Three-loop vertices with massive particles

Loops and Legs, 2024

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Lutherstadt Wittenberg, April 18, 2024

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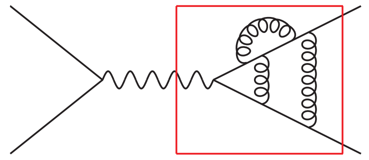
Outline

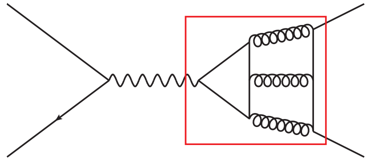
- Introduction
- $\bar{B} \to X_s\gamma$ in the SM
  - Low-energy effective theory
  - Perturbative calculations
- The heavy-to-light form factor
- Summary and Outlook

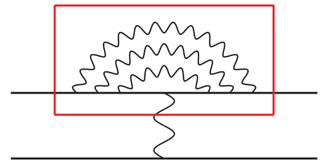
Introduction

Motivation

- Form factors are basic building blocks for many physical observables:
  - $t\,\bar{t}$ production at hadron and $e^+\,e^-$ colliders
  - $\mu\,e$ scattering
  - Higgs production and decay
  - ...
- Form factors exhibit an universal infrared behavior.
- In flavor physics non-diagonal form factors are important (especially $b \to s$, $b \to u$ and $b \to c$ transitions).





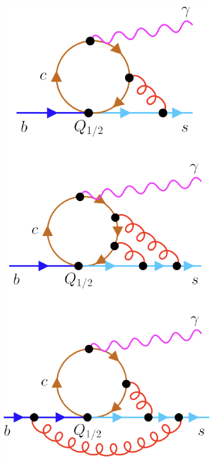


$\bar{B} \to X_s\gamma$ is interesting to search (or constraint) new physics in the quark sector:

- $b \to s\gamma$ is forbidden at tree-level in the SM.
- The dominant contributions in the SM come from weak decays.

⇒ The SM rate is small.

⇒ The decay is sensitive to new physics.



Status of $B \to X_s\gamma$

Experimental:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.49 \pm 0.19)}_{\pm 5.4\%} \times 10^{-4}$$

- CLEO, BaBar and Belle measurements combined by PDG and HFLAV [arXiv:2206.07501] .

Theoretical:

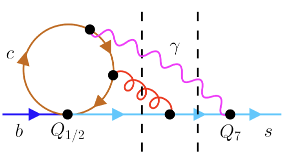
[Misiak, Rehman, Steinhauser '20]

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.40 \pm 0.17)}_{\pm 5.0\%} \times 10^{-4}$$

Breakdown of the error: m_c-interpolation

$$\pm 5\% = \sqrt{(\pm 3\%)^2 + (\pm 3\%)^2 + (\pm 2.5\%)^2}$$

higher orders parametric and non-perturbative



$\bar{B} \to X_s\gamma$ in the SM

$\bar{B} \to X_s\gamma$ in the SM

Determination of $\bar{B} \to X_s\gamma$ in the SM:

$$\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = \mathcal{B}(\bar{B} \to X_c e\bar{\nu})_{\text{exp}} \left|\frac{V_{ts}^* V_{tb}}{V_{cb}}\right|^2 \frac{6\alpha}{\pi C} \left[P(E_0) + N(E_0)\right]$$

- semileptonic phase-space factor: [Alberti, Gambino, Healey, Nandi '14] [1]

$$C = \left|\frac{V_{ub}}{V_{cb}}\right|^2 \frac{\Gamma(\bar{B} \to X_c e\bar{\nu})}{\Gamma(\bar{B} \to X_u e\bar{\nu})}$$

- $P(E_0)$: perturbative contributions

$$P(E_0) \sim \Gamma(b \to X_s^p\gamma) = \Gamma(b \to s\gamma) + \Gamma(b \to sg\gamma) + \Gamma(b \to sq\bar{q}\gamma) + \ldots \approx 96\%$$

- $N(E_0)$: non perturbtative contributions $\approx 4\%$

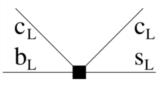
[1] See also the talk by Matteo Fael.

Effective Hamiltonian

- At low energies we want to work in the effective theory to resum large logarithmic contributions: $\left(\alpha_s \ln m_W^2/m_b^2\right)^n$
- For $b \to s\gamma$ (when neglecting NLO EW and CKM suppressed effects) we have:

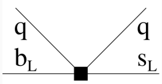
$$\mathcal{L} = \mathcal{L}_{QCD\times QED}(u,d,s,c,b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) Q_i + \dots$$

$Q_{1,2}$ $\quad (\bar{s}\Gamma_i c)(\bar{c}\Gamma_i' b)$



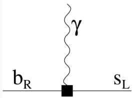
$|C_i(m_b)| \sim 1$

$Q_{3,4,5,6}$ $\quad (\bar{s}\Gamma_i b)\sum_q (\bar{q}\Gamma_i' q)$



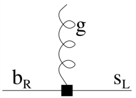
$|C_i(m_b)| < 0.07$

Q_7 $\quad \frac{e m_b}{16\pi^2} \bar{s}_L \sigma^{\mu\nu} b_R F_{\mu\nu}$



$|C_7(m_b)| \sim 0.3$

Q_8 $\quad \frac{g m_b}{16\pi^2} \bar{s}_L \sigma^{\mu\nu} T^a b_R G^a_{\mu\nu}$



$|C_8(m_b)| \sim 0.15$

Effective Hamiltonian

$$\Gamma(b \to X_s^p \gamma) = \frac{G_F^2 m_b^5 \alpha}{3} 2\pi^4 \left|V_{ts}^* V_{tb}\right|^2 \underbrace{\sum_{i,j=1}^{8} C_i(\mu_b) C_j(\mu_b) G_{ij}}_{\sim P(E_0)}$$

Three steps for the calculation:

1. Calculate the Wilson coefficients $C_i(\mu_0)$ at the hard scale $\mu_0 = m_W$.
2. Derive the renormalization group equations and anomalous dimensions γ_{ij} in the effective theory to evolve down to the low scale $\mu = m_b$:

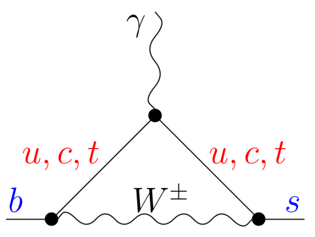
$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_i(\mu) = \sum_j \gamma_{ij}(\mu) \cdot C_j(\mu)$$

3. Evaluate the matrix elements $G_{ij}(m_b)$ in the effective theory.

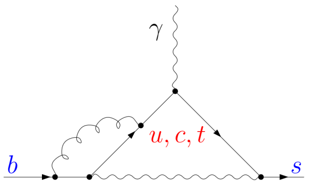
Effective Hamiltonian

Wilson coefficients at hard scale: for expample $C_7(m_W)$

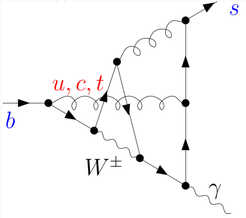
LO [Inami, Lim '81, ...]



NLO [Adel, Yao '93, ...]



NNLO [Misiak, Steinhauser '04]



Effective Hamiltonian

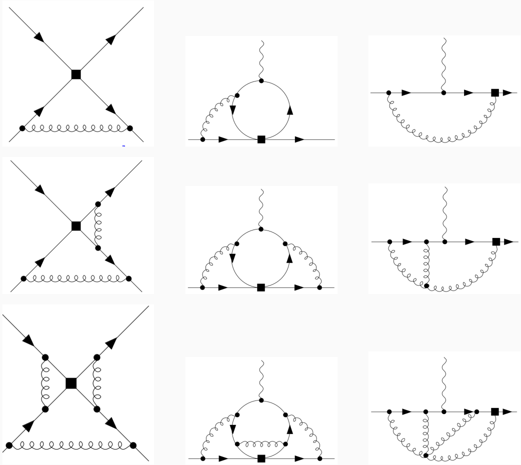
Anomalous dimensions: $\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_i(\mu) = \sum_j \gamma_{ij}(\mu) \cdot C_j(\mu)$

Z_{22} Z_{27} Z_{28}

LO

NLO

NNLO



[Gaillard, Lee '74; Altarelli, Maiani '74] [Grinstein et al '90]

[Shifmann et al '78; Grigjanis et al '88]

[Altarelli et al '81; Buras, Weisz '90] [Chetryrkin, Misiak, Münz '97]

[Misiak, Münz '95]

[Gorbahn, Haisch '04] [Czakon, Haisch, Misiak '06] [Gorbahn, Haisch, Misiak '05]

NNLO corrections give -4% correction to the branching ratio

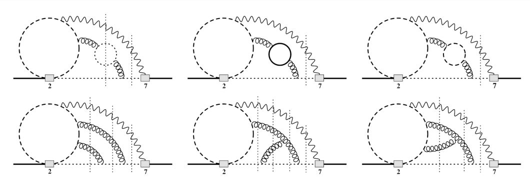
NNLO corrections to $B \to X_s\gamma$

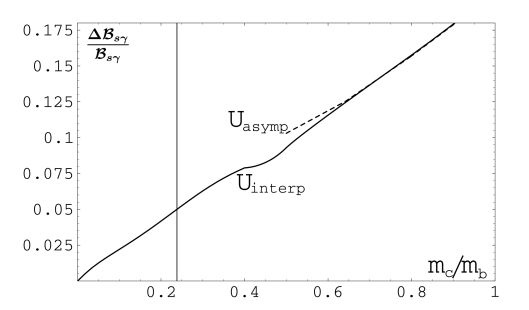
$$\Gamma(b \to X_s^p\gamma) = \frac{G_F^2 m_b^5 \alpha}{3} 2\pi^4 \left|V_{ts}^* V_{tb}\right|^2 \underbrace{\sum_{i,j=1}^{8} C_i(\mu_b) C_j(\mu_b) G_{ij}}_{\sim P(E_0)}$$

Status:

- NLO is known completely. [Greub, Hurth, Wyler '96; Ali, Greub '91-'95; Buras, Czarnecki, Urban, Misiak '02; Pott '95]
- NNLO:
  - G_{77} and G_{78} are known completely. [Blokland et al '05; Melnikov, Mitov '05; Asatrian et al. '06-'10]
  - For numerically small contributions the two body contributions are known, the rest is approximated using BLM.
  - G_{17} and G_{27} interpolated between $m_c = 0$ and $m_c \to \infty$.

Calculation of G_{27} at NNLO



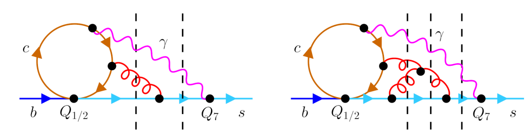


- Perturbative calculation can be done by considering diagrams with operator insertions and unitarity cuts.
- Calculation for $m_c \to \infty$: [Misiak, Steinhauser '06, '10]
- Calculation for $m_c = 0$: [Czakon, Fiedler, Huber, Misiak, Schutzmeier, Steinhauser '15]
- Calculation of terms proportional to n_f for arbitrary values of m_c: [Misiak, Rehmann, Steinhauser '20]

$\Rightarrow$ Interpolation to physical m_c/m_b introduces $\pm 3\%$ error in final result.

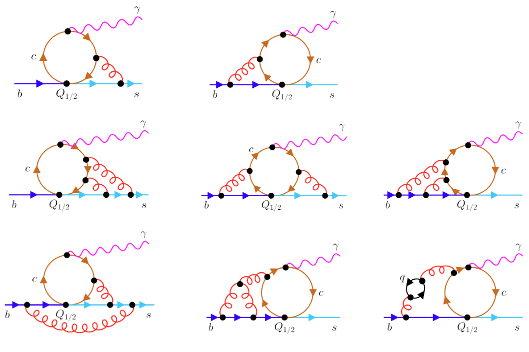
Two Body contributions to G_{27} at NNLO

[Fael, Lange, KS, Steinhauser '23]



Two Body contributions:

- Interpret the cut diagrams as vertex corrections:

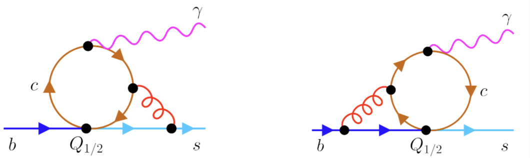


Two Body contributions to G_{27} at NNLO

Calculation of the vertex: $b(p_b) \to s(p_s) + \gamma(q_\gamma)$, $p_b^2 = m_b^2,\ p_s^2 = q_\gamma^2 = 0$

$$M^\mu = \bar{u}_s(p_s) P_R \left(t_1 \frac{q_\gamma^\mu}{m_b} + t_2 \frac{p_b^\mu}{m_b} + t_3\, \gamma^\mu \right) u_b(p_b)$$

- We find 30 (591) diagrams at 2-(3-)loop level.
- We use `qgraf`, `tapir`, `exp` and `calc` for the generation of the amplitude.
- We find masters 14 (479) master integrals with `Kira` [Klappert, Lange, Maierhöfer, Usovitsch '20] .



Master integrals at NLO

Master integrals at 2-loop:

- We solve the differential equation in the variable $z = m_b/m_c$.
- Boundary conditions can be computed using a large-mass expansion in m_c $(z \to 0)$.
- We use the algorithm of [Ablinger, Blümlein, Marquard, Rana, Schneider '18] to solve the differential equation:
  1. Decouple blocks of the differential equations with `OreSys` [Gerhold, Schneider '02] and `Sigma` [Schneider '02] .
  2. Factorize and solve the differential equations with the help of `HarmonicSums` [Ablinger '10] .
- We find the following alphabet:

$$\frac{1}{z}, \qquad \frac{1}{1 \pm z}, \qquad \frac{1}{2 \pm z}, \qquad \sqrt{4 - z^2}$$

Master integrals at 2-loop:

- We find the following alphabet:

$$\frac{1}{z}, \qquad \frac{1}{1 \pm z}, \qquad \frac{1}{2 \pm z}, \qquad \sqrt{4 - z^2}$$

- We perform an analytic continuation to $x = 1/z < 1/4$.
- For iterated integrals containing the square root letters we change back to the variable:

$$w = \frac{1 - \sqrt{1 - 4x^2}}{1 + \sqrt{1 - 4x^2}}, \qquad \text{with} \qquad x = \frac{m_c}{m_b}, \quad z = \frac{m_b}{m_c}$$

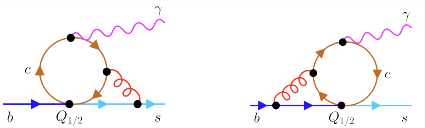
$\Rightarrow$ We can express the final result in terms of harmonic polylogarithms of argument x and w only.

Two Body contributions to G_{27} at NLO

New analytic results at 2-loop:

$$G_{27}^{(1),2P} = -\frac{92}{81\epsilon} + f_0(z) + \epsilon f_1(z) + \mathcal{O}(\epsilon^2),$$

$$\begin{aligned}
f_0 = C_F \Bigg[& -\frac{971 + 1916w + 1602w^2 + 1916w^3 + 971w^4}{162(1+w)^4} + \frac{2wH_0(w)^3}{3(1+w)^2} \\
& + \frac{8w(27 + 57w + 26w^2 + 7w^3 + 5w^4)H_0(w)}{27(1+w)^5} - \frac{16w(2 + 3w + 2w^2)H_0(x)^3}{9(1+w)^4} \\
& - \frac{2w(-1 - 2w + 4w^2 + 6w^3 + 3w^4)H_0(w)^2}{3(1+w)^6} - \frac{8w(1+w^2)H_{0,0,-1}(w)}{3(1+w)^4} \\
& - \frac{8(5 + 29w + 54w^2 + 29w^3 + 5w^4)H_{-1}(w)}{27(1+w)^4} + \frac{16w^2(3 + 13w + 15w^2 + 6w^3)H_{0,-1}(w)}{9(1+w)^6} \\
& - \frac{16w(1 - \sqrt{w} + w)}{9(1+w)^6}(3 + 8w + 8w^2 + 3w^3 + 2\sqrt{w} + 3w^{3/2} + 2w^{5/2})H_{1,0}(x) \\
& + \frac{16w(1 + \sqrt{w} + w)}{9(1+w)^6}(3 + 8w + 8w^2 + 3w^3 - 2\sqrt{w} - 3w^{3/2} - 2w^{5/2})H_{-1,0}(x) \\
& + \frac{16w(3 + 9w + 13w^2 + 9w^3 + 3w^4)H_{-1,0}(w)}{9(1+w)^6} - \frac{8w\zeta_3}{(1+w)^2} \\
& - \frac{32w(3 + 9w + 13w^2 + 9w^3 + 3w^4)H_{-1,-1}(w)}{9(1+w)^6} - \frac{16w(2 + 3w + 2w^2)H_{0,1,0}(x)}{3(1+w)^4} \\
& + \frac{16w(2 + 3w + 2w^2)H_{0,-1,0}(x)}{3(1+w)^4} - \frac{16w(1 + 3w + w^2)H_{-1,0,0}(w)}{3(1+w)^4} \\
& + \pi^2 \Bigg(-\frac{2w}{27(1+w)^6}(15 + 60w + 94w^2 + 84w^3 + 27w^4 - 12\sqrt{w} - 36w^{3/2} \\
& - 36w^{5/2} - 12w^{7/2}) - \frac{2w(3 + 8w + 3w^2)H_0(w)}{3(1+w)^4} + \frac{8w(5 + 12w + 5w^2)H_{-1}(w)}{9(1+w)^4} \Bigg) \Bigg]
\end{aligned}$$



with

$$x = \frac{m_c}{m_b},$$

$$w = \frac{1 - \sqrt{1 - 4x^2}}{1 + \sqrt{1 - 4x^2}}$$

Master integrals at NNLO

Master integrals at 3-loop:

- We use the 'expand-and-match' method already successfully applied in other projects:
  - massive (diagonal) form factors [Fael, Lange, KS, Steinhauser '22]
  - massive operator matrix elements [Ablinger, Behring, Blümlein, De Freitas, Manteuffel, Schneider, KS '24]
  - ...
- The basic steps are given by:
  1. Calculate initial values of the master integrals.
     Here we use AMFlow [Liu, Ma '22] at $x = m_c/m_b = 1/5$.
  2. Construct symbolic expansions around $x = 1/5, 1/10, 0$ by inserting an ansatz into the differential equation and solve a large linear system of equations in terms of a small number of initial conditions with `Kira`.
  3. Use the initial boundary value or to obtain an expansion around $x = 1/5$.
  4. To obtain the next expansion use the previous expansion to obtain boundary values.

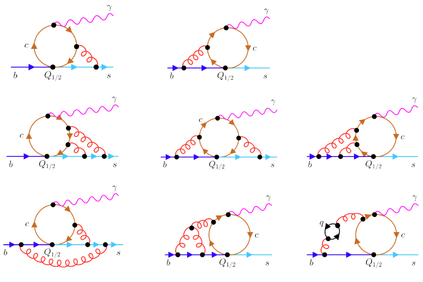
$\Rightarrow$ We obtain a precise semi-analytic result for $0 < m_c/m_b < 1/5$.

We agree with a partial result obtained in [Greub, Asatrian, Saturnino, Wiegand '23]

No gluons connecting to b quark lines were considered.

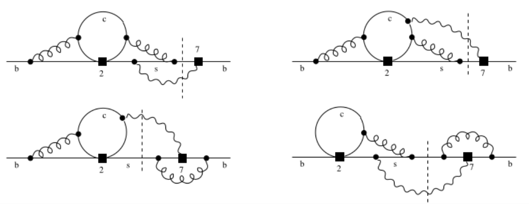
Two Body contributions to G_{27} at NNLO

$$
\begin{aligned}
\mathrm{Re}(t_2^{Q_1}) = n_l\Bigg\{ & -\frac{0.643804}{\epsilon^2} - \frac{6.31123}{\epsilon} - 27.9137 + x^2\Bigg[\frac{1}{\epsilon}\Big(2.107l_x^3 + 3.16049l_x^2 - 27.8263l_x \\
& - 11.7523\Big) - 7.37449l_x^4 + 3.51166l_x^3 + 25.8566l_x^2 - 201.543l_x - 247.57\Bigg]\Bigg\} \\
& + n_c\Bigg\{ -\frac{0.643804}{\epsilon^2} - \frac{6.31123}{\epsilon} - 27.9137 + x^2\Bigg[\frac{1}{\epsilon}\Big(2.107l_x^3 + 3.16049l_x^2 - 24.6658l_x \\
& - 9.61098\Big) - 7.37449l_x^4 + 12.9931l_x^3 + 54.3011l_x^2 - 224.155l_x - 335.398\Bigg]\Bigg\} \\
& + n_b\Bigg\{ -\frac{0.643804}{\epsilon^2} - \frac{6.25499}{\epsilon} - 14.2846 + x^2\Bigg[\frac{1}{\epsilon}\Big(2.107l_x^3 + 3.16049l_x^2 - 27.8263l_x \\
& - 11.7523\Big) - 5.26749l_x^4 + 23.7497l_x^2 - 104.437l_x - 132.539\Bigg]\Bigg\} \\
& - \frac{2.0192}{\epsilon^3} + \frac{87.3997}{\epsilon} + 256.363 + \frac{8.17904}{\epsilon^2} + x\left(\frac{374.314}{\epsilon} - 1497.26l_x + 669.332\right) \\
& + x^2\Bigg[\frac{1}{\epsilon^2}\Big(4.21399l_x^3 + 6.32099l_x^2 - 55.6525l_x - 23.5046\Big) + \frac{1}{\epsilon}\Big(- 13.6955l_x^4 \\
& - 36.8724l_x^3 - 209.669l_x^2 + 1407.45l_x + 233.132\Big) + 27.8123l_x^5 + 142.222l_x^4 \\
& + 402.206l_x^3 - 2492.03l_x^2 + 7662.75l_x + 8375.85\Bigg], \qquad (21)
\end{aligned}
$$



Two Body contributions to G_{27} at NNLO

[Czaja, Czakon, Huber, Misiak, Niggetiedt, Rehmann, KS, Steinhauser '23]

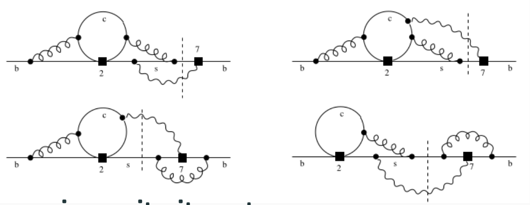


Independent calculation using unitarity cuts:

- Consider b quark propagators with operator insertions and consider all two body cuts.
    - The reductions to master integrals are done with `Fire` [Smirnv, Chuharev '19] and `Kira` [Klappert, Lange, Maierhöfer, Usovitsch '20] .
    - For the two body contributions we need to evaluate 447 master integrals.
- The master integrals are evaluated at a physical point with AMFLow.
- We cross-checked the boundary conditions for $z \to \infty$.
- We find agreement between the two approaches at the level of 10^{-15}.

Two Body contributions to G_{27} at NNLO

[Czaja, Czakon, Huber, Misiak, Niggetiedt, Rehmann, KS, Steinhauser '23]

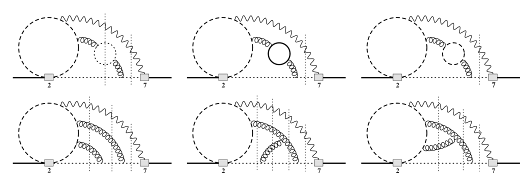


Independent calculation using unitarity cuts:

- Consider b quark propagators with operator insertions and consider all two body cuts.

$$
\begin{aligned}
\Delta_{30}\hat{G}_{27}^{(2)2P}(z=0.04) \;\simeq\;& \frac{0.181070}{\epsilon^3} - \frac{6.063805}{\epsilon^2} - \frac{34.087329}{\epsilon} - 127.624515 \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.093615}{\epsilon} + 10.984004\right) n_b \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.185427}{\epsilon} + 19.194053\right) n_c \\
&+ \left(\frac{0.482853}{\epsilon^2} + \frac{4.135795}{\epsilon} + 19.647238\right) n_l \,, \\
\Delta_{30}\hat{G}_{17}^{(2)2P}(z=0.04) \;\simeq\;& -\frac{1}{6}\Delta_{30}\hat{G}_{27}^{(2)2P}(z=0.04) + \frac{0.987654}{\epsilon^2} + \frac{6.383643}{\epsilon} + 34.077780
\end{aligned}
$$

Towards the $b \to s\gamma$ at NNLO without interpolation



Outlook:

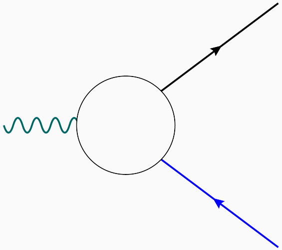
- Finish the calculation of the three and four body cuts:
    1. IBP reductions for all cuts are done.
    2. Boundary conditions for $z \to \infty$ calculated and cross-checked.

$\Rightarrow$ The calculation will enable a more precise prediction of $b \to s\gamma$ in the Standard Model.

The heavy-to-light form factor

The heavy-to-light form factor

- The heavy-to-light form factors are important for
  - heavy quark decays.
  - top quark production.
  - muon decays.
  - ..



$$X(q) \to Q(q_1) + q(q_2)$$

$$q_1^2 = m_b^2,\ q_2^2 = 0, \qquad q^2 = s = \hat{s} \cdot m_b^2$$

vector : $\quad j_\mu^v = \overline{\psi}_Q \gamma_\mu \psi_q \qquad \Gamma_\mu^v = F_1^v(s)\gamma_\mu - \frac{i}{2m_b} F_2^v(s)\sigma_{\mu\nu} q^\nu$

axial-vector : $\quad j_\mu^a = \overline{\psi}_Q \gamma_\mu \gamma_5 \psi_q \qquad \Gamma_\mu^a = F_1^a(s)\gamma_\mu\gamma_5 - \frac{1}{2m_b} F_2^a(s) q_\mu \gamma_5$

scalar : $\quad j^s = m_b \overline{\psi}_Q \psi_q \qquad \Gamma^s = m_b F^s(s)$

pseudo-scalar : $\quad j^p = i m_b \overline{\psi}_Q \gamma_5 \psi_q \qquad \Gamma^p = i m_b F^p(s) \gamma_5$

Previous Calculations

NNLO – neglecting light fermion mass

[Bonciani, Ferroglia '08] [Asatrian, Greub, Pecjak '08]

[Beneke, Huber, Ki '09] [Bell '09]

NNLO – non-vanishing light fermion mass

master integrals: [Chen '18]

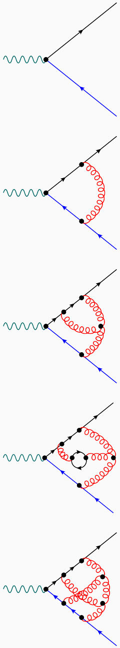
amplitude and small mass expansion: [Engel, Gnendiger, Signer, Ulrich '19]

NNNLO – neglecting light fermion mass

leading color masters: [Chen, Wang '18]

leading color amplitudes: [Datta, Rana, Ravindran, Sarkar '23]

this talk: full (semi-analytic) results at NNNLO



Technical Details

- Generate diagrams with `QGRAF`. [Nogueira '93]
- Use `FORM` [Ruijl, Ueda, Vermaseren '17] for Lorentz, Dirac and color algebra. [Ritbergen, Schellekens, Vermaseren '98]
- Map the output to predefined integral families with `q2e`/`exp` [Harlander, Seidensticker, Steinhauser '97-'99] and `tapir` [Gerlach, Herren, Lang '22] .
- Reduce the scalar integrals to masters with `Kira`. [Klappert, Lange, Maierhöfer, Usovitsch, Uwer '17,'20]
  - We ensure a good basis where denominators factorize in ϵ and $\hat{s}$ with `ImproveMasters.m`. [Smirnov, Smirnov '20]
- Establish differential equations in variable $\hat{s}$ using `LiteRed`. [Lee '12,'14]

	complete	leading-color	n_h^2	n_l
families	47	13	5	11
masters	429	71	9	50

Calculation of the master integrals

- The result can be split up in several color factors:

$$F_i^{(3)} = T_F^2 n_h^2 F_i^{n_h^2,(3)} + T_F^2 n_l^2 F_i^{n_l^2,(3)} + T_F^2 n_h n_l F_i^{n_h n_l,(3)} + C_A T_F n_l F_i^{C_A n_l,(3)} + C_F T_F n_l F_i^{C_F n_l,(3)}$$
$$+ n_c^3 F_i^{\mathrm{lc},(3)} + C_A T_F n_h F_i^{C_A n_h,(3)} + C_F T_F n_h F_i^{C_F n_h,(3)} + \ldots$$

- The red color factors can be solved in terms of iterated integrals, again with the algorithm of [Ablinger, Blümlein, Marquard, Rana, Schneider '18] .
- We calculate initial values with AMFLow at $\hat{s} = 0$ and use PSLQ for analytic reconstruction.
- We find the same letters as has been found up to 2-loop:

$$\boxed{\frac{1}{x}, \qquad \frac{1}{1 \pm x}, \qquad \frac{1}{2 + x}}$$

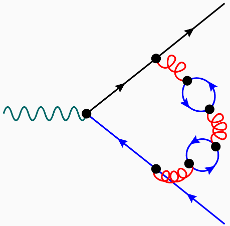
- The other integrals are solved using 'expand-and-match' starting at $\hat{s} = 0$.

Results

[Fael, Huber Lange, Müller, KS, Steinhauser '23]

Preliminary

$$
\begin{aligned}
\hat{F}_i^{n_l^2,(3)} = & -\frac{322979}{6561} - \frac{442\pi^4}{1215} + \pi^2\left(-\frac{2692}{243} - \frac{608}{81}H_1 - \frac{64}{27}H_{0,1}\right. \\
& \left. - \frac{128}{27}H_{1,1}\right) - \frac{8(1026+4919\hat{s})H_1}{729\hat{s}} - \frac{16(18+203\hat{s})H_{0,1}}{81\hat{s}} \\
& - \frac{32(18+203\hat{s})H_{1,1}}{81\hat{s}} - \frac{608}{27}H_{0,0,1} - \frac{1216}{27}H_{0,1,1} \\
& - \frac{1216}{27}H_{1,0,1} - \frac{2432}{27}H_{1,1,1} - \frac{64}{9}H_{0,0,0,1} - \frac{128}{9}H_{0,0,1,1} \\
& - \frac{128}{9}H_{0,1,0,1} - \frac{256}{9}H_{0,1,1,1} - \frac{128}{9}H_{1,0,0,1} - \frac{256}{9}H_{1,0,1,1} \\
& - \frac{256}{9}H_{1,1,0,1} - \frac{512}{9}H_{1,1,1,1} - \frac{64}{243}(130+63H_1)\zeta(3)
\end{aligned}
$$



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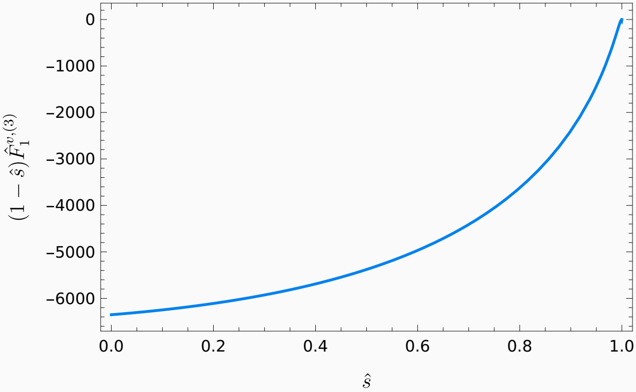

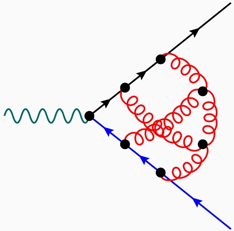

vector : $j^v_\mu = \overline{\psi}_Q \gamma_\mu \psi_q$

$$\Gamma^v_\mu = F^v_1(s)\gamma_\mu - \frac{i}{2m_b} F^v_2(s)\sigma_{\mu\nu}q^\nu$$

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- $b \to s\gamma$ is an important precision probe of the Standard Model.
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- We have calculated full charm mass effects at NLO and NNLO:
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$$\pm 5\% = \sqrt{(\pm 3\%)^2 + (\pm 3\%)^2 + (\pm 2.5\%)^2}$$

higher orders m_c-interpolation parametric and non-perturbative

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higher orders ~~m_c-interpolation~~ parametric and non-perturbative

Backup

Status of $B \to X_s\gamma$

Experimental:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.49 \pm 0.19)}_{\pm 5.4\%} \times 10^{-4}$$

Theoretical: [Misiak, Rehman, Steinhauser '20]

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.40 \pm 0.17)}_{\pm 5.0\%} \times 10^{-4}$$

Breakdown of the error: m_c-interpolation

$$\pm 5\% = \sqrt{(\pm 3\%)^2 + (\pm 3\%)^2 + (\pm 2.5\%)^2}$$

higher orders parametric and non-perturbative

Status of $B \to X_s\gamma$

In the future:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.49 \pm 0.09)}_{\pm 2.6\%} \times 10^{-4}$$

Theoretical:

[Misiak, Rehman, Steinhauser '20]

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{exp}}_{E_\gamma > 1.6\text{GeV}} = \underbrace{(3.40 \pm 0.13)}_{\pm 3.9\%} \times 10^{-4}$$

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higher orders

parametric and non-perturbative

Non-perturbative effects

- The matrix elements also receive non-perturbative contributions.
- The most important effects come from photons coupling to light quarks.
- Effects can be described using SCET and non-local soft matrix elements (shape functions). [Benzke, Lee, Neubert, Paz '10]
- Moments of the shape functions can be related to HQET parameters. [Gunawardana, Paz '19]

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For example:

$$\Lambda_{17} = \frac{2}{3} \mathrm{Re} \int_{-\infty}^{+\infty} \frac{\mathrm{d}w_1}{w_1} \left[1 - F\left(\frac{m_c^2}{m_b w_1} \right) + \frac{m_b w_1}{12 m_c^2} \right] h_{17}(\omega_1) \; , \qquad \int_{-\infty}^{\infty} \mathrm{d}\omega_1 \, h_{17} = \frac{2}{3} \mu_G^2 \; , \; \ldots$$

Non-perturbative effects

- Some non-perturbative effects can be estimated by data driven approaches, e.g. the $Q_7 - Q_8$ interference:

$$\Gamma[B^- \to X_s\gamma] \sim A + B\,Q_u + C\,Q_d + DQ_s, \quad \Gamma[\bar{B}^0 \to X_s\gamma] \sim A + B\,Q_d + C\,Q_u + DQ_s$$

- Isospin averaged: $\Gamma \sim A + \frac{1}{2}(B + C)(Q_u + Q_d) + DQ_s = A + \delta\Gamma_{78}$
- Isospin asymmetry: $\Delta_{0-} \sim \frac{C-B}{2\Gamma}(Q_u - Q_d)$

$$\boxed{\frac{\delta\Gamma_{78}}{\Gamma} \sim \frac{Q_u + Q_d}{Q_d - Q_u}\left[1 + \underbrace{\pm 0.3}_{SU_F(3)\text{ breaking}}\right]\Delta_{0-}}$$

- Belle [arXiv:1807.04236] : $\Gamma_{0-} = \left(-0.48 \pm 1.49 \pm 0.97 \pm 1.15\right)\%$

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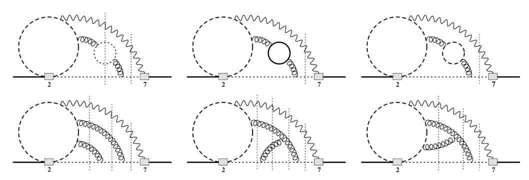
- Belle II expects a factor of 4 improvement.

Towards the $b \to s\gamma$ at NNLO without interpolation



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1. Generate all diagrams and express the amplitudes in terms of four-loop two-scale scalar integrals with unitarity cuts.

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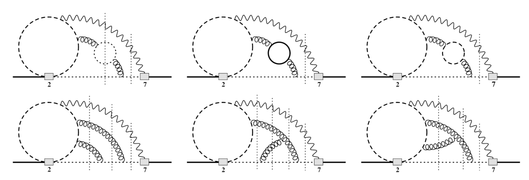

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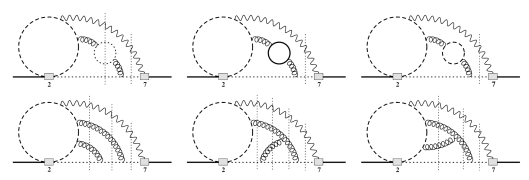

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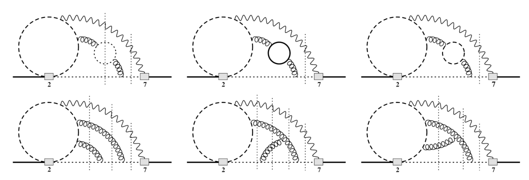

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4. Solve the master integrals numerically with boundary values obtained for $z \to \infty$.