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Revealing Hidden Regions and Forward Scattering

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Gardi, Herzog, Ma [To Appear]

Gardi, Herzog, Ma, Schlenk [2211.14845]

Heinrich, Jahn, Kerner, Langer, Magerya,
Olsson, Põldaru, Schlenk, Villa
[2108.10807, 2305.19768]

Outline

Introduction

Feynman & Lee-Pomeransky representation

Method of Regions (MoR)

Hidden regions due to cancellation

Integrals with Pinch Singularities

Finding integrals with pinch singularities for *generic* kinematics

Evaluating such integrals in parameter space

MoR and Hidden Regions due to Cancellation

On-Shell & Forward Scattering

Introduction

Parameter Space

Can exchange integrals over loop momenta for integrals over parameters

Feynman Parametrisation

$$[\mathrm{d}\boldsymbol{\alpha}] = \prod_{e\in G} \frac{\mathrm{d}\alpha_e}{\alpha_e} \qquad \boldsymbol{\alpha}^{\nu} = \prod_{e\in G} \alpha_e^{\nu_e}$$

$$I(\boldsymbol{s}) = \frac{\Gamma(\nu - LD/2)}{\prod_{e\in G}\Gamma(\nu_e)} \int_0^\infty [d\boldsymbol{\alpha}]\,\boldsymbol{\alpha}^{\nu}\delta\left(1 - H(\boldsymbol{\alpha})\right) \frac{\left[\mathscr{U}(\boldsymbol{\alpha})\right]^{\nu-(L+1)D/2}}{\left[\mathscr{F}(\boldsymbol{\alpha};\boldsymbol{s})\right]^{\nu-LD/2}}$$

$\mathscr{U}, \mathscr{F}$ homogeneous polynomials of degree L and $L+1$

Lee-Pomeransky Parametrisation

$$I(\boldsymbol{s}) = \frac{\Gamma(D/2)}{\Gamma\left((L+1)\,D/2 - \nu\right)\prod_{e\in G}\Gamma(\nu_e)} \int_0^\infty [d\boldsymbol{x}]\,\boldsymbol{x}^{\nu}\left(\mathscr{G}(\mathbf{x},\boldsymbol{s})\right)^{-D/2}$$

$$\mathscr{G}(\mathbf{x};\boldsymbol{s}) = \mathscr{U}(\mathbf{x}) + \mathscr{F}(\mathbf{x};\boldsymbol{s})$$

Lee, Pomeransky 13

Method of Regions

Consider expanding an integral about some limit:
$p_i^2 \sim \lambda Q^2$, $p_i \cdot p_j \to \lambda Q^2$ or $m^2 \sim \lambda Q^2$ for $\lambda \to 0$

Issue: integration and series expansion do not necessarily commute

Method of Regions

$$I(\mathbf{s}) = \sum_R I^{(R)}(\mathbf{s}) = \sum_R T_{\mathbf{t}}^{(R)} I(\mathbf{s})$$

1. Split integrand up into regions (R)
2. Series expand each region in λ
3. Integrate each expansion over the whole integration domain
4. Discard scaleless integrals (= 0 in dimensional regularisation)
5. Sum over all regions

Smirnov 91; Beneke, Smirnov 97; Smirnov, Rakhmetov 99; Pak, Smirnov 11; Jantzen 2011; …

Finding Regions

Assuming all c_i have the same sign we rescale $\boldsymbol{s} \to \lambda^{\boldsymbol{\omega}} \boldsymbol{s}$ $s_i \to \lambda^{\omega_i} s_i$

Newton Polytope

$$I \sim \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t \to \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i} \lambda^{r_{i,N+1}}\right)^t \to \mathcal{N}^{N+1}$$

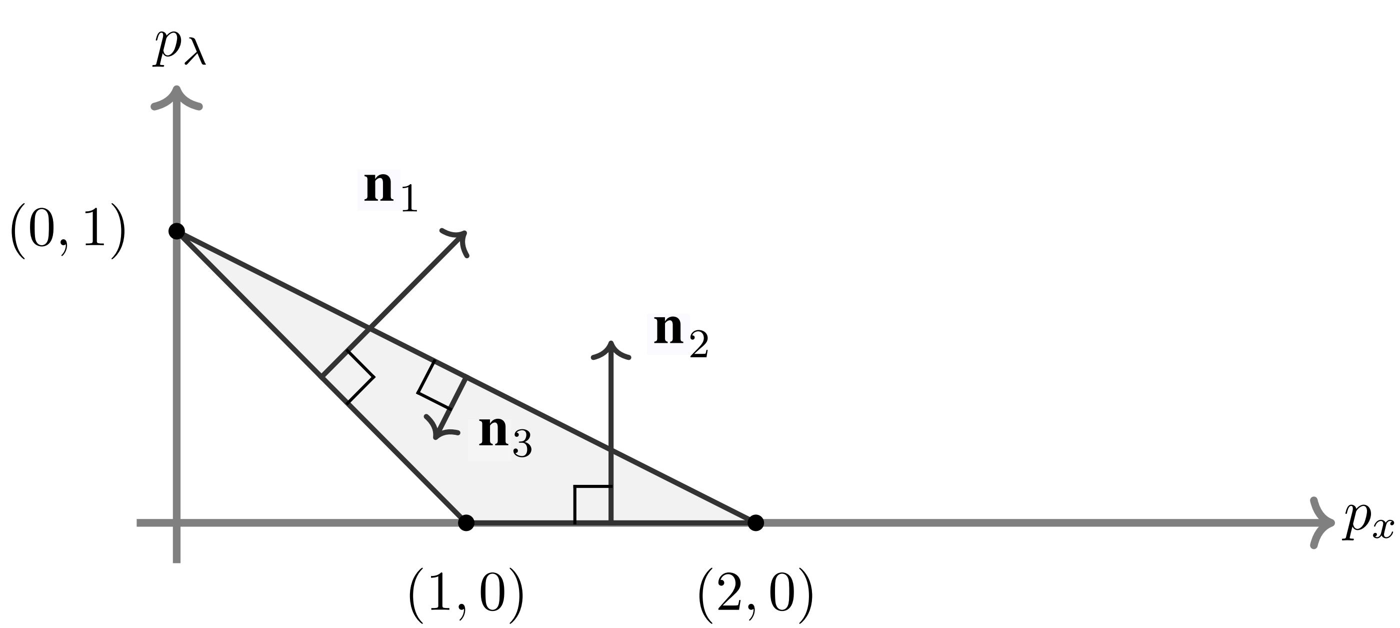
Normal vectors w/ positive λ component define change of variables $\mathbf{n}_f = (v_1, \ldots, v_N, 1)$

$$\boldsymbol{x} = \lambda^{\mathbf{n}_f} \mathbf{y}\,, \qquad \lambda \to \lambda$$

Pak, Smirnov 10; Semenova, A. Smirnov, V. Smirnov 18

Example

$p(x, \lambda) = \lambda + x + x^2$



$1{,}2 \in F^+$

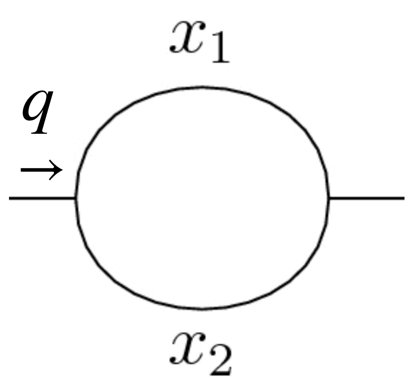
$3 \notin F^+$

Original integral I may then be approximated as $I = \sum_{f \in F^+} I^{(f)} + \ldots$

Regions due to Cancellation

What happens if c_i have different signs?

Consider a 1-loop massive bubble at *threshold* $y = m^2 - q^2/4 \to 0$



$$I = \Gamma(\epsilon)\int d\alpha_1 d\alpha_2 \frac{\delta(1-\alpha_1-\alpha_2)(\alpha_1+\alpha_2)^{-2+2\epsilon}}{\left(\mathscr{F}_{\text{bub}}(\alpha_1,\alpha_2;q^2,y)\right)^{\epsilon}}$$

$$\mathscr{F}_{\text{bub}} = \frac{q^2}{4}(\alpha_1 - \alpha_2)^2 + y(\alpha_1+\alpha_2)^2$$

Can split integral into two subdomains $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$ then remap

$$\begin{aligned} \alpha_1 &= \alpha_1'/2 \\ \alpha_2 &= \alpha_2' + \alpha_1'/2 \end{aligned} \;:\quad \mathscr{F}_{\text{bub},1} \to \frac{q^2}{4}\alpha'^2_2 + y(\alpha'_1+\alpha'_2)^2 \quad \text{(for first domain)}$$

Jantzen, A. Smirnov, V. Smirnov 12

Before split: only **hard** region found ($\alpha_1 \sim y^0, \alpha_2 \sim y^0$)

After split: also **potential** region found ($\alpha_1 \sim y^0, \alpha_2 \sim y^{1/2}$)

Regions due to Cancellation

Various tools attempt to find such re-mappings using **linear** changes of variables

ASY/FIESTA Jantzen, A. Smirnov, V. Smirnov 12

Check all pairs of variables (α_1, α_2) which are part of monomials of opposite sign

For each pair, try to build linear combination $\alpha_1 \to b\alpha_1', \alpha_2 \to \alpha_2' + b\alpha_1'$ s.t negative monomial vanishes

Repeat until all negative monomials vanish **or** warn user

ASPIRE Ananthanarayan, Pal, Ramanan, Sarkar 18; B. Ananthanarayan, Das, Sarkar 20

Consider Gröbner basis of $\left\{\mathscr{F}, \partial\mathscr{F}/\alpha_1, \partial\mathscr{F}/\alpha_2, \ldots\right\}$ (i.e. $\mathscr{F}$ and Landau equations)

Eliminate negative monomials with linear transformations $\alpha_1 \to b\alpha_1', \alpha_2 \to \alpha_2' + b\alpha_1'$

This is not enough to straightforwardly expose all regions in parameter space

Integrals with Pinch Singularities

Landau Equations

Polynomials $\mathscr{U}, \mathscr{F}$ can vanish (gives singularities) for some $\alpha_i \to 0$ (end-point)

Additionally, due to signs in $\mathscr{F}$ it can vanish due to cancellation of terms

Avoid poles on real axis by deforming contour (roughly speaking…):

$$\alpha_k \to \alpha_k - i\varepsilon_k(\boldsymbol{\alpha})$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) \to \mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) - i\sum_k \varepsilon_k \frac{\partial \mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_k} + \mathcal{O}(\varepsilon^2)$$

If $\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = 0$ and $\partial\mathscr{F}(\boldsymbol{\alpha};\mathbf{s})/\partial\alpha_j = 0\ \forall j$ simultaneously, contour will vanish exactly where the deformation is required, above conditions are just the Landau equations

Landau Equations (parameter space):

$$1)\quad \mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = 0$$

$$2)\quad \alpha_j \frac{\partial \mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_j} = 0 \quad \forall j$$

Leading: $\alpha_j \neq 0 \forall j$

Solutions are *pinched surfaces* of the integral where IR divergences may arise

Looking for Trouble: Algorithm

Generally, solutions of the Landau equations depend on $\mathbf{s}$.
Let us restrict our search to solutions with *generic* kinematics

$$\mathcal{F} = -\sum_i s_i \left[f_i(\boldsymbol{\alpha}) - g_i(\boldsymbol{\alpha})\right] = \sum_i \mathcal{F}_{i,-} + \mathcal{F}_{i,+}$$

$$\mathcal{F}_{i,-} = -\, s_i\, f_i(\boldsymbol{\alpha}), \quad \mathcal{F}_{i,+} = s_i\, g_i(\boldsymbol{\alpha}), \quad f_i(\boldsymbol{\alpha}), g_i(\boldsymbol{\alpha}) \geq 0$$

Algorithm (finds integrals which *potentially* have a pinch for massless case)

For each s_i:

1) Compute $\mathcal{F}_{i,-}$, $\mathcal{F}_{i,+}$

2) If $\mathcal{F}_{i,-} = 0$ or $\mathcal{F}_{i,+} = 0 \to$ **Exit (no cancellation)**

3) If $\partial\mathcal{F}_{i,-}/\partial\alpha_j = 0$ or $\partial\mathcal{F}_{i,+}/\partial\alpha_j = 0$ set $\alpha_j = 0 \to$ Goto 1

Else $\to$ **Exit (potential cancellation)**

Much more sophisticated algorithms for solving Landau equations exist

(E.g.) Mizera, Simon Telen 21; Fevola, Mizera, Telen 23

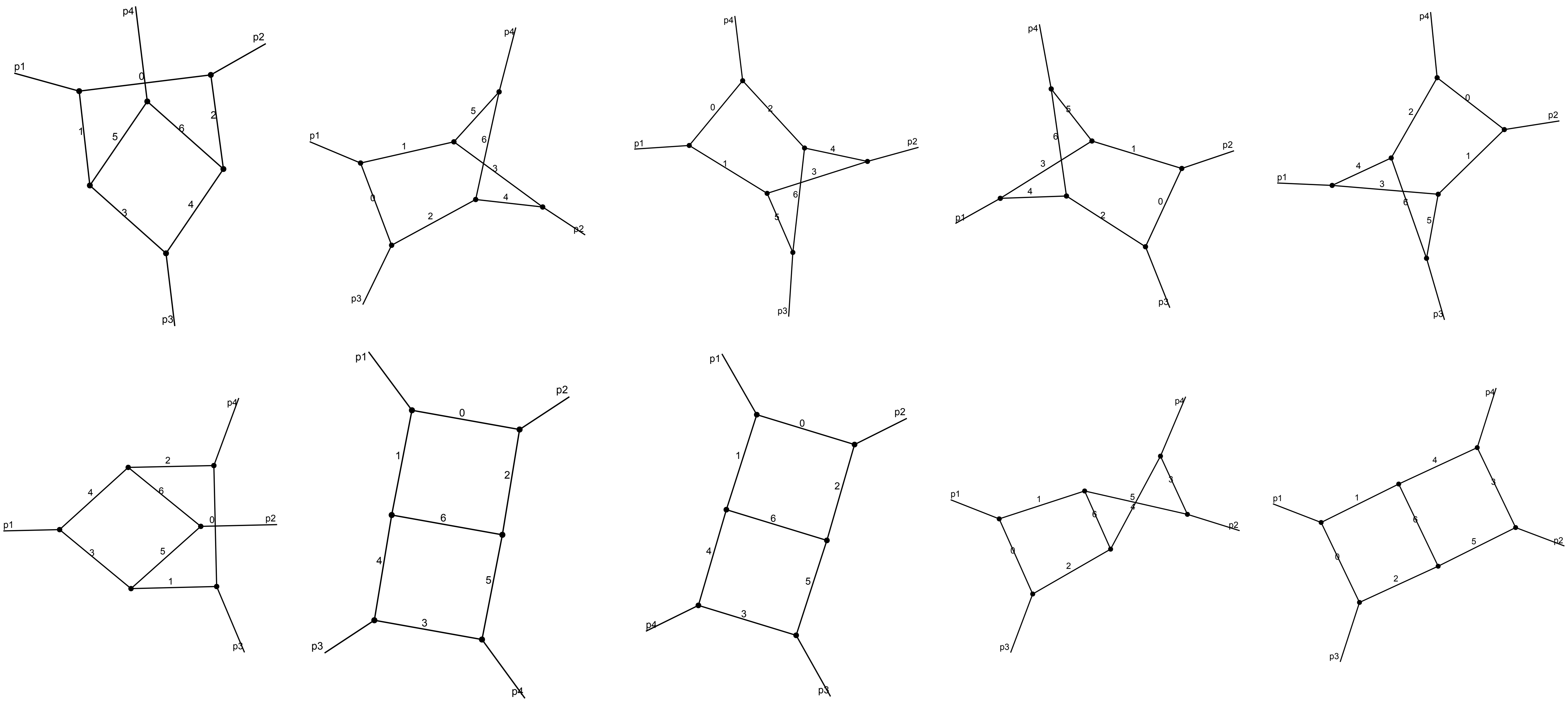
(See also) Gambuti, Kosower, Novichkov, Tancredi 23

Looking for Trouble: 1- & 2-loops

We considered massless 4-point scattering amplitudes ($s_{23} = -s_{12} - s_{13}$)

@1-loop: found no candidates (trivially)

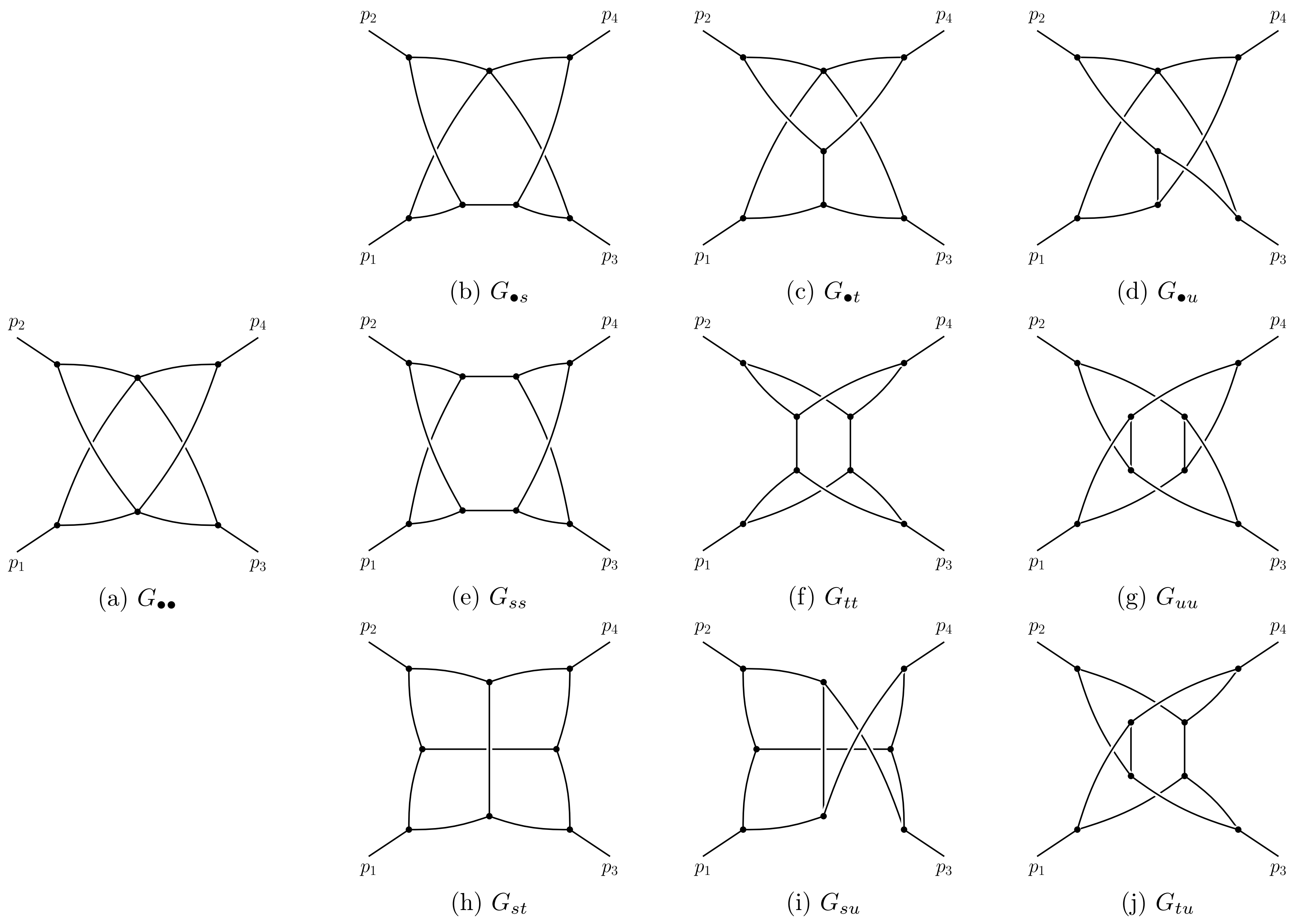
@2-loop:



+ … no candidates (!)

Looking for Trouble: 3-loops

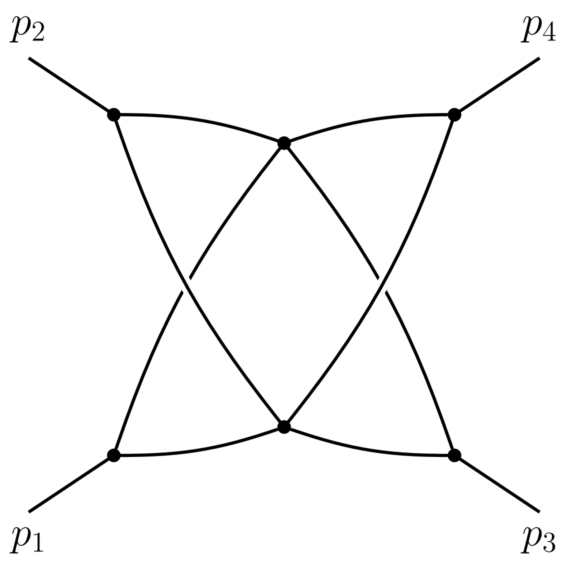
@3-loop: finally some interesting candidates



The complete set of corresponding master integrals for generic s_{12}, s_{13} are known

Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21;

Interesting Example



$$= \int_0^\infty \mathrm{d}x_0 \ \ldots \mathrm{d}x_7 \, \frac{\mathscr{U}(\mathbf{x})^{4\epsilon}}{\mathscr{F}(\mathbf{x};\mathbf{s})^{2+3\epsilon}} \delta(1-x_7)$$

$$\mathscr{U}(\alpha) = \alpha_0\alpha_2\alpha_4 + \alpha_0\alpha_2\alpha_5 + \alpha_0\alpha_2\alpha_6 + (29 \ \text{terms})$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\ (\alpha_1\alpha_4 - \alpha_0\alpha_5)\,(\alpha_3\alpha_6 - \alpha_2\alpha_7) - s_{13}\ (\alpha_1\alpha_2 - \alpha_0\alpha_3)\,(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

$$\frac{\partial \mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_0} = s_{12}\ \alpha_5(\alpha_3\alpha_6 - \alpha_2\alpha_7) + s_{13}\ \alpha_3(\alpha_5\alpha_6 - \alpha_4\alpha_7),$$

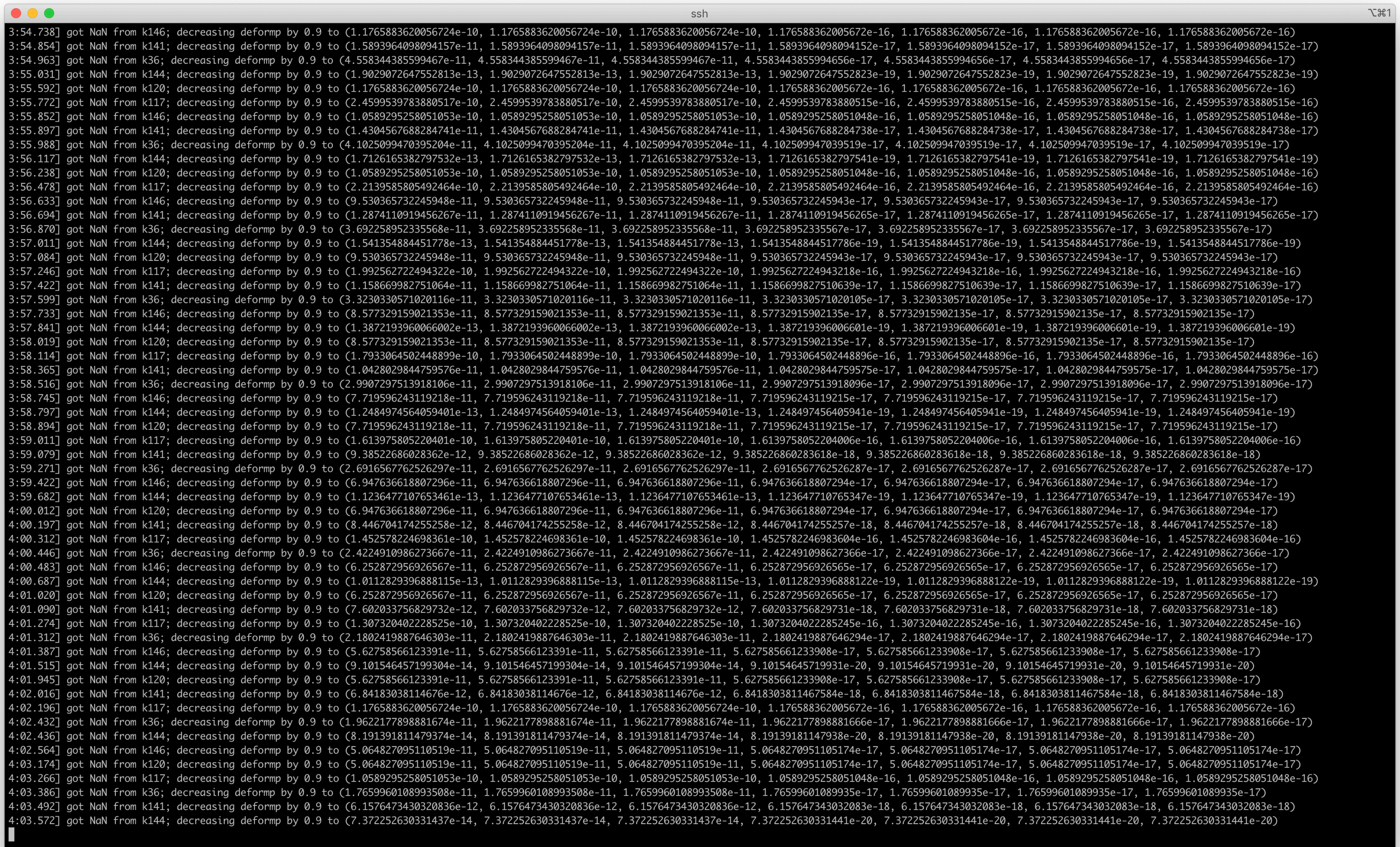
$$\vdots$$

$$\frac{\partial \mathscr{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial \alpha_7} = s_{12}\ \alpha_2(\alpha_1\alpha_4 - \alpha_0\alpha_5) + s_{13}\ \alpha_4(\alpha_1\alpha_2 - \alpha_0\alpha_3)$$

Can have a leading Landau singularity with *generic kinematics* (arbitrary s_{12}, s_{13}) when each factor of $\mathscr{F}$ vanishes!

Interesting Example

Let's try to compute this with sector decomposition (pySecDec)

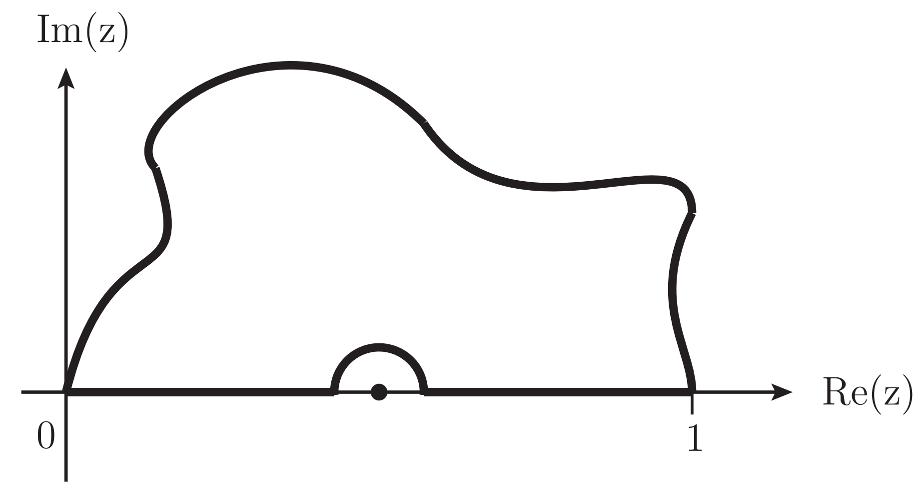


Fails to find contour…

Contour Deformation

Feynman integral (after sector decomp):

$$I \sim \int_0^1 [\mathrm{d}\boldsymbol{\alpha}]\, \boldsymbol{\alpha}^{\nu} \frac{[\mathcal{U}(\boldsymbol{\alpha})]^{N-(L+1)D/2}}{[\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})]^{N-LD/2}}$$



Deform integration contour to avoid poles on real axis

Feynman prescription $\mathcal{F} \to \mathcal{F} - i\delta$ tells us how to do this

Expand $\mathcal{F}(z = \boldsymbol{\alpha} - i\boldsymbol{\tau})$ around $\boldsymbol{\alpha}$, $\mathcal{F}(z) = \mathcal{F}(\boldsymbol{\alpha}) - i \sum_j \tau_j \frac{\partial \mathcal{F}(\boldsymbol{\alpha})}{\partial \alpha_j} + \mathcal{O}(\tau^2)$

Choose $\tau_j = \lambda_j \alpha_j (1 - \alpha_j) \frac{\partial \mathcal{F}(\boldsymbol{\alpha})}{\partial \alpha_j}$ with small constants $\lambda_j > 0$

Soper 99; Binoth, Guillet, Heinrich, Pilon, Schubert 05; Nagy, Soper 06; Anastasiou, Beerli, Daleo 07, 08; Beerli 08; Borowka, Carter, Heinrich 12; Borowka 14;…

Contour Deformation

But for this class of examples $\mathcal{F}(\boldsymbol{\alpha})$ and all $\partial\mathcal{F}(\boldsymbol{\alpha})/\partial\alpha_i$ vanish at the same point inside the integration domain

→ *pinch* singularity

Example

$$\begin{aligned}
\mathcal{F}(\boldsymbol{\alpha};\mathbf{s}) &= -s_{12}\ (\alpha_1\alpha_4-\alpha_0\alpha_5)\,(\alpha_3\alpha_6-\alpha_2\alpha_7) - s_{13}\ (\alpha_1\alpha_2-\alpha_0\alpha_3)\,(\alpha_5\alpha_6-\alpha_4\alpha_7),\\
\frac{\partial\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_0} &= s_{12}\ \alpha_5(\alpha_3\alpha_6-\alpha_2\alpha_7) + s_{13}\ \alpha_3(\alpha_5\alpha_6-\alpha_4\alpha_7),\\
&\ \vdots\\
\frac{\partial\mathcal{F}(\boldsymbol{\alpha};\mathbf{s})}{\partial\alpha_7} &= s_{12}\ \alpha_2(\alpha_1\alpha_4-\alpha_0\alpha_5) + s_{13}\ \alpha_4(\alpha_1\alpha_2-\alpha_0\alpha_3)
\end{aligned}$$

vanish for

$$\alpha_2 = \frac{\alpha_0\alpha_3}{\alpha_1}, \qquad \alpha_4 = \frac{\alpha_0\alpha_5}{\alpha_1}, \qquad \alpha_6 = \frac{\alpha_0\alpha_7}{\alpha_1}\,.$$

Resolution

The problem is that we have monomials with different signs…

Asy2.1 PreResolve->True

```
-bash                                                                ⌥⌘1
MACTHXJONES:fiesta sj$ cat diagram2636.m
Get["asy2.1.m"];

Print["Diagram2636"];
result = AlphaRepExpand[{k1,k2,k3}, {k1^2, k2^2,k3^2, (k1+p1)^2, (k2+p2)^2, (k3+p3)^2,
(k1+k2+k3)^2, (k1+k2+k3+p1+p2+p3)^2}, {p1^2->0, p2^2->0, p3^2->0, p1*p2->s12/2, p1*p3->
s13/2, p2*p3->-s12/2-s13/2}, {s12 -> 1, s13 -> -1/5}, PreResolve->True] (* 3-loop box *
)
Print[result];
Print["=========="];
Exit[];

MACTHXJONES:fiesta sj$ wolframscript -file diagram2636.m
Diagram2636
Asy2.1
Variables for UF: {k1, k2, k3, p1, p2, p3}
WARNING: preresolution failed
{}
==========
MACTHXJONES:fiesta sj$
```

Correctly identifies that iterated linear changes of variables are not sufficient to resolve the singularity and reports that pre-resolution has failed

Resolution

1) Rescale parameters to *linearise* singular surfaces

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = -s_{12}\,(\alpha_1\alpha_4 - \alpha_0\alpha_5)\,(\alpha_3\alpha_6 - \alpha_2\alpha_7) - s_{13}\,(\alpha_1\alpha_2 - \alpha_0\alpha_3)\,(\alpha_5\alpha_6 - \alpha_4\alpha_7)$$

$$\alpha_0 \to \alpha_0\alpha_1,\ \alpha_2 \to \alpha_2\alpha_3,\ \alpha_4 \to \alpha_4\alpha_5,\ \alpha_6 \to \alpha_6\alpha_7$$

$$\mathscr{F}(\boldsymbol{\alpha};\mathbf{s}) = \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_4-\alpha_0)(\alpha_6-\alpha_2) - s_{13}(\alpha_2-\alpha_0)(\alpha_6-\alpha_4)\right]$$

2) Split the integral by imposing $\alpha_i \geq \alpha_j \geq \alpha_k \geq \alpha_l$

$$\begin{aligned} \alpha_0 &\to \alpha_0+\alpha_2+\alpha_4+\alpha_6, \\ \alpha_2 &\to \alpha_2+\alpha_4+\alpha_6, \\ \alpha_4 &\to \alpha_4+\alpha_6, \\ \alpha_6 &\to \alpha_6 \end{aligned} \qquad +\text{perms}$$

$$\begin{aligned} \mathscr{F}_1(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_0+\alpha_2)(\alpha_2+\alpha_4) - s_{13}(\alpha_0)(\alpha_4)\right] \\ \mathscr{F}_2(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2)(\alpha_0+\alpha_2+\alpha_6) + s_{13}(\alpha_0)(\alpha_6)\right] \\ &\vdots \\ \mathscr{F}_{24}(\boldsymbol{\alpha};\mathbf{s}) &= \alpha_1\alpha_3\alpha_5\alpha_7\left[-s_{12}(\alpha_2+\alpha_4)(\alpha_4+\alpha_6) - s_{13}(\alpha_2)(\alpha_6)\right] \end{aligned}$$

All coefficients of s_{12}, s_{13} now have definite sign

Result

Can now obtain results numerically ($s_{12} = 1,\ s_{13} = -1/5$)

$$I_1 = \epsilon^{-4}\left[(-3.8842800687 + 5.2359902003j) \pm (4.458 \cdot 10^{-6} + 3.638 \cdot 10^{-6}j)\right] + \ldots$$
$$I_2 = \epsilon^{-4}\left[(-7.9291803033 + 20.943767810j) \pm (9.149 \cdot 10^{-5} + 1.061 \cdot 10^{-4}j)\right] + \ldots$$
$$I_3 = \epsilon^{-4}\left[(18.5195704502 - 15.707988011j) \pm (5.897 \cdot 10^{-5} + 5.897 \cdot 10^{-5}j)\right] + \ldots$$
$$I_4 = \epsilon^{-4}\left[(-13.294034089) \pm (2.068 \cdot 10^{-5})\right] + \ldots$$
$$I_5 = \epsilon^{-4}\left[(12.7432949988 - 23.561968275j) \pm (1.605 \cdot 10^{-5} + 1.415 \cdot 10^{-5}j)\right] + \ldots$$
$$I_6 = \epsilon^{-4}\left[(-4.0702330904) \pm (2.018 \cdot 10^{-6})\right] + \ldots$$

Agrees with analytic result

$$\begin{aligned} I &= 4\ (I_1 + I_2 + I_3 + I_4 + I_5 + I_6) \\ &= \epsilon^{-4}\left[8.340\mathbf{55} - 52.36\mathbf{08}j\right] + \mathcal{O}(\epsilon^{-3}) \\ I_{\text{analytic}} &= \epsilon^{-4}\left[8.3400403922 - 52.3598775598j\right] + \mathcal{O}(\epsilon^{-3}) \end{aligned}$$

Note: even after resolution this integral is slow to compute numerically, possible to vastly improve performance by avoiding contour deformation entirely

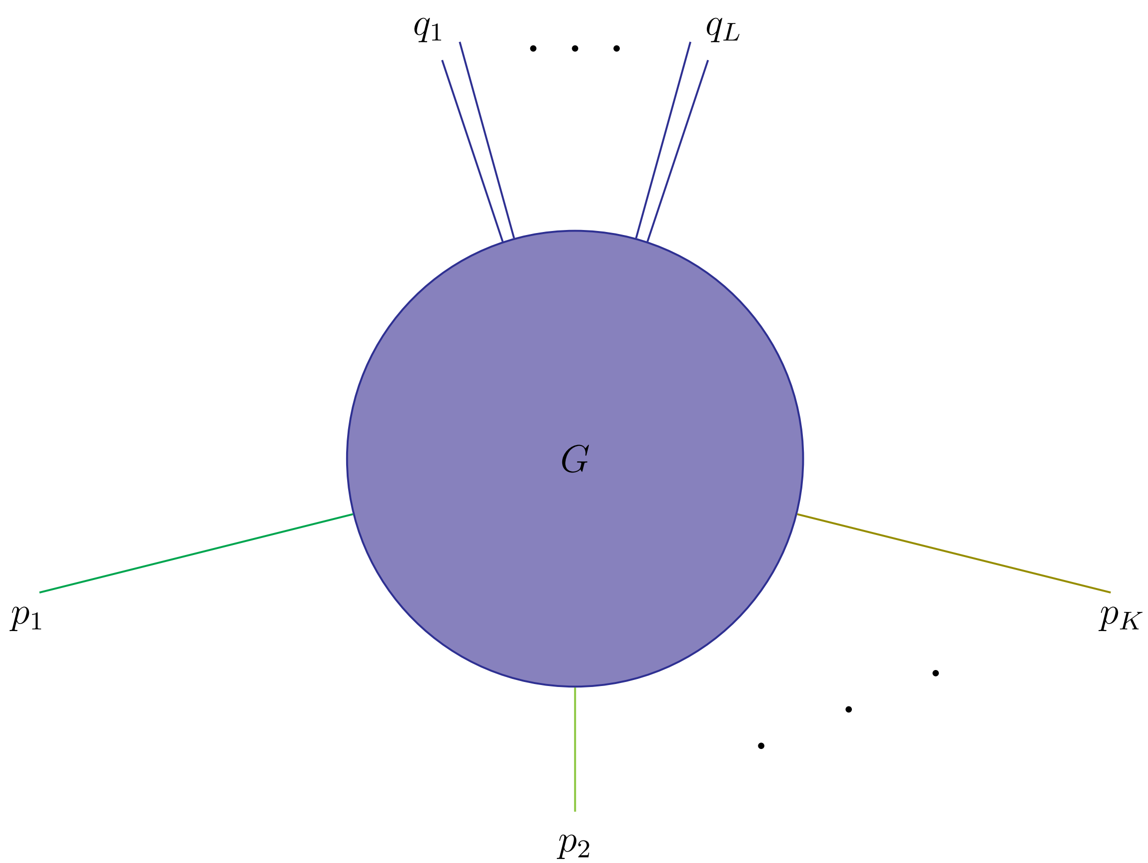
→See talk of Tom Stone

MoR and Hidden Regions

On-Shell Expansion

On-shell expansion provides a way to explore emergence of IR singularities starting from an object free of IR singularities (off-shell Green's function)

Consider an arbitrary loop, $(K + L)$-leg wide-angle scattering graph



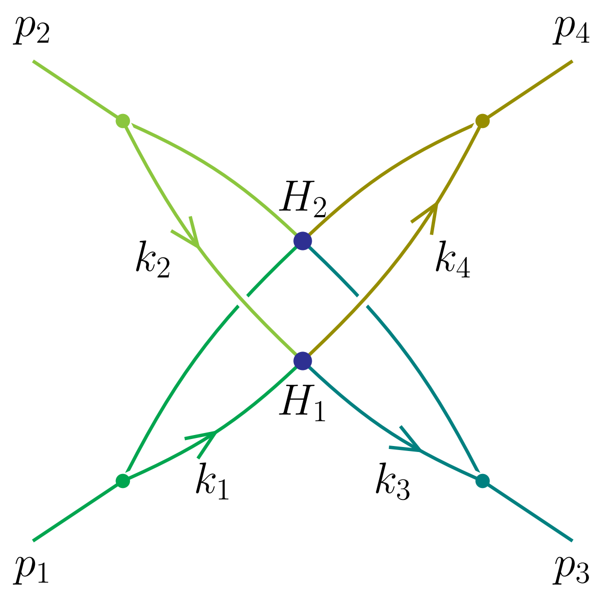
on-shell: $p_i^2 \sim \lambda Q^2 \quad (i = 1, \ldots, K)$,

off-shell: $q_j^2 \sim Q^2 \quad (j = 1, \ldots, L)$,

wide-angle: $p_k \cdot p_l \sim Q^2 \quad (k \neq l)$.

Cancellations of the type just observed lead to new regions that are *hidden* in the straightforward Newton polytope approach as they do not originate from an end-point singularity

On-Shell Expansion



Consider a collinear/jet configuration

$$p_i^2 = \lambda Q^2, \quad p_i \cdot v_i \sim \lambda Q, \quad p_i \cdot \bar{v}_i \sim Q, \quad p_i \cdot v_{i\perp} \sim \sqrt{\lambda} Q$$

Let us introduce a fourth (extra) loop momentum and consider the mode with all k_i collinear to p_i

$$k_i^\mu = Q\left(\xi_i v_i^\mu + \lambda \kappa_i \bar{v}_i^\mu + \sqrt{\lambda}\tau_i u_i^\mu + \sqrt{\lambda}\nu_i n^\mu\right)$$

Botts, Sterman 89

Momentum conservation at H_1 vertex ($k_1 + k_2 = k_3 + k_4$) implies not all ξ_i are independent:

$$\xi_2 = \xi_1 - \frac{1}{2}\sqrt{\lambda}\cos^2(\theta)\left(\tan\left(\frac{\theta}{2}\right)\Delta\tau - \cot\left(\frac{\theta}{2}\right)\Sigma\tau\right) + \lambda(\kappa_2 - \kappa_1),$$

$$\xi_3 = \xi_1 + \frac{1}{2}\sqrt{\lambda}\tan\left(\frac{\theta}{2}\right)\Delta\tau + \lambda(\kappa_2 - \kappa_4),$$

$$\xi_4 = \xi_1 - \frac{1}{2}\sqrt{\lambda}\cot\left(\frac{\theta}{2}\right)\Sigma\tau + \lambda(\kappa_2 - \kappa_3)\,.$$

$$\Delta\tau \equiv \tau_1 + \tau_2 - \tau_3 - \tau_4$$
$$\Sigma\tau = \tau_1 + \tau_2 + \tau_3 + \tau_4$$

On-Shell Expansion

Now let us analyse the leading behaviour of this integrand for small λ,

1) Loop measure can be expressed as $\int d^Dk_1 d^Dk_2 d^Dk_3 = Q^{3D} \int \prod_{i=1}^{3} d\xi_i d\kappa_i d\tau_i d\nu_i$
2) Trade large components of k_2, k_3 for small components of k_4, $\{\xi_2, \xi_3\} \to \{\kappa_4, \tau_4\}$

   Jacobian of transformation: $\det\left(\frac{\partial(\xi_2, \xi_3)}{\partial(\kappa_4, \tau_4)}\right) = \lambda^{3/2}\cos(\theta)\cot(\theta)$

Overall obtain the following scaling:

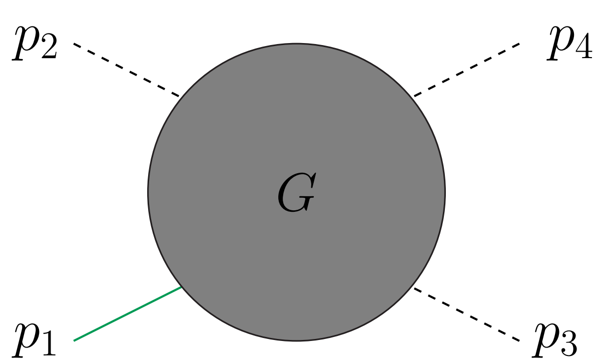
$$\int \prod_{i=1}^{3} d\xi_i \; d\kappa_i d\tau_i d\nu_i \sim \int_0^1 d\xi_1 \underbrace{\left(\int \prod_{i=1}^{3} (\lambda d\kappa_i)(\lambda^{\frac{1}{2}} d\tau_i)(\lambda^{\frac{1}{2}} d\nu_i)^{1-2\epsilon}\right)}_{\lambda^{6-3\epsilon}} \int d\kappa_4 d\tau_4 \; \underbrace{\det\left(\frac{\partial(\xi_2, \xi_3)}{\partial(\kappa_4, \tau_4)}\right)}_{\lambda^{3/2}}$$

Expect this region to scale as $\boxed{\mu = 6 - 3\epsilon + \frac{3}{2} - 8 = -\frac{1}{2} - 3\epsilon}$

Scaling of collinear propagators

On-Shell Expansion

Directly applying MoR in parameter space, we do not see this region…



$I \sim$

$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	**order**
$(-2,-1,-2,-1,-2,-1,-2,-1;1)$	-6ϵ
$(-1,-2,-1,-2,-1,-2,-1,-2;1)$	-6ϵ
$(-1,-1,-1,0,-1,0,-1,0;1)$	$1-3\epsilon$
$(-1,-1,0,-1,0,-1,0,-1;1)$	$1-3\epsilon$
$(-1,-1,0,0,0,0,0,0;1)$	$-\epsilon$
$(0,0,0,0,0,0,0,0;1)$	0

Dissecting the polytope according to our resolution procedure eliminates monomials of different sign, we now see the region in each of the 24 new polytopes

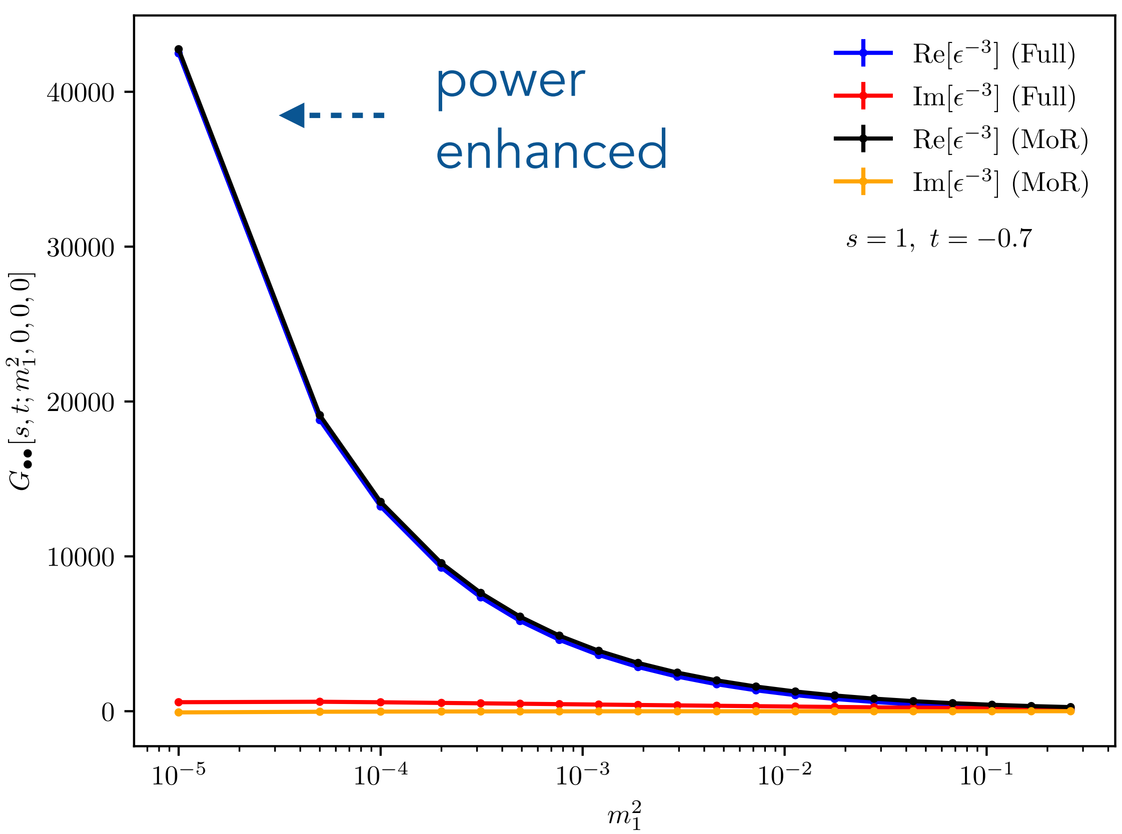
$I_1 \sim$

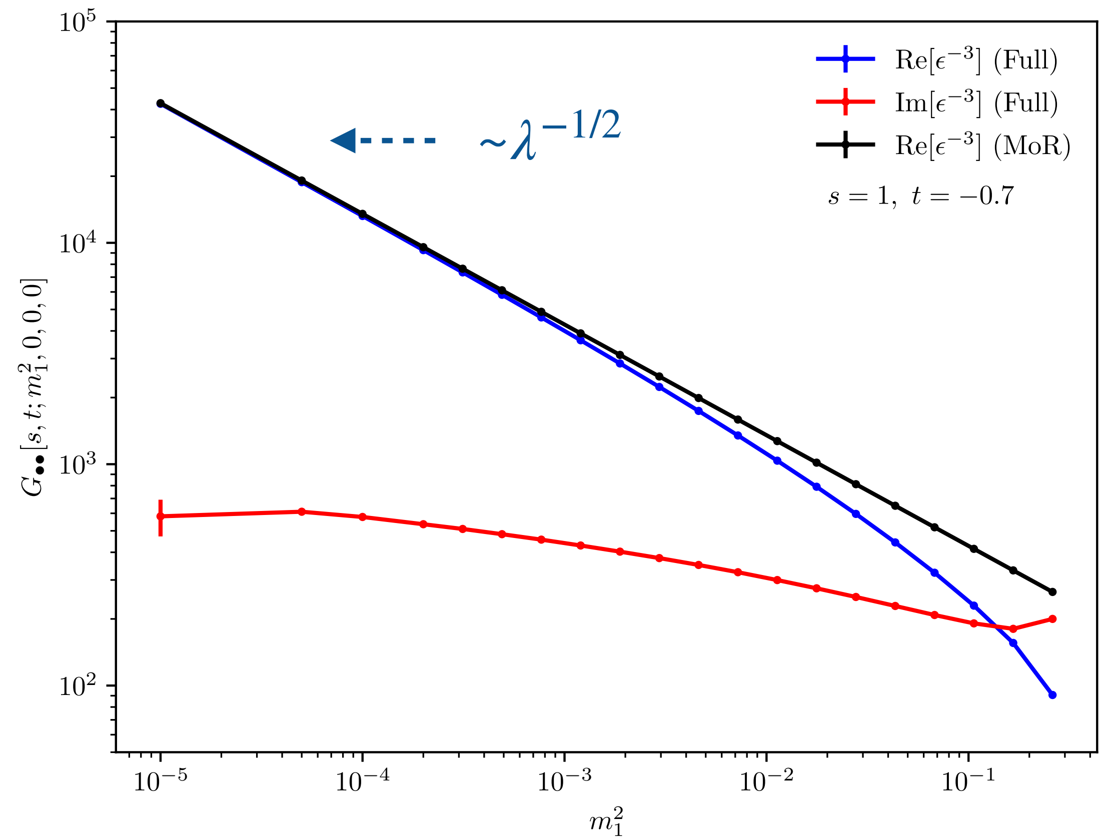
$\boldsymbol{v}_{\mathrm{R}}$ $(y_0, x_1, y_2, x_3, y_4, x_5, y_6, x_7)$	$\boldsymbol{v}_{\mathrm{R}}$ $(x_0, x_1, \ldots, x_7)$	**order**
$(1/2,-1,1/2,-1,1/2,-1,0,-1;1)$	$(-2,-2,-2,-2,-2,-2,-2,-2;2)$	$-1/2-3\epsilon$
$(0,-1,1,-1,1,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	-3ϵ
$(1,-1,1,-1,0,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	-3ϵ
$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$(-2,-1,-2,-1,-2,-1,-2,-1;1)$	-6ϵ
$(1,-2,1,-2,1,-2,1,-2;1)$	$(-1,-2,-1,-2,-1,-2,-1,-2;1)$	-6ϵ
$(0,-1,0,0,0,0,0,0;1)$	$(-1,-1,0,0,0,0,0,0;1)$	$-\epsilon$
$(0,0,0,0,0,0,0,0;1)$	$(0,0,0,0,0,0,0,0;1)$	0

$\mu = -\dfrac{1}{2} - 3\epsilon$

On-Shell Expansion

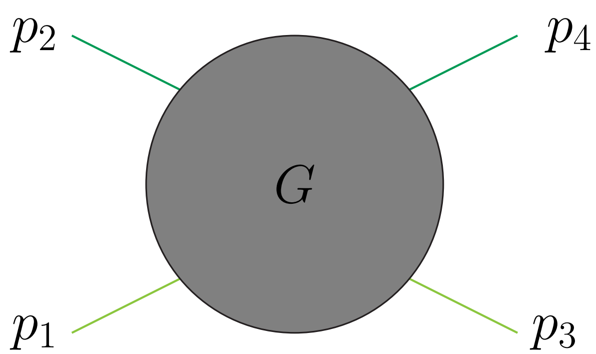
Use MoR on each of the split integrals $I_1, \ldots, I_{24}$ and summing only the leading region for each split (with $\mu = -1/2 - 3\epsilon$)





See strong numerical evidence that the split integrals (MoR) reproduce the leading behaviour of the full integral in the limit $p_1^2 \to 0$

Forward Scattering



Inserting $\theta \sim \sqrt{\lambda}$ into the Botts-Sterman analysis leads to one of the loop momenta becoming Glauber:

$$k_4^\mu - k_2^\mu = k_1^\mu - k_3^\mu \sim Q(\lambda, \lambda; \sqrt{\lambda})$$

We obtain $\mu = -1 - 3\epsilon$

Alternatively, can expand known analytic result in the foward limit $x = -s_{13}/s_{12}$

Henn, Mistlberger, Smirnov, Wasser 20; Bargiela, Caola, von Manteuffel, Tancredi 21;

$$I(s_{12}, s_{13}; \epsilon) = s_{12}^{-2-3\epsilon}\mathcal{J}(x;\epsilon), \quad \mathcal{J}(x;\epsilon) \sum_{n=-4}^{\infty} \mathcal{J}^{(n)}(x)\,\epsilon^n = \sum_{n=-4}^{\infty}\sum_{k=-1}^{\infty} \mathcal{J}^{(n,k)}(L)\,x^k\,\epsilon^n$$

$L = \log(x)$

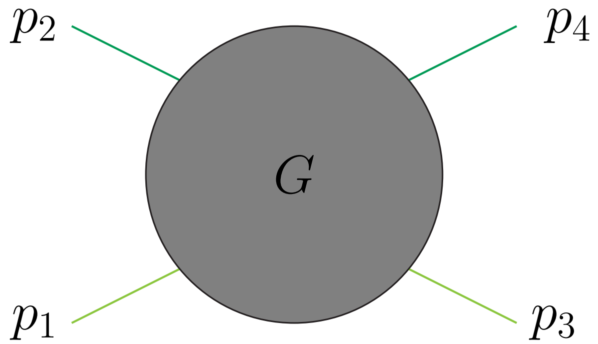
$$\mathcal{J}(x;\epsilon) = \mathrm{LP}\left\{I_{\mathrm{XX}}\right\}(L;\epsilon) + \mathcal{O}(x^0)$$

$$\mathrm{LP}\left\{\mathcal{J}\right\}(L;\epsilon) = i\pi x^{-1-3\epsilon}\left(-\frac{8}{3\epsilon^4} + \frac{16}{\epsilon^3} + \frac{2\left(\pi^2 - 144\right)}{3\epsilon^2} - \frac{4\left(-58\zeta(3) + 3\pi^2 - 432\right)}{3\epsilon}\right.$$
$$\left. + \frac{1}{60}\left(-27840\zeta(3) + 71\pi^4 + 1440\pi^2 - 207360\right) + \cdots\right),$$

gives $\mathcal{J}(x;\epsilon) \sim x^{-1-3\epsilon}$

Forward Scattering

Directly applying MoR in parameter space, no region with correct scaling…



$I \sim$

$\boldsymbol{v}_{\mathrm{R}}\ (x_0, x_1, \ldots, x_7)$	**order**
$(-1,-1,-1,0,-1,-1,-1,0;1))$	-3ϵ
$(-1,-1,0,-1,-1,-1,0,-1;1)$	-3ϵ
$(-1,0,-1,-1,-1,0,-1,-1;1)$	-3ϵ
$(0,-1,-1,-1,0,-1,-1,-1;1)$	-3ϵ
$(0,0,0,0,0,0,0,0;1)$	0

After resolution, in some polytopes we now directly see the leading region observed in the analytic result!

$I_1 \sim$

$\boldsymbol{v}_{\mathrm{R}}\ (y_0, x_1, y_2, x_3, y_4, x_5, y_6, x_7)$	$\boldsymbol{v}_{\mathrm{R}}\ (x_0, x_1, \ldots, x_7)$	**order**
$(0,-1,0,-1,0,-1,1,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$-1-3\epsilon$
$(1,-1,0,-1,0,-1,0,-1;1)$	$(-1,-1,-1,-1,-1,-1,-1,-1;1)$	$-1-3\epsilon$
$(-1,0,0,-1,-1,0,0,-1;1)$	$(-1,0,-1,-1,-1,0,-1,-1;1)$	-3ϵ
$(0,0,0,0,0,0,0,0;1)$	$(0,0,0,0,0,0,0,0;1)$	0

Conclusion

Pinched Feynman Integrals

- Studied an integral with a *pinched* contour independent of kinematics
- Found a resolution procedure to remove the pinch
- Can obtain stable numerical results only after removing pinch

MoR

- Expect regions can appear due to cancelling monomials either generically or at particular kinematic points
- Have characterised some such regions for on-shell expansion and forward scattering @ 3-loops

Outlook

- General/automated procedure to resolve these pinches/cancellations?
- New ways to analyse/compute Feynman integrals?

Thank you for listening!

Backup

Sector Decomposition

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^{N+1}_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \frac{[\mathscr{U}(\boldsymbol{x})]^{N-(L+1)D/2}}{[\mathscr{F}(\boldsymbol{x},\mathbf{s}) - i\delta]^{N-LD/2}} \delta(1 - H(\boldsymbol{x}))$$

Singularities

1. UV/IR singularities when some $x \to 0$ simultaneously $\Longrightarrow$ Sector Decomposition
2. Thresholds when $\mathscr{F}$ vanishes inside integration region $\Longrightarrow$ $i\delta$

Sector decomposition

Find a local change of coordinates for each singularity that factorises it (blow-up)

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t$$

$$\mathcal{N}(I) = \mathrm{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

Normal vectors incident to each extremal vertex define a local change of variables*

Kaneko, Ueda 10

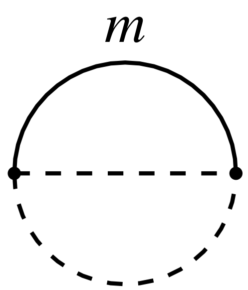
$$x_i = \prod_{f \in S_j} y_f^{\langle \mathbf{n}_f, \mathbf{e}_i \rangle}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_0^1 \left[\mathrm{d}\mathbf{y}_f\right] \underbrace{\prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle - t a_f}}_{\text{Singularities}} \left(\underbrace{c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f}}_{\text{Finite}} \right)^t$$

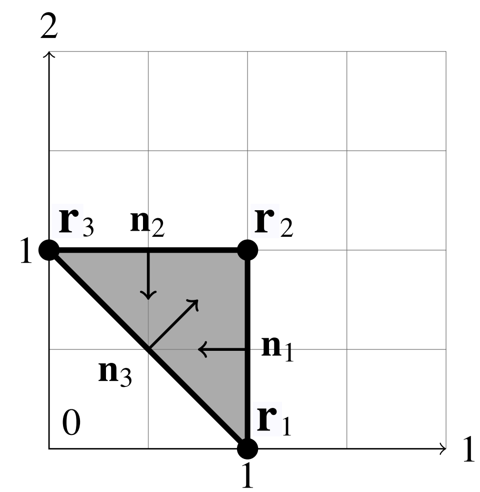
Singularities **Finite**

*If $|S_j| > N$, need triangulation to define variables (simplicial normal cones $\sigma \in \Delta^T_{\mathcal{N}}$)

Sector Decomposition in a Nutshell

$I =$  $= -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon} \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{\left(x_1^1 x_2^0 + x_1^1 x_2^1 + x_1^0 x_2^1\right)^{2-\varepsilon}}.$

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\mathcal{N}(I) =$ 

$$= \quad \begin{array}{lll} \mathbf{n}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \mathbf{n}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \mathbf{n}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ a_1 = \quad 1 & a_2 = \quad 1 & a_3 = \quad -1 \end{array}$$

For each vertex make the local change of variables

e.g. $\mathbf{r}_1$: $x_1 = y_1^{-1} y_3^1,\ x_2 = y_1^0 y_3^1$, $\mathbf{r}_2 : x_1 = y_1^{-1} y_2^0,\ x_2 = y_1^0 y_2^{-1}$, $\mathbf{r}_3 : x_1 = y_2^0 y_3^1, x_2 = y_2^{-1} y_3^1$

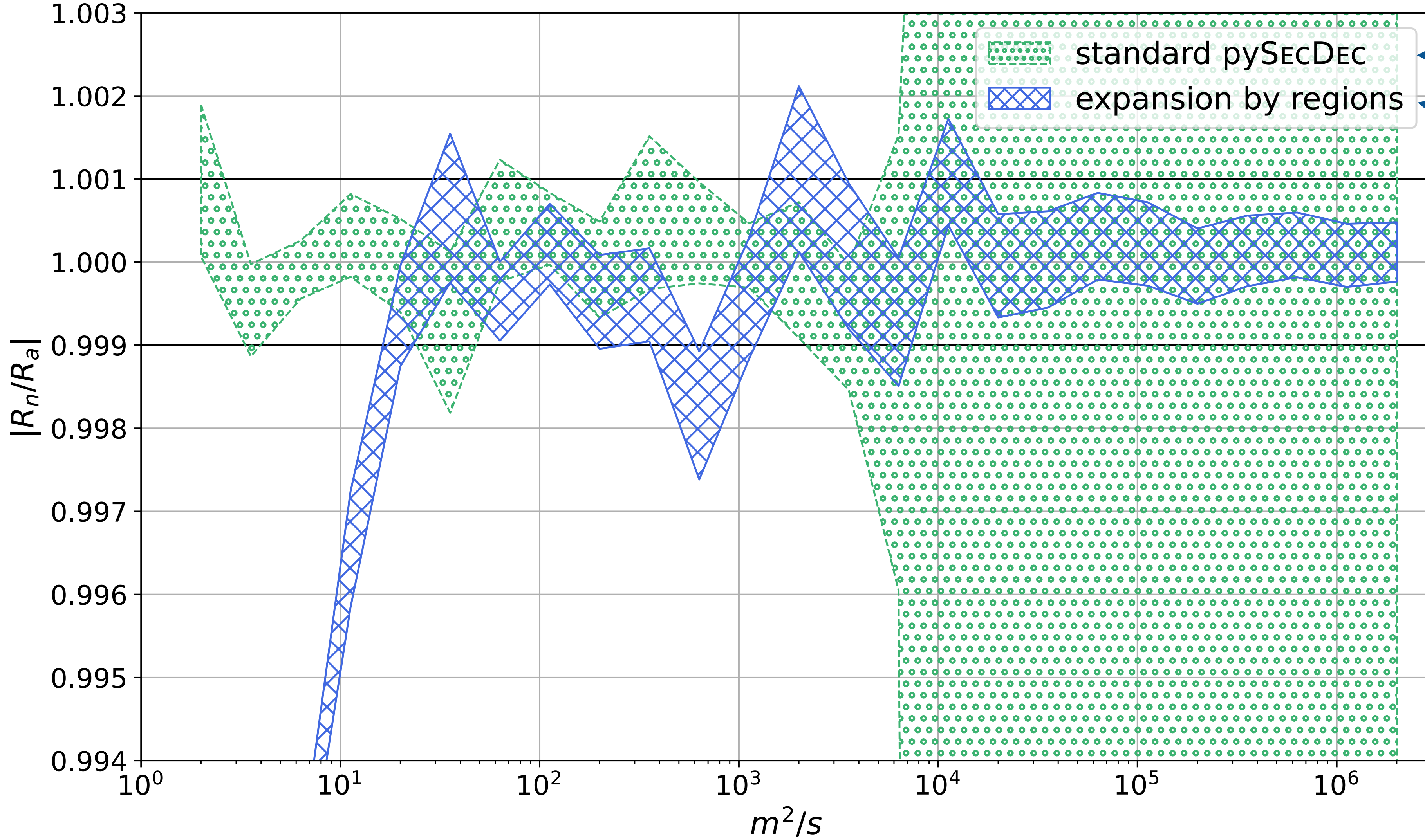
$$I = -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon} \int_0^1 \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \frac{y_1^{-\varepsilon} y_2^{-\varepsilon} y_3^{-1+\varepsilon}}{(y_1+y_2+y_3)^{2-\varepsilon}} \left[\delta(1-y_2)+\delta(1-y_3)+\delta(1-y_1)\right]$$

Schlenk 2016

Applications

Applying Expansion by Regions

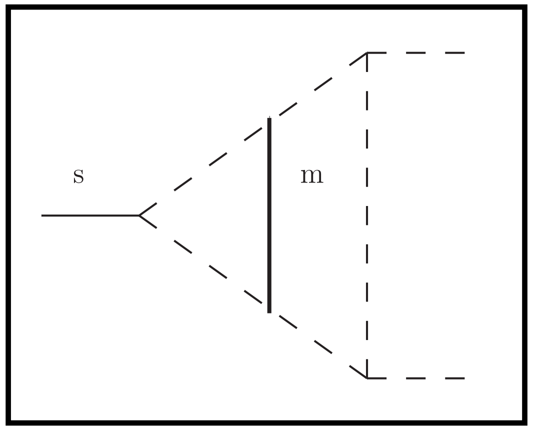
Ratio of the finite $\mathcal{O}(\epsilon^0)$ piece of numerical result R_n to the analytic result R_a



(at large m^2/s)

~1 day

~2 min



For large ratio of scales (m^2/s) the EBR result is **faster** & **easier** to integrate

Additional Regulators

MoR subdivides $\mathcal{N}(I) \to \{\mathcal{N}(I^R)\} \implies$ new (internal) facets $F^{\text{int.}}$

New facets can introduce spurious singularities not regulated by dim reg

Lee Pomeransky Representation:

$$\mathcal{N}(I^{(R)}) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_{\mathbb{R}^N_{>0}} \left[\mathrm{d}\mathbf{y}_f \right] \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle + \frac{D}{2} a_f} \left(c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f} \right)^{-\frac{D}{2}}$$

If $f \in F^{\text{int}}$ have $a_f = 0$ need analytic regulators $\boldsymbol{\nu} \to \boldsymbol{\nu} + \boldsymbol{\delta\nu}$

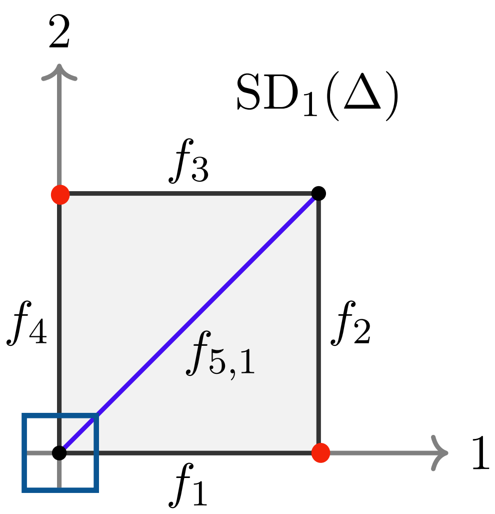
Heinrich, Jahn, SJ, Kerner, Langer, Magerya, Põldaru, Schlenk, Villa 21; Schlenk 16

Additional Regulators (II)

Toy Example:

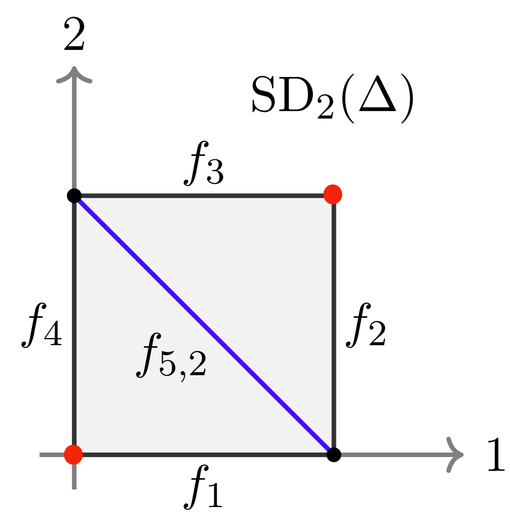
$$P_1(x,\lambda) = 1 + \lambda x_1 + x_1 x_2 + \lambda x_2$$

need analytic regulator



$$P_2(x,\lambda) = \lambda + x_1 + \lambda x_1 x_2 + x_2$$

ok!



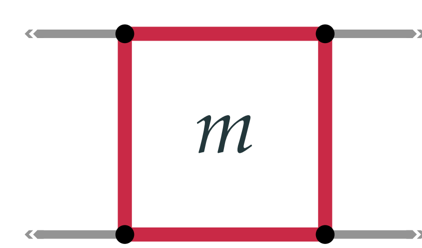
pySecDec can find the constraints on the analytic regulators for you

`extra_regulator_constraints()`:

$$\nu_2 - \nu_4 \neq 0, \quad \nu_1 - \nu_3 \neq 0$$

`suggested_extra_regulator_exponent()`:

$$\{\delta\nu_1, \delta\nu_2, \delta\nu_3, \delta\nu_4\} = \{0, 0, \eta, -\eta\}$$



Small m expansion

Lee-Pomeransky and MoR

Building Bridges: LP $\leftrightarrow$ Propagator Scaling

Region vectors in momentum space and Lee-Pomeransky space are related, we can see this using Schwinger parameters $\tilde{x}_e$

$$\frac{1}{D_n^{\nu_e}} = \frac{1}{\Gamma(\nu_e)} \int_0^\infty \frac{\mathrm{d}\tilde{x}_e}{\tilde{x}_e}\, \tilde{x}_e^{\nu_e}\, e^{-\tilde{x}_e D_e} \text{, with } x_e \propto \tilde{x}_e$$

$$(D_1^{-1}, \ldots, D_N^{-1}) \sim (\tilde{x}_1, \ldots, \tilde{x}_N) \sim (x_1, \ldots, x_N)$$

Example: 1-loop form factor

$$\begin{aligned}
\text{Hard}: &\quad (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^0, \lambda^0), &\quad (x_1, x_2, x_3) \sim (\lambda^0, \lambda^0, \lambda^0)\\
\text{Collinear to } \mathrm{p}_1: &\quad (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1}), &\quad (x_1, x_2, x_3) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1})\\
\text{Collinear to } \mathrm{p}_2: &\quad (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1}), &\quad (x_1, x_2, x_3) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1})\\
\text{Soft}: &\quad (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2}), &\quad (x_1, x_2, x_3) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2})
\end{aligned}$$

Can connect the regions in mom. space with those we determine geometrically

Next step: automatically find (Sudakov decomposed) loop momentum scalings compatible with region vectors WIP w/ Yannick Ulrich

Building Bridges: Landau ↔ Regions

The **Landau equations** give the necessary conditions for an integral to diverge

$$1)\quad \alpha_e l_e^2(k,p,q) = 0 \qquad \forall e \in G$$

$$2)\quad \frac{\partial}{\partial k_a^\mu}\mathscr{D}(k,p,q;\alpha) = \frac{\partial}{\partial k_a^\mu}\sum_{e\in G}\alpha_e\left(-l_e^2(k,p,q) - i\varepsilon\right) = 0 \qquad \forall a \in \{1,\ldots,L\}$$

Solutions are *pinched surfaces* of the integral where IR divergences may arise

Idea is to explore the *neighbourhood of a pinched surface,* defined by

$$1)\quad \alpha_e l_e^2(k,p,q) \sim \lambda^p \qquad \forall e \in G, \quad \text{with} \quad p \in \{1,2\}$$

$$2)\quad \frac{\partial}{\partial k_a^\mu}\mathscr{D}(k,p,q;\alpha) \lesssim \lambda^{1/2} \qquad \forall a \in \{1,\ldots,L\}$$

with the goal of further understanding the connection between

Solutions of the Landau equations ↔ Regions

Gardi, Herzog, Ma, Schlenk 22

Method of Regions (Details/Examples)

Geometric Method

In Feynman parameter space, there is a **geometric method** for finding regions

Pak, Smirnov 10

Each region will be defined by a **region vector** $\mathbf{v} = (v_1, \ldots, v_N; 1)$, in each region we will perform a change of variables $x_i \to \lambda^{v_i} x_i$ and series expand about $\lambda = 0$

Let us start by considering some polynomial

$$P(\mathbf{x}, \lambda) = \sum_{i=1}^{m} c_i \, x_1^{r_{i,1}} \cdots x_N^{r_{i,N}} \, \lambda^{r_{i,N+1}}$$

c_i - non-negative coefficients

x_i - integration variables

λ - small parameter

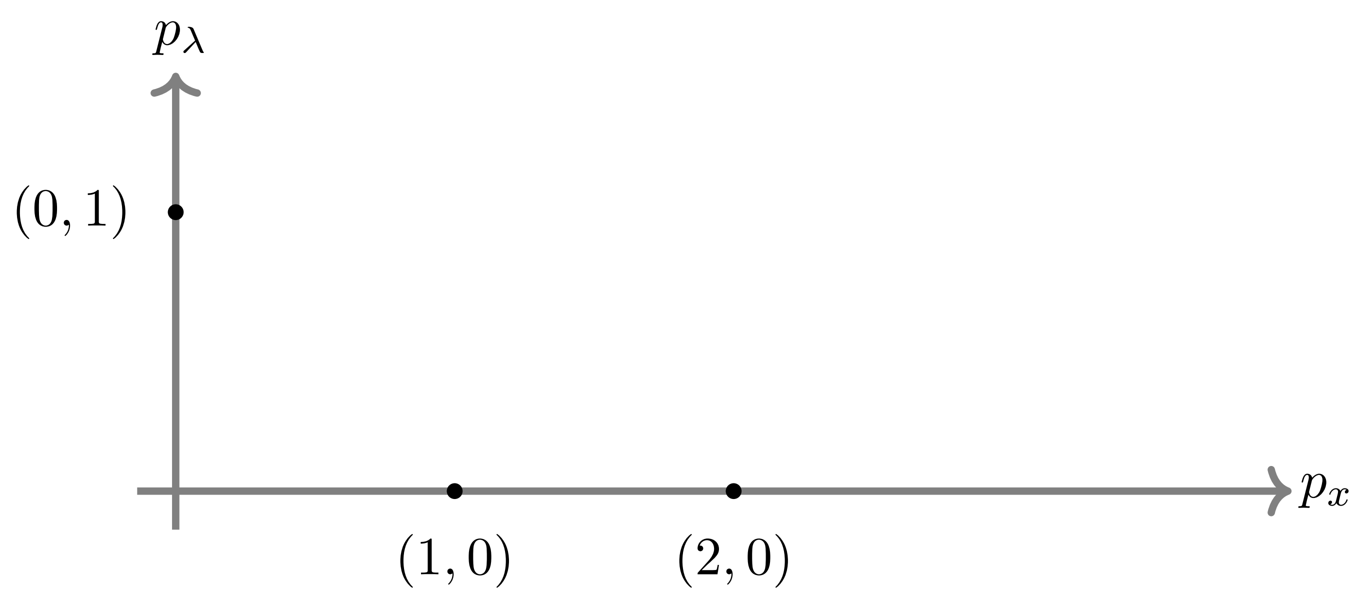
$\mathbf{r}_i = (r_{i,1}, \ldots, r_{i,N+1}) \in \mathbb{N}^{N+1}$ - exponent vectors

Geometric Method

Ignoring, for now, the coefficients c_i we can introduce a simple but useful picture for such polynomials:

- For each variable x_i or λ draw an orthogonal axis
- For each monomial, draw a dot at position $\mathbf{r}_i$

Example: $P(x, \lambda) = \lambda + x + x^2$ has exponent vectors $\mathbf{r}_1 = (0,1), \mathbf{r}_2 = (1,0), \mathbf{r}_3 = (2,0)$



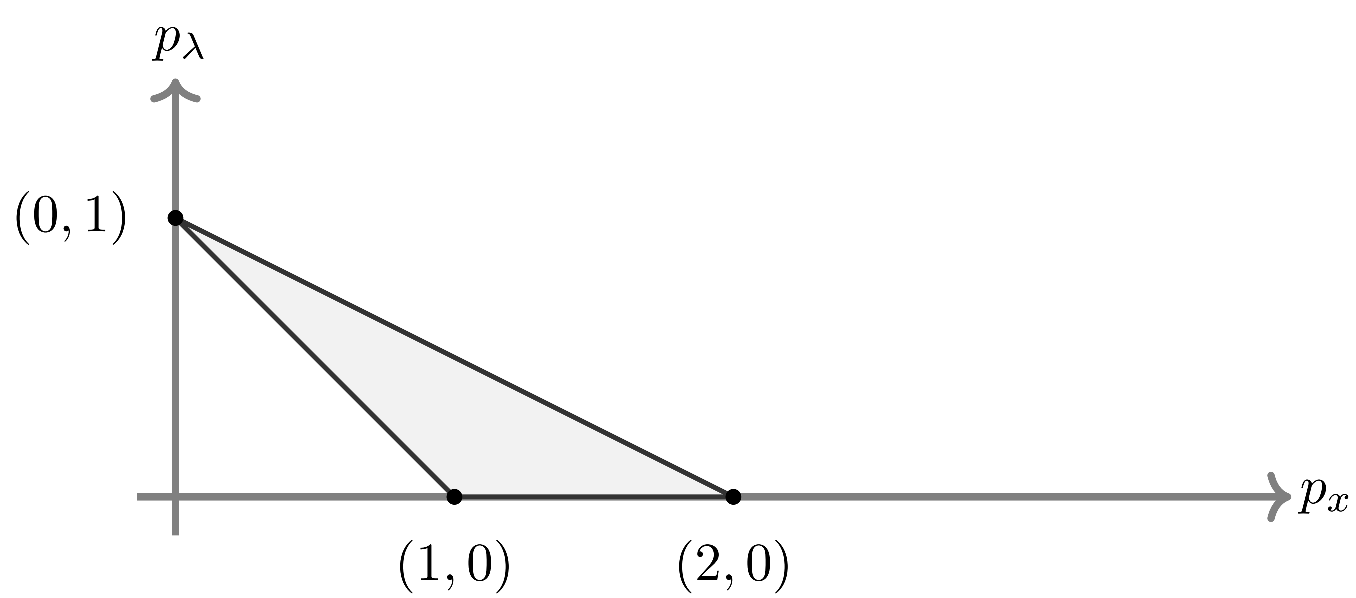
Geometric Method

We may define a **Newton polytope** of the polynomial, this is the convex hull of the exponent vectors:

$$\Delta = \text{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \left\{ \sum_j \alpha_j \mathbf{r}_j \,|\, \alpha_j \geq 0 \wedge \sum_j \alpha_j = 1 \right\}$$

Example: $P(x, \lambda) = \lambda + x + x^2$ has exponent vectors

$\mathbf{r}_1 = (0,1), \mathbf{r}_2 = (1,0), \mathbf{r}_3 = (2,0)$



Geometric Method

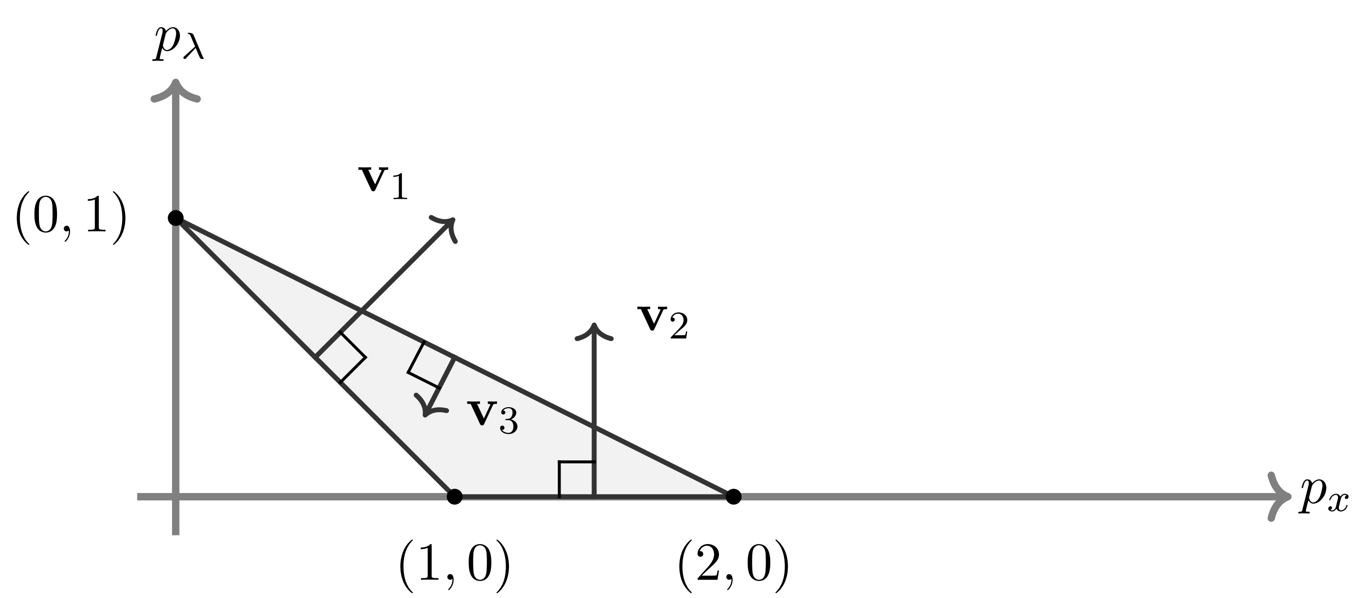
Alternatively, this polytope can also be described as the intersection of half spaces:

$$\Delta = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^{N+1} \mid \langle \mathbf{m}, \mathbf{v}_f \rangle + a_f \geq 0 \right\}$$

F - set of polytope facets, $a_f \in \mathbb{Z}$

$\mathbf{v}_f$ - inward-pointing normal vectors for each facet (co-dimension 1 face)

Several public tools exist for computing Newton polytopes/convex hulls and their representation in terms of facets exist, e.g. **Normaliz** and **Qhull**



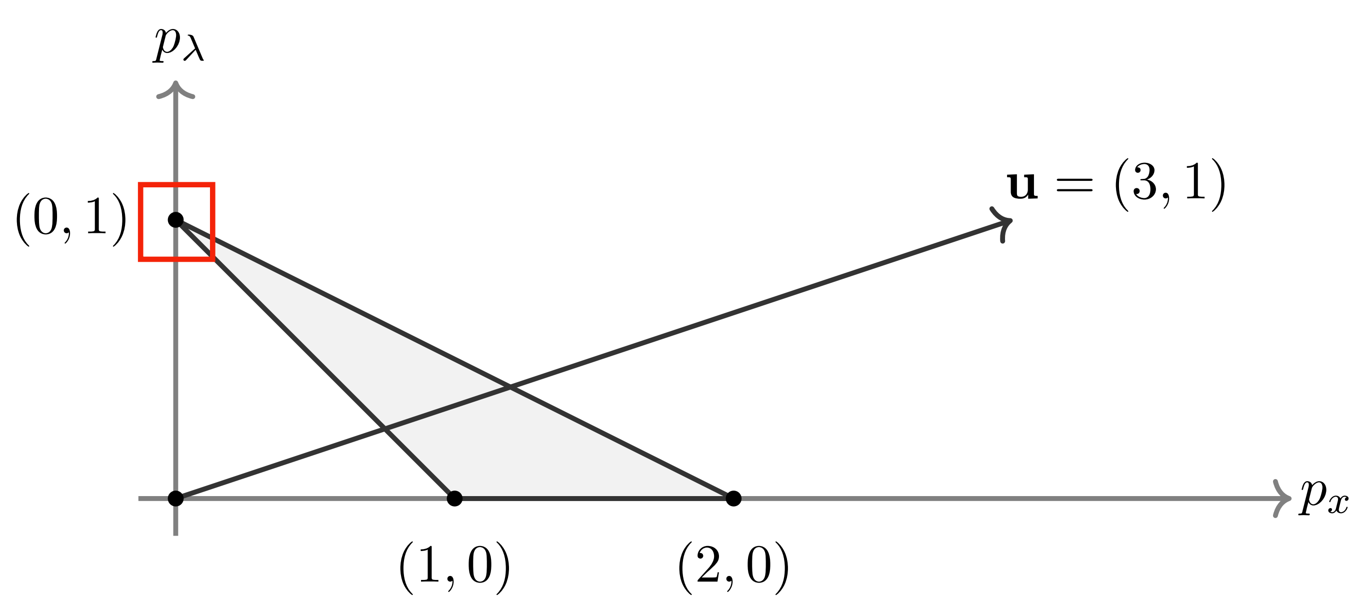
Geometric Method

Next, let us define a vector $\mathbf{u}$ such that $x_i = \lambda^{u_i}$ with $u_{N+1} = 1$ for each point $\mathbf{x}$ in the integration domain, we can write:

$$P(\mathbf{u}, \lambda) = \sum_{i=1}^{m} c_i \lambda^{\langle \mathbf{r}_i, \mathbf{u} \rangle}$$

Since $\lambda \ll 1$, the largest term in the polynomial has the smallest $\langle \mathbf{r}_i, \mathbf{u} \rangle$
Note that we can have several points with the same projection on $\mathbf{u}$, i.e. we can have several largest terms

Example: $P(x, \lambda) = \lambda + x + x^2$ with $\mathbf{u} = (3,1)$ gives $P(\mathbf{u}, \lambda) = \underline{\lambda} + \lambda^3 + \lambda^6$



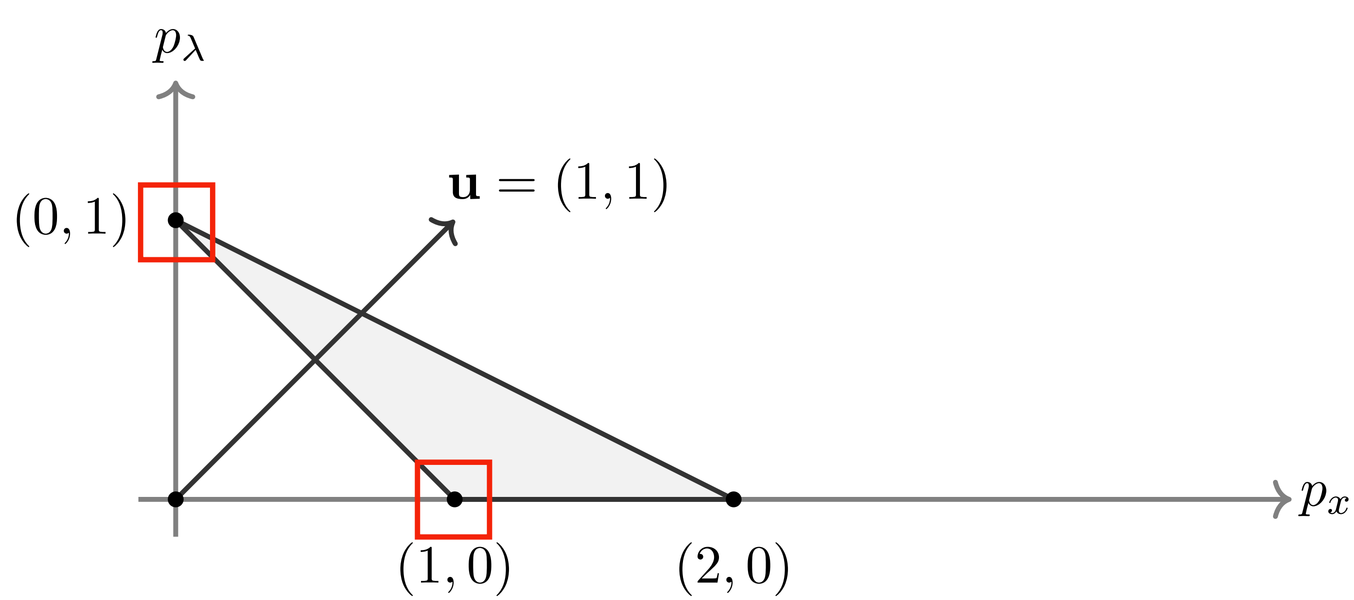
Geometric Method

Next, let us define a vector $\mathbf{u}$ such that $x_i = \lambda^{u_i}$ with $u_{N+1} = 1$ for each point $\mathbf{x}$ in the integration domain, we can write:

$$P(\mathbf{u}, \lambda) = \sum_{i=1}^{m} c_i \lambda^{\langle \mathbf{r}_i, \mathbf{u} \rangle}$$

Since $\lambda \ll 1$, the largest term in the polynomial has the smallest $\langle \mathbf{r}_i, \mathbf{u} \rangle$
Note that we can have several points with the same projection on $\mathbf{u}$, i.e. we can have several largest terms

Example: $P(x, \lambda) = \lambda + x + x^2$ with $\mathbf{u} = (1,1)$ gives $P(\mathbf{u}, \lambda) = \underline{\lambda + \lambda} + \lambda^2$



Expanding Regions

Rewrite our polynomial as: $P(\mathbf{x}) = Q(\mathbf{x}) + R(\mathbf{x})$
With $Q(\mathbf{x})$ defined such that it contains all of the lowest order terms in λ

The binomial expansion of
$P(\mathbf{x})^m = Q(\mathbf{x})^m \left(1 + \frac{R(\mathbf{x})}{Q(\mathbf{x})}\right)^m$ converges for $\mathbf{x} = \lambda^{\mathbf{u}}$ if $R(\mathbf{x})/Q(\mathbf{x}) < 1$

Some observations:

- An expansion with region vector $\mathbf{v}$ converges at a point $\mathbf{u}$ if the terms with minimum $< \mathbf{r}_i, \mathbf{u} >$ are contained in the terms with minimum $< \mathbf{r}_i, \mathbf{v} >$
- For any $\mathbf{u}$ the vertices with the smallest $< \mathbf{r}_i, \mathbf{u} >$ must be part of some facet F
- Since $u_{N+1} > 0$, the lowest order terms for any $\mathbf{u}$ must lie on a facet whose inwards pointing normal vector has a positive $(N+1)$-th component, let us call the set of such facets F^+ or lower facets

Claim: regions are defined by vectors normal to the facets in F^+, the integrand in each region consists of the monomials lying on the facet

Scaleless Integrals

Scaleless integrals seem to play quite an interesting role

Momentum space

In dimensional regularisation, **scaleless integrals are 0**

$$I(\{k_i\}_a, \{ck_i\}_b) = c^q\, I(\{k_i\}) \implies I(\{k_i\}) = 0, \quad \{k_i\} = \{k_i\}_a \cup \{k_i\}_b$$

Where $c \neq 1$ and $q \neq 0$ is some scaling dimension

Feynman parameter space

$(\mathscr{U}\mathscr{F})\left(c^{\mathbf{u}}\mathbf{x}\right) = c^q(\mathscr{U}\mathscr{F})(\mathbf{x}), \quad \mathbf{u} \neq n\mathbf{1}, \quad n \in \mathbb{R}$

Geometrical view

For Δ built from $\mathscr{U} + \mathscr{F}$

$$\dim(\Delta) = \dim(\mathbf{x}) \iff I \text{ scaleful}$$
$$\dim(\Delta) < \dim(\mathbf{x}) \iff I \text{ scaleless}$$

Important consequences:

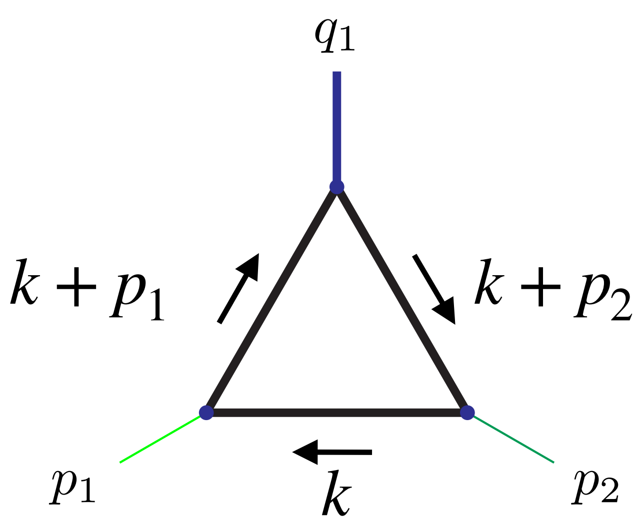
Faces of co-dimension > 1 are scaleless

``Region" vectors not normal to a facet give scaleless integrals

Overlap contributions i.e. rescaling by two region vectors, are scaleless

Triangle Example

Consider the on-shell limit $p_1^2 \sim p_2^2 \sim \lambda q_1^2$ for $\lambda \to 0$



$$I = i\pi^{D/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k+p_1)^2(k+p_2)^2(k^2)}$$

$$p_1 = (p_1^+, p_1^-, p_1^\perp) \sim Q(\lambda, 1, \lambda^{\frac{1}{2}})$$

$$p_2 \sim Q(1, \lambda, \lambda^{\frac{1}{2}})$$

1) Split integrand up into regions

$$\text{Hard}: k_H^\mu \sim (1,1,1)\,Q$$

$$\text{Collinear to } \mathrm{p}_1 : k_{J_1}^\mu \sim (\lambda, 1, \lambda^{\frac{1}{2}})\,Q$$

$$\text{Collinear to } \mathrm{p}_2 : k_{J_2}^\mu \sim (1, \lambda, \lambda^{\frac{1}{2}})\,Q$$

$$\text{Soft}: k_S^\mu \sim (\lambda, \lambda, \lambda)\,Q$$

2) Series expand each region in λ

$$I_H = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k^2 + 2k^+ \cdot p_1^-)(k^2 + 2k^- \cdot p_2^+)(k^2)}$$

$$I_{C_1} = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k+p_1)^2(2k^- \cdot p_2^+)(k^2)}$$

$$I_{C_2} = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(2k^- \cdot p_1^+)(k+p_2)^2(k^2)}$$

$$I_S = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(2k^+ \cdot p_1^- + p_1^2)(2k^- \cdot p_2^+ + p_2^2)(k^2)}$$

Analysis follows:
Becher, Broggio, Ferroglia 14

Triangle Example

3-5) Integrate each expansion over the whole integration domain, discard scaleless, sum

$$I_H = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_{C_1} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_{C_2} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_2^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_S = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{P_2^2 P_1^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{P_2^2 P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I = I_H + I_{C_1} + I_{C_2} + I_S = \frac{1}{Q^2}\left(\ln\frac{Q^2}{P_2^2}\ln\frac{Q^2}{P_1^2} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right)$$

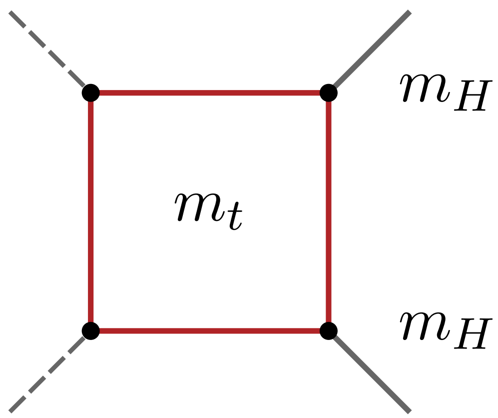
This reproduces the expected result, but why does this work (and does it always)?

1) How did we **find all the regions**?

2) Did we not **double-count** when integrating over the whole domain ?

pySecDec: EBR Box Example

Example: 1-loop massive box expanded for small $m_t^2 \ll s, |t|$



Requires the use of analytic regulators

Can regulate spurious singularities by adjusting propagators powers

$$G_4 = \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{d^D k}{i\pi^{D/2}} \frac{1}{[k^2 - m_t^2]^{\delta_1} [(k+p_1)^2 - m_t^2]^{\delta_2} [(k+p_1+p_2)^2 - m_t^2]^{\delta_3} [(k-p_4)^2 - m_t^2]^{\delta_4}}$$

Can keep $\delta_1, \ldots, \delta_4$ symbolic or $\delta_1 = 1 + n_1/2, \delta_2 = 1 + n_1/3, \ldots$ and take $n_1 \to 0^+$

Output region vectors:

$\mathbf{v}_1 = (0,0,0,0,1)$

$\mathbf{v}_2 = (-1, -1,0,0,1)$

$\mathbf{v}_3 = (0,0, -1, -1,1)$

$\mathbf{v}_4 = (-1,0,0, -1,1)$

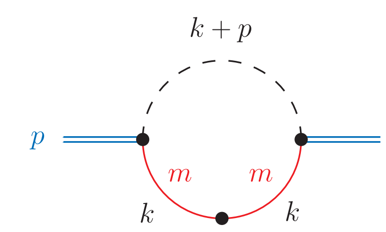
$\mathbf{v}_5 = (0, -1, -1,0,1)$

Result: s = 4.0, t = -2.82843, $m_t^2 = 0.1, m_h^2 = 0)$

$$I = -1.30718 \pm 2.7 \cdot 10^{-6} + (1.85618 \pm 3.0 \cdot 10^{-6})\, i$$

$$+\mathcal{O}\left(\epsilon, n_1, \frac{m_t^2}{s}, \frac{m_t^2}{t}\right)$$

Transform the expression for the full integral:



$$F = \int_{k \in D_h} \mathrm{D}k\, I + \int_{k \in D_s} \mathrm{D}k\, I = \sum_i \int_{k \in D_h} \mathrm{D}k\, T_i^{(h)} I + \sum_j \int_{k \in D_s} \mathrm{D}k\, T_j^{(s)} I$$

$$= \sum_i \left(\int_{k \in \mathbb{R}^d} \mathrm{D}k\, T_i^{(h)} I - \sum_j \int_{k \in D_s} \mathrm{D}k\, T_j^{(s)} T_i^{(h)} I \right) + \sum_j \left(\int_{k \in \mathbb{R}^d} \mathrm{D}k\, T_j^{(s)} I - \sum_i \int_{k \in D_h} \mathrm{D}k\, T_i^{(h)} T_j^{(s)} I \right)$$

The expansions commute: $\boxed{T_i^{(h)} T_j^{(s)} I = T_j^{(s)} T_i^{(h)} I \equiv T_{i,j}^{(h,s)} I}$

$\Rightarrow$ **Identity:** $$F = \underbrace{\sum_i \int \mathrm{D}k\, T_i^{(h)} I}_{\boldsymbol{F^{(h)}}} + \underbrace{\sum_j \int \mathrm{D}k\, T_j^{(s)} I}_{\boldsymbol{F^{(s)}}} - \underbrace{\sum_{i,j} \int \mathrm{D}k\, T_{i,j}^{(h,s)} I}_{\boldsymbol{F^{(h,s)}}}$$

All terms are integrated over the whole integration domain $\mathbb{R}^d$ as prescribed for the expansion by regions $\Rightarrow$ location of boundary Λ between D_h, D_s is irrelevant.

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The general formalism (details)

Identities as in the examples are generally valid, under some conditions.

Consider

- a (multiple) integral $F = \int \mathrm{D}k\, I$ over the domain D (e.g. $D = \mathbb{R}^d$),
- a set of N regions $R = \{x_1, \ldots, x_N\}$,
- for each region $x \in R$ an expansion $T^{(x)} = \sum_j T_j^{(x)}$ which converges absolutely in the domain $D_x \subset D$.

Conditions

- $\bigcup_{x \in R} D_x = D \qquad [D_x \cap D_{x'} = \emptyset \; \forall x \neq x']\,.$
- Some of the expansions commute with each other. Let $R_\mathsf{c} = \{x_1, \ldots, x_{N_\mathsf{c}}\}$ and $R_\mathsf{nc} = \{x_{N_\mathsf{c}+1}, \ldots, x_N\}$ with $1 \leq N_\mathsf{c} \leq N$. Then: $T^{(x)} T^{(x')} = T^{(x')} T^{(x)} \equiv T^{(x,x')} \; \forall x \in R_\mathsf{c}\,,\; x' \in R\,.$
- Every pair of non-commuting expansions is invariant under some expansion from R_c: $\forall x_1', x_2' \in R_\mathsf{nc},\; x_1' \neq x_2',\; \exists x \in R_\mathsf{c} : \; T^{(x)} T^{(x_2')} T^{(x_1')} = T^{(x_2')} T^{(x_1')}\,.$
- $\exists$ regularization for singularities, e.g. dimensional (+ analytic) regularization. $\hookrightarrow$ All expanded integrals and series expansions in the formalism are well-defined.

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The general formalism (2)

Under these conditions, the following **identity** holds: $\left[F^{(x,\ldots)} \equiv \sum_{j,\ldots} \int \mathrm{D}k\, T^{(x,\ldots)}_{j,\ldots} I\right]$

$$\boxed{F = \sum_{x \,\in\, R} F^{(x)} - \sum_{\{x'_1,x'_2\} \,\subset\, R}^{\langle R_\mathrm{c}+1\rangle} F^{(x'_1,x'_2)} + \ldots - (-1)^n \sum_{\{x'_1,\ldots,x'_n\} \,\subset\, R}^{\langle R_\mathrm{c}+1\rangle} F^{(x'_1,\ldots,x'_n)} + \ldots + (-1)^{N_\mathrm{c}} \sum_{x' \,\in\, R_\mathrm{nc}} F^{(x',x_1,\ldots,x_{N_\mathrm{c}})}}$$

where the sums run over subsets $\{x'_1,\ldots\}$ containing at most one region from R_nc.

Comments

- This identity is exact when the expansions are summed to all orders. ✓
  Leading-order approximation for $F \rightsquigarrow$ dropping higher-order terms.
- It is independent of the regularization (dim. reg., analytic reg., cut-off, infinitesimal masses/off-shellness, ...) as long as all individual terms are well-defined.
- Usually regions & regularization are chosen such that multiple expansions $F^{(x'_1,\ldots,x'_n)}$ $(n \geq 2)$ are scaleless and vanish.
  [✓ if each $F_0^{(x)}$ is a *homogeneous* function of the expansion parameter with *unique scaling*.]
- If $\exists\, F^{(x'_1,x'_2,\ldots)} \neq 0 \rightsquigarrow$ relevant overlap contributions ($\rightarrow$ "zero-bin subtractions").
  They appear e.g. when avoiding analytic regularization in SCET. e.g. Manohar, Stewart '06; Chiu, Fuhrer, Hoang, Kelley, Manohar '09; ...

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