N(z) = P_{-N-1}(z)$$

$$\widehat{\gamma}(N) \simeq \widehat{\gamma}(-N-1)$$

The integrand "almost" antisymmetric at $N \to -N-1$ **(** $N = -\frac{1}{2} + i\rho$ **)**

$$\widehat{\gamma} \sim 1/(N(N+1))^k \qquad\qquad \Omega_{a_1,\ldots,a_k}(N)\ ??$$

- $$h_{\vec{m}}(\tau) = \int_C \frac{dN}{2\pi i}(2N+1)\Omega_{\vec{m}}(N)P_N(z)\,, \qquad\qquad z = \frac{1+\tau}{1-\tau}$$

- **Use recurrence relation for Legendre functions**

$$(2N+1)P_N(z) = \frac{d}{dz}\Big(P_{N+1}(z) - P_{N-1}(z)\Big)$$

$$h_{\vec{m}}(\tau) = -\frac{d}{dz}\int_C \frac{dN}{2\pi i}P_N(z)\Big(\Omega_{\vec{m}}(N+1) - \Omega_{\vec{m}}(N-1)\Big).$$

- $$\Omega_{\vec{m}}(N+1) - \Omega_{\vec{m}}(N-1) = (2N+1)\left(\sum_{k=2}^{p} r_k(\eta)\Omega_{m_k,\ldots,m_p}(N) + r(\eta)\right)$$

$\eta = \frac{1}{N(N+1)}$

$$r_k(\eta)\Omega_{m_k,\ldots,m_p}(N) \mapsto H_k \otimes h_{m_k,\ldots,m_p}$$

Examples

$$\Omega_3 = S_3 - \zeta_3,$$

$$\mathcal{H}_3 = -\frac{1}{2}\frac{\bar\tau}{\tau}\mathrm{H}_1,$$

$$\Omega_{3,1} = S_{3,1} - \frac{1}{2}S_4 - \frac{3}{10}\zeta_2^2$$

$$\mathcal{H}_{3,1} = \frac{1}{4}\frac{\bar\tau}{\tau}\left(\mathrm{H}_{11} + \mathrm{H}_{10}\right)$$

$$\Omega_{-2,-2} = S_{-2,-2} - \frac{1}{2}S_4 + \frac{1}{2}\zeta_2 S_{-2} + \frac{1}{8}\zeta_2^2,$$

$$\mathcal{H}_{-2,-2} = \frac{1}{4}\frac{\bar\tau}{\tau}\mathrm{H}_{11},$$

$$\Omega_{1,3,1} = S_{1,3,1} - \frac{1}{2}S_{1,4} - \frac{1}{2}S_{4,1} + \frac{1}{4}S_5 - \frac{3}{10}\zeta_2^2 S_1 + \frac{3}{4}\zeta_5$$

$$\mathcal{H}_{1,3,1} = -\frac{1}{8}\frac{\bar\tau}{\tau}\left(\mathrm{H}_{20} + \mathrm{H}_{110} + \mathrm{H}_{21} + \mathrm{H}_{111}\right),$$

$$\Omega_{-2} = (-1)^N\left[S_{-2} + \frac{\zeta_2}{2}\right],$$

$$\mathcal{H}_{-2} = \frac{1}{2}\bar\tau,$$

$$\Omega_{-2,1} = (-1)^N\left[S_{-2,1} - \frac{1}{2}S_{-3} + \frac{1}{4}\zeta_3\right],$$

$$\mathcal{H}_{-2,1} = -\frac{1}{4}\bar\tau(\mathrm{H}_1 + \mathrm{H}_0),$$

$$\widehat{\gamma}(N) = 2\Gamma_{\textsf{cusp}}(a)S_1(N) + A(a) + \Delta\widehat{\gamma}(N)\,,$$

$\Gamma_{\textsf{cusp}}(a)$ **is known at four-loops:**

Henn, Korchemsky, Mistlberger, 2020,

von Manteuffel, Panzer, Schabinger, 2020

$$\Delta\widehat{\gamma}(N) = \gamma_+(N) + (-1)^{N-1}\gamma_-(N)$$

$$h_+^{(3)} = C_F N_c^2 4\Big[\mathrm{H}_{13} + \mathrm{H}_{112} - \mathrm{H}_{120} - \mathrm{H}_{1110} + 2\mathrm{H}_4 - 2\mathrm{H}_{30} - 2\mathrm{H}_{210} + 2\mathrm{H}_{22}$$
$$+ \left(\frac{8}{3} - \frac{2}{\tau}\right)\Big(\mathrm{H}_{20} - \mathrm{H}_3 + \mathrm{H}_{110} - \mathrm{H}_{12}\Big) - \frac{5}{4}\Big(\mathrm{H}_{10} + \mathrm{H}_{11}\Big) + \frac{2}{3\tau}\Big(\mathrm{H}_{10} - \mathrm{H}_2\Big) - \frac{5}{2}\mathrm{H}_0$$
$$+ \left(\frac{115}{72} + \zeta_2 + \frac{1}{\tau}\right)\mathrm{H}_1 - \frac{44}{5}\zeta_2^2 - \frac{22}{3}\zeta_3 + \frac{436}{9}\zeta_2 - \frac{4783}{27}\Big]$$

$$\bar{h}_-^{(3)} = -8C_F\Big\{\mathrm{H}_{120} + \mathrm{H}_{22} - \mathrm{H}_{1110} - \mathrm{H}_{112} - 2\mathrm{H}_{121} + 2\mathrm{H}_{211} - 4\mathrm{H}_{1111} + \tau\mathrm{H}_{20}$$
$$+ \left(\frac{13}{6} - \tau\right)\mathrm{H}_{110} + \Big(\frac{1}{2} - 2\tau\Big)\mathrm{H}_{12} + \Big(\frac{5}{2} - 2\tau\Big)\mathrm{H}_{21} + \left(\frac{43}{6} - 6\tau\right)\mathrm{H}_{111}$$
$$- \left(\zeta_2 - \frac{13}{6}\tau\right)\mathrm{H}_{10} - \Big(3 + \zeta_2 + \frac{3}{2}\tau\Big)\mathrm{H}_2 + \left(\frac{236}{9} + \frac{2}{3}\tau\right)\mathrm{H}_{11} - \zeta_2\tau\mathrm{H}_0$$
$$+ \left(\frac{53}{6} + \zeta_2 + 3\zeta_3 + \frac{134}{9}\tau + \zeta_2\tau\right)\mathrm{H}_1 + \frac{11}{6} + 3\left(\zeta_2 + \zeta_3 - \frac{1}{2}\right)\tau\Big\}$$

$\gamma^{\mathsf{SYM}}(N)$, **Kotikov, Lipatov, Velizhanin, Beccaria,...**

For the kernels we find $h_1 = \bar{h}_1 = 0$,

$$h_2 = 8\frac{\bar{\tau}}{\tau}\mathrm{H}_1\,, \qquad \bar{h}_2 = -8\bar{\tau}\mathrm{H}_1$$

and

$$h_3 = -16\frac{\bar{\tau}}{\tau}\Big(4\,\mathrm{H}_{111} + \mathrm{H}_{21} + \mathrm{H}_{12} + \mathrm{H}_{110}\Big)\,,$$

$$\bar{h}_3 = 16\bar{\tau}\Big(4\mathrm{H}_{111} + 3\Big(\mathrm{H}_{21} + \mathrm{H}_{12}\Big) - \mathrm{H}_{110} + \mathrm{H}_{20} - \zeta_2\mathrm{H}_0\Big).$$

Conclusions

- **We constructed the transformation:** $S_\alpha(\mathrm{H},\Delta) \mapsto S_\alpha(0)$, $\mathrm{H}(a) \mapsto \widehat{\mathrm{H}}(a)$**:**

$$\mathrm{H}(a) = \mathrm{V}^{-1}(a)\,\widehat{\mathrm{H}}(a)\,\mathrm{V}(a) \qquad\qquad \gamma(N) = \widehat{\gamma}\left(N + \bar{\beta}(a) + \frac{1}{2}\gamma(N)\right)$$

  $\mathrm{V}(a)$ **intertwines the"deformed" and canonical symmetry generators.**

- **Developed a procedure to restore the kernels from anomalous dimensions** $\widehat{\gamma}_N \mapsto \widehat{\omega}(\tau)$**.**

Thank you for your attention