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N3LO corrections to zero-jettiness soft function

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Plan of the talk

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Motivation

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- Differential calculation require a good handle on IR divergences, many schemes at NNLO
- Slicing scheme seems feasible at N3LO due to complexity of subtraction schemes

$$\sigma(O) = \int_0 \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau} = \int_0^{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau} + \int_{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\sigma(O)}{\mathrm{d}\tau}$$

  - q_T subtraction scheme [Catani,Grazzini'07]
  - N-jettiness subtraction scheme [Boughezal et al.'15][Gaunt et al.'15]

- SCET factorization theorem motivates us to consider jettiness as convenient slicing variable

$$\lim_{\tau\to 0} \mathrm{d}\sigma(O) = B_\tau \otimes B_\tau \otimes S_\tau \otimes H_\tau \otimes \mathrm{d}\sigma_{\mathrm{LO}}$$

Difficulties with jettiness variable

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- 0-jettiness for the colorless final state is complicated due to the presence of min-function

$$\tau = \sum_{i=1}^{m} \min_{q\in\{n,\bar{n}\}} \left[\frac{2q\cdot k_i}{n\cdot\bar{n}}\right] = \sum_{i=1}^{m} \min\{\alpha_i,\beta_i\}$$

- Definition which is more friendly for PS integration generates different configurations

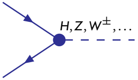
$$\delta\left(\tau - \sum_{i=1}^{m}\min\{\alpha_i,\beta_i\}\right) = \delta(\tau-\alpha_1-\alpha_2-\ldots)\theta(\beta_1-\alpha_1)\theta(\beta_2-\alpha_2)\ldots$$
$$+\delta(\tau-\alpha_1-\beta_2-\ldots)\theta(\beta_1-\alpha_1)\theta(\alpha_2-\beta_2)\ldots$$

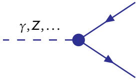
Sudakov decomposition

$$k_i = \frac{\alpha_i}{2}n + \frac{\beta_i}{2}\bar{n} + k_{i,\perp}, \quad k_i\cdot n = \beta_i, \quad k_i\cdot\bar{n} = \alpha_i, \quad n\cdot\bar{n} = 2$$

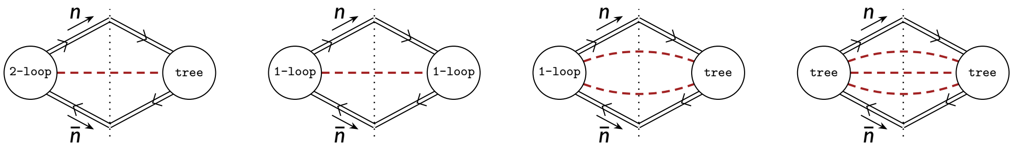
What is actually calculated?

- 0-jettiness in hadronic collisions is equal to Thrust or 2-jettiness in e^+e^- annihilation or Higgs decay





- The limit $\tau \to 0$ corresponds to the soft limit of the squared amplitude - eikonal rules
- Need to include all possible real and virtual corrections to the amplitude squared



- Possible to combine different measurement function terms into unique configurations
- Perform integration over highly non-trivial region - all kinds of divergencies are possible

Predictions from RGE and NNLO results

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- General form of the bare result up to NNNLO, our goal: $S^{(3)}$

$$S_B = \delta(\tau) + \frac{\alpha_s}{\tau^{1+2\varepsilon}} S^{(1)} + \frac{\alpha_s^2}{\tau^{1+4\varepsilon}} S^{(2)} + \frac{\alpha_s^3}{\tau^{1+6\varepsilon}} S^{(3)} + \mathcal{O}(\alpha_s^4)$$

- NNLO [Monni,Gehrmann,Luisoni'11] [Kelley,Schabinger,Schwartz,Zhu'11][Baranowski'20]
- Multiplicatively renormalizable in Laplace space after α_s UV renormalization

$$\tilde{S}_B(u\mu) = \int_0^\infty \mathrm{d}\tau e^{-u\tau} S_B(\tau, \alpha_{s,B} \to \alpha_s(\mu)), \quad \tilde{S}_{\mathrm{ren}} = Z_S \tilde{S}_B$$

- All poles in $S^{(3)}$ can be fixed from RGE and NNLO result with known anomalous dimensions

$$\left(\frac{\partial}{\partial \log \mu} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right) \log \tilde{S}_{\mathrm{ren}} = -4\Gamma_{\mathrm{cusp}} \log(u\mu) - 2\Gamma_{\mathrm{soft}}$$

Color structures decomposition

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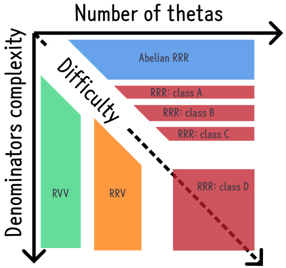
Full result splitting according to the number of soft emissions

$$S_{N^3LO} = S_{\mathrm{RRR}} + S_{\mathrm{RRV}} + S_{\mathrm{RVV}}$$

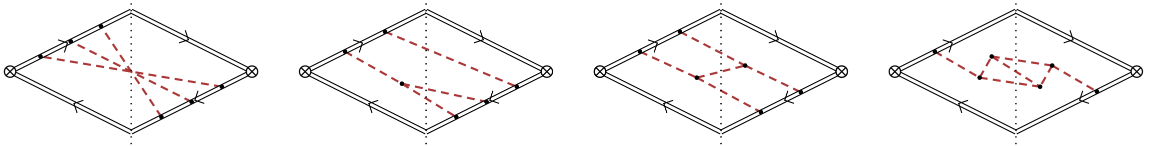
	C_R^3	$C_R^2 n_f T_F$	$C_R^2 C_A$	$C_R(n_f T_F)^2$	$C_R C_F n_f T_F$	$C_R C_A n_f T_F$	$C_R C_A^2$
S_{RRR}	+	+	+		+	?	?
S_{RRV}			+	+	+	+	+
S_{RVV}						+	+
S_{N^3LO}	+	+	+	+	+	+	+
Poles	✔	✔	✔	✔	✔	?	?

Required expansions up to higher orders in ε for N^2LO soft function are known [Baranowski'20]

Relative complexity of ingredients



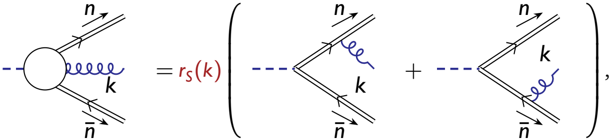
- For each soft emission we have one θ-function in the measurement function making integration more complicated
- Most complicated denominators in RRR case make direct integration impossible
- Most complicated one-loop sub-integrals in the RRV case make direct integration impossible
- Unregulated divergencies in the RRR case



A $\quad B \sim \frac{1}{k_1 \cdot k_2}$ $\quad C \sim \frac{1}{(k_1 \cdot k_2)(k_1 \cdot k_3)}$ $\quad D \sim \frac{1}{(k_1 + k_2 + k_3)^2}$

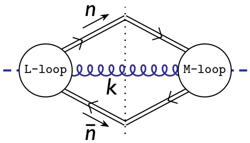
RVV corrections

Two-loop corrections $r_S^{(2)}$ to single gluon emission soft current are known exactly in ε [Duhr, Gehrmann'13]



$$r_S(k) = 1 + \sum_{l=1}^{\infty} A_s^l \left[\frac{-(n\cdot\bar{n})}{2(k\cdot n)(k\cdot\bar{n})} \right]^{l\varepsilon} r_S^{(l)}$$

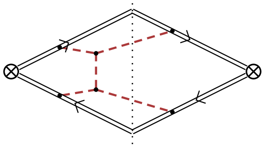
$$w_{L,M}(k) = \mathrm{Re}\left[J_L^\dagger(k) J_M(k)\right] =$$



Two contributions from different hemisphere emissions need to be integrated, $S_g^{(3)} = s_{2,0} + s_{1,1} + s_{0,2}$

$$s_{l,m} = \int \frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \delta^+(k^2) \left[\delta(1 - k\cdot n)\theta(k\cdot\bar{n} - k\cdot n) + \delta(1 - k\cdot\bar{n})\theta(k\cdot n - k\cdot\bar{n})\right] w_{l,m}(k)$$

One-loop corrections with double emission



- RRV squared amplitudes generated from scratch
- Results for one-loop soft current are known [Zhu'20][Czakon et al.'22]
- RRV result for gg final state is known [Chen,Feng,Jia,Liue'22]
- Recalculation including $q\bar{q}$ final state [Baranowski et al.'24]

Multi-loop calculations inspired approach

- Reduction to the minimal set of master integrals
- Differential equations from IBP reduction

Configuration specific measurement functions

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- We need to integrate squared amplitude in the soft limit with constraints from δ and θ functions
- δ and θ arguments are linear in momenta scalar products $\rightarrow$ similarity with propagators
- Only two different configurations nn and $n\bar{n}$ contribute

Same hemisphere

- RRV: α_1, α_2 unconstrained

$$\mathrm{d}\Phi^{nn}_{\theta\theta} = \delta(\tau - \beta_1 - \beta_2)\theta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2)$$

Different hemispheres

- RRV: α_1, β_2 unconstrained

$$\mathrm{d}\Phi^{n\bar{n}}_{\theta\theta} = \delta(\tau - \beta_1 - \alpha_2)\theta(\alpha_1 - \beta_1)\theta(\beta_2 - \alpha_2)$$

- Well known technique: reduction of integrals with δ-functions [Anastasiou,Melnikov'02]
- New technique: reduction of integrals with θ-functions [Baranowski,Delto,Melnikov,Wang'21]

Modified reverse unitarity

- In dimensional regularisation system of IBP equation can be constructed by integration under integral sign

$$\int \mathrm{d}^d l \frac{\partial}{\partial l_\mu}\left[v_\mu \cdot f\left(\{l\}\right)\right], \qquad \frac{\partial}{\partial k\cdot\bar{n}}\theta(k\cdot\bar{n} - k\cdot n) = \delta(k\cdot\bar{n} - k\cdot n)$$

- IBP for integrals with θ-functions generate new auxiliary topologies, partial fractioning required

$$\frac{\theta(k\cdot\bar{n}-k\cdot n)}{(k\cdot\bar{n})^a(k\cdot n)^b} \to \frac{\delta(k\cdot\bar{n}-k\cdot n)}{(k\cdot\bar{n})^a(k\cdot n)^b}$$

– RRR $\underbrace{\theta\theta\theta}_{\text{Level 3}} \to \underbrace{\delta\theta\theta+\theta\delta\theta+\theta\theta\delta}_{\text{Level 2}} \to \underbrace{\delta\delta\theta+\delta\theta\delta+\theta\delta\delta}_{\text{Level 1}} \to \underbrace{\delta\delta\delta}_{\text{Level 0}}$

– RRV $\underbrace{\theta\theta}_{\text{Level 2}} \to \underbrace{\delta\theta+\theta\delta}_{\text{Level 1}} \to \underbrace{\delta\delta}_{\text{Level 0}}$

Details of IBP reduction

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- Many integrals can be mapped on a small unique set, same symmetries as loop integral [Pak'11]

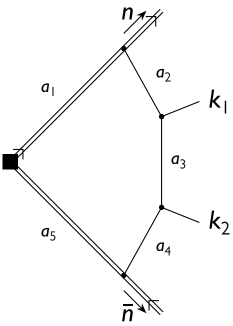
$$\theta(A) \to \frac{1}{A - m_\theta^2}, \quad \delta(A) \to \frac{1}{A - m_\delta^2}$$

- User-defined system reduction option available in `Kira` [Klappert et al.'21]

- Complexity of master integrals after IBP reduction
  - RRR: number of θ-functions, complexity of gluon propagators
  - RRV: number of θ-functions, complexity of one-loop integral

- No $\theta \ldots \theta$ master integrals for both RRR and RRV same hemisphere configurations

RRV master integrals calculation



- Number of MIs after IBP reduction of both configurations in RRV case

$\delta\delta$	$\delta\theta + \theta\delta$	$\theta\theta$
8	36	15

- Direct integration possible, except pentagon and box with $a_3 = 0$
- DE in auxiliary parameters for most complicated integrals

Original integrals from DE solution

Different strategy compared to RRR case: instead of $I = \lim_{z \to z_0} J(z)$ now $I = \int \mathrm{d}z J(z)$

RRV master integrals from differential equations

- For $\delta\delta$ integrals we introduce auxiliary paremeter x and solve DE system $\partial_x J(x) = M(\varepsilon, x)J(x)$

$$I_{\delta\delta} = \int \mathrm{d}\,(k_1 \cdot k_2)\, f(k_1 \cdot k_2) = \int_0^1 \mathrm{d}x \int \mathrm{d}\,(k_1 \cdot k_2)\,\delta(k_1 \cdot k_2 - \frac{x}{2}) f(k_1 \cdot k_2) = \int_0^1 J(x)\mathrm{d}x$$

- For $\delta\theta$ and $\theta\delta$ we use integral representation for θ-function and solve DE system $\partial_z J(z) = M(\varepsilon, z)J(z)$

$$\theta(b-a) = \int_0^1 b\delta(zb-a)\mathrm{d}z, \quad I_{\delta\theta} = \int_0^1 J(z)\mathrm{d}z$$

- For $\theta\theta$ integrals PDE system in two variables z_1, z_2, no IBP reduction with θ-functions needed

$$I_{\theta\theta} = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2 J(z_1, z_2)$$

Differential equations in canonical form

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- For all auxiliary integrals it is possible to find alternative basis of integrals, such ε dependence of the DE system matrix factorizes completely: $M(\varepsilon) \to \varepsilon A$
- Straightforward solution for integrals in canonicaal basis in terms of GPLs
- Simpler boundary conditions fixing due to known general form of expansion near singular points

$$g(z) = z^{a_1+b_1\varepsilon}\left(c_1 + \mathcal{O}(z)\right) + z^{a_2+b_2\varepsilon}\left(c_2 + \mathcal{O}(z)\right) + \ldots$$

- Construction of subtraction terms to remove endpoint singularities in final integration

$$\int_0^1 J(z)\mathrm{d}z = \int_0^1 \underbrace{\left[J(z) - z^{a_i+b_i\varepsilon}j_0(z) - (1-z)^{a_k+b_k\varepsilon}j_1(z)\right]}_{\varepsilon-\text{expanded}}\mathrm{d}z + \int_0^1 \underbrace{\left(z^{a_i+b_i\varepsilon}j_0(z) - (1-z)^{a_k+b_k\varepsilon}j_1(z)\right)}_{\varepsilon-\text{exact}}\mathrm{d}z$$

Results RRV

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- Results for separate configurations contain $\mathrm{Li}_4(1/2)$, but sum of two configurations has ζ_n only with maximal transcendental weight 6
- As the consequence of the exponentiation of soft emissions in Abelian gauge theories C_R^3 in S_{gg} and $C_R^2 n_f T_F$ in S_{qq} are absent
- From N^3LO RRV + N^2LO result its possible extract large n_f contribution to the renormalized soft function

Non-logarithmic part of the Laplace space result

$$\tilde{S}^{(3)}_{\mathrm{nl}} = C_R \left(n_f T_F\right)^2 \left[\frac{265408}{6561} - \frac{400}{243}\pi^2 - \frac{51904}{243}\zeta_3 + \frac{328}{1215}\pi^4 + \mathcal{O}\left(\frac{1}{n_f}\right)\right]$$

Triple real emissions

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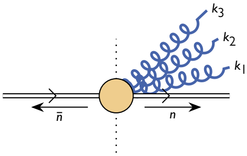
Recalculated input for eikonal factor with partial fractioning and topology mapping

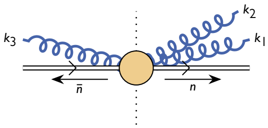
- $ggg = ggg + gc\bar{c}$, coincides with known expression in physical gauge [Catani,Colferai,Torrini'19]
- $gq\bar{q}$ in agreement with [Del Duca,Duhr,Haindl,Liu'23]

Same hemisphere



$\delta(\tau - \beta_1 - \beta_2 - \beta_3)$

Different hemispheres



$\delta(\tau - \beta_1 - \beta_2 - \alpha_3)$

- Same hemisphere result for ggg final state is known [Baranowski et al.'22]

Difficulty no. 1

Integrals unregulated dimensionally

- Not all integrals appearing during IBP reduction are regulated dimensionally, initial integrals are regular
- Possible solution is to introduce additional regulator: $\mathrm{d}\Phi^{nn\bar{n}}_{f_1 f_2 f_3} \to \mathrm{d}\Phi^{nn\bar{n}}_{f_1 f_2 f_3} (k_1 \cdot n)^{\nu} (k_2 \cdot n)^{\nu} (k_3 \cdot \bar{n})^{\nu}$

$$I \sim \int_0^1 \mathrm{d}z \frac{1}{z} z^{\varepsilon} z^{-\varepsilon} (\dots) \quad \longrightarrow \quad I(\nu) \sim \int_0^1 \mathrm{d}z \frac{1}{z^{1+\nu}} (\dots)$$

- Same complexity level topologies $\partial_k (k \cdot n)^{\nu} \sim (k \cdot n)^{\nu} \frac{\nu}{(k \cdot n)}$, new denominator needs partial fractioning

Reduction of ν-regulated integrals

Approaches to ν-dependent IBP reduction problem (IBP with ν is available)

1. Direct ν-dependent reduction with additional variable
   - ✘ Time consuming and not flexible especially if basis change needed
   - ✔ Minimal set of master integrals and full ν-dependent solution

2. Filtering - remove from the IBP system all equations with potentially divergent integrals
   - ✔ Very fast compared to the full ν-dependent reduction
   - ✘ Potentially unreduced integrals, needs divergencies analysis for all integrals in the IBP system

3. Expansion - rewrite IBP system as a new system for $1/\nu$ expansion coefficients of integrals
   - ✔ Fast reduction with control of divergencies
   - ✘ Additional divergencies of integrals from intermediate steps can appear

Importance of a good master integrals basis

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- From analysis of possible divergencies we consider ansatz $J_a = \sum_{k=k_0}^{\infty} J_a^{(k)} \nu^k$ with $k_0 = -1$
- Solution of the IBP reduction problem for regular in $1/\nu$ integrals I_a has the form

$$I_a^{(0)} = R_{ab} J_b^{(0)} + D_{ab} \tilde{J}_b^{(-1)}$$

- We require a good basis to fullfill following conditions:
  - Coefficients in front of master integrals are free from $1/\nu$ poles
  - Each master integral is a member of only one set J_b or $\tilde{J}_b$
  - Candidates to the set J_b can be found from $\nu = 0$ reduction
- Regular integrals $J_b^{(0)}$ are calculated in a standard way, calculation of needed divergent parts $\tilde{J}_b^{(-1)}$ is simplified, since only specific regions contribute

Difficulty no. 2

DE for RRR integrals with auxiliary mass

- Integrals for both *nnn* and $nn\bar{n}$ configurations with denominator $1/k_{123}^2$ are difficult to calculate
- Integrals are single scale, auxiliary parameter needed to construct DE system $I \to J(m^2)$
- Our solution is to make the most complicated propagator massive $\frac{1}{(k_1+k_2+k_3)^2} \to \frac{1}{(k_1+k_2+k_3)^2+m^2}$
- Calculation of boundary conditions simplifies in the limit $m^2 \to \infty$
- Result for integrals of our interest from the solution for $J(m^2)$ in the limit $m^2 \to 0$

Difficulties of the chosen strategy:

- Both points $m^2 \to 0$ and $m^2 \to \infty$ are singular points of the DE system
- Solution of the DE for integrals with massive denominator possible only numerically

Details of the DE solution

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- Larger DE system size with ~ 650 equations for $nn\bar{n}$ configuration compared to ~ 150 for nnn
- Needed to calculate all contributing regions into boundary conditions in the $m^2 \to \infty$ limit

$\sim (m^2)^0$	$\sim (m^2)^{-\varepsilon}$	$\sim (m^2)^{-2\varepsilon}$
$1/m^2$	$\alpha_i \sim m^2$	$\alpha_i, \alpha_j \sim m^2$

- For each large parameter $\alpha_i \sim m^2$ we remove $\theta \Longrightarrow$ additional IBP reduction of BC integrals possible
- Numerical solution of the DE system as a sequence of series expansions [Liu et al.'18][Chen et al.'22]

Difficulty no. 3

Direct integration of master integrals and boundary conditions

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- We have calculated ~ 130 integrals without $1/k_{123}^2$ denominator and ~ 100 boundary conditions by direct integration with `HyperInt` [Panzer'15]
- Summary of used techniques
  1. Change variables to satisfy all constraints from δ and θ functions
  2. Perform as many integrations as possible in terms of ${}_2F_1$ and F_1 functions with known argument transforms
  3. Do remaining integrations in terms of ${}_pF_q$ functions if possible
  4. For final integral representation with minimal number of integrations and minimal set of divergencies we construct subtraction terms
  5. Integrand with all divergencies subtracted is expanded in ε and integrated term by term with `HyperInt`
  6. Subtraction terms are integrated in the same way

Remaining steps to complete calculation

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- Numerical checks
  - Extensive numerical checks of separate amplitude terms and/or master integrals are needed for RRR integrals with $1/k_{123}^2$.
  - Very complicated due to high degree of divergencies for both sector decomposition and MB approaches.

- More orders in ε
  - Once agreement for highest poles is found we need to produce higher ε-order expansions from DE
  - Also highly nontrivial since deep expansions for boundaries are needed due to spurious poles.

- More digits of precision
  - When all needed ε-expansions are known we need to solve DE numerically to high precision to reconstruct analytical result.
  - Reconstruction of each configuration independently is a good check of the obtained result. Basis can be complicated, e.g. GPLs with sixth-roots of unity alphabet as in the same-hemisphere case.

Conclusion

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- Developed efficient reduction techniques for integrals with Heaviside θ-functions applicable for phase-space integrals with loops and additional regulators
- Developed powerfull approach for calculation of the most complicated RRR integrals by solving differential equations in auxuliary parameter numerically with high precision and calculated all needed boundary conditions
- Recalculated one-loop corrections for two gluon emission contribution to the soft function together with a new part for the quark pair emission
- We are ready to produce final numbers for renormalized N^3LO zero-jettiness soft function
- Extensive numerical checks of the most complicated triple emission contributions are needed

Thank you for attention!