Top Decay at Next-to-Next-to-Next-to-Leading Order in QCD

陈龙 (Long Chen)

山东大学 (Shandong University)

Loops and Legs in Quantum Field Theory (LL2024)

April 14-19, 2024 Wittenberg

Based on: [2309.01937 and work in progress]
In collaboration with **Xiang Chen**, **Xin Guan** and **Yan-Qing Ma**

Why Study the Top Quark

- Heaviest fundamental particle in SM

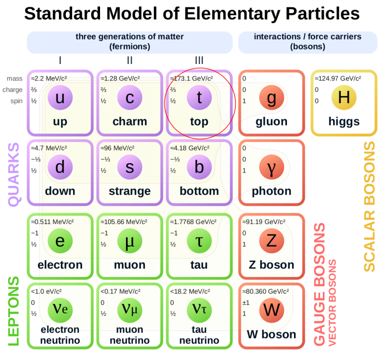
$$\boxed{m_t = 172.69 \pm 0.30\,\mathrm{GeV}}$$

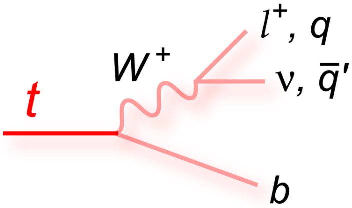
- Precision test of SM mechanism, and probs for possible BSM physics

- Decay exclusively to $b + W$ before hadronization:

$$\boxed{\Gamma_t = 1.4\,\mathrm{GeV} \gg \Lambda_{\mathrm{QCD}}}$$

- Convergence of the perturbative QCD series (e.g. renormalon issue)



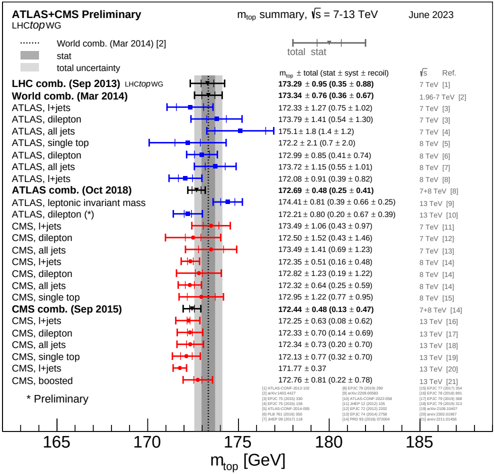


Top Quark Mass m_t and Decay Width Γ_t

- **PDG average for m_t:**
  172.69 ± 0.30 **GeV**

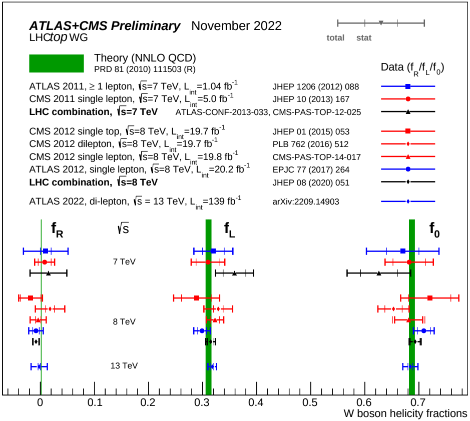
- **Current best measurement for Γ_t:**
  $1.36 \pm 0.02(\text{stat.})^{+0.14}_{-0.11}(\text{syst.})\ \text{GeV}.$

- **Experimental uncertainties anticipated at future colliders:**
  $20 \sim 26\ \text{MeV}$



The W-helicity fractions in Top Decay

W from $t \to b + W^+ + X_{\text{QCD}}$ is polarized even if the t-quark is unpolarized



- The current best measurements: $f_0 = 0.684 \pm 0.005\,(\text{stat.}) \pm 0.014\,(\text{syst.})$, $f_L = 0.318 \pm 0.003\,(\text{stat.}) \pm 0.008\,(\text{syst.})$ and $f_R = -0.002 \pm 0.002\,(\text{stat.}) \pm 0.014\,(\text{syst.})$.
- Notoriously difficult to be predicted theoretically to high precision

Much Theoretical Work Done So Far

Given the key role played by the top-quark both in SM precision test and searching for BSM, there have been vast amount of works done in literature regarding $t \to b + W^+ + X_{\text{QCD}}$.

- **The inclusive Γ_t**
  - Up to NNLO in QCD: [Jezabek etc 88; Czarnecki etc 90; Li etc 90; Czarnecki etc 98; Chetyrkin etc 99; Fischer etc 01; Blokland etc 04'05;......; Czarnecki etc 10; Meng etc 22; Chen etc 22]

    @**NNNLO** in QCD: [**LC**, Chen, Guan, Ma 23; Chen, Li, Li, Wang, Wang, Wu 23] [→ See also talk by J. Wang]
    [Datta, Rana, Ravindran, Sarkar 23 (only virtuals)]
  - NLO Electroweak: [Denner Sack 91; Eilam, Mendel Migneron Soni 91]

- **W-helicities $f_{\text{L,R,0}}$**
  - @NNLO in QCD: [Czarnecki, Korner, Piclum 10; Gao, Li, Zhu 12; Brucherseifer, Caola, Melnikov 13; Czarnecki, Groote Korner ,Piclum 18]

    @**NNNLO** in QCD: [**LC**, Chen, Guan, Ma 23]
  - NLO Electroweak: [Do, Groote, Korner, Mauser 02]

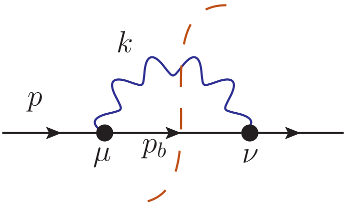
- **Differential results**
  - QCD:

    @NLO [Fischer, Groote, Korner, Mauser 01; Brandenburg, Si, Uwer 02; Bernreuther, Gonzalez , Mellei 14; Kniehl, Nejad 21]
    @NNLO [Gao, Li, Zhu 12; Brucherseifer, Caola, Melnikov 13; Campbell, Neumann, Sullivan 20]
    @**NNNLO** in QCD: [**LC**, Chen, Guan, Ma 23]

Cut Diagrams for Top Decay Width

Γ_t in terms of the **semi-inclusive** $\mathcal{W}_{tb}^{\mu\nu}$

$$\Gamma_t = \frac{1}{2\,m_t} \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1} 2E}\, \mathcal{W}_{tb}^{\mu\nu} \sum_{\lambda}^{L,R,0} \varepsilon^*_\mu(k,\lambda)\, \varepsilon_\nu(k,\lambda)\,,$$



$$\mathcal{W}_{tb}^{\mu\nu}(p,k) = W_1\, g^{\mu\nu} + W_2\, p^\mu p^\nu + W_3\, k^\mu k^\nu + W_4 \left(p^\mu k^\nu + k^\mu p^\nu\right) + W_5\, i\epsilon^{\mu\nu\rho\sigma}\, p_\rho\, k_\sigma\,,$$

Selection Criteria: the cut diagrams of t-quark self-energy function with exactly one (cut) W propagator interacting with the *external* t-quark plus (up to 3) QCD loops

Amplitude-squared Level Helicity Projectors

$$\Gamma_t = \frac{1}{2\,m_t} \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1} 2E}\, \mathcal{W}_{tb}^{\mu\nu} \sum_{\lambda}^{L,R,0} \varepsilon_\mu^*(k,\lambda)\, \varepsilon_\nu(k,\lambda)\,,$$

The W-polarization-sum rule:

$$\sum_{\lambda}^{L,R,0} \varepsilon_\mu^*(k,\lambda)\, \varepsilon_\nu(k,\lambda) = g^{\mu\nu} - k^\mu k^\nu / m_W^2 \equiv \mathcal{P}_{\mathrm{tot}}^{\mu\nu}$$

Projectors for polarized-W in the squared amplitude: [Fischer, Groote, Korner, Mauser 01]

$$\begin{aligned}
\varepsilon^{*\,\mu}(k,\lambda_0)\, \varepsilon^\nu(k,\lambda_0) &= \frac{\left(m_W^2 p^\mu - p\cdot k k^\mu\right)\left(m_W^2 p^\mu - p\cdot k k^\mu\right)}{m_t^2\, m_W^2\, \vec{k}^2} \equiv \mathcal{P}_{\mathrm{L}}^{\mu\nu} \\
\varepsilon^{*\,\mu}(k,\lambda_L)\, \varepsilon^\nu(k,\lambda_L) &= \left(\mathcal{P}_{\mathrm{tot}}^{\mu\nu} - \mathcal{P}_{\mathrm{L}}^{\mu\nu} + \mathcal{P}_{\mathrm{A_{FB}}}^{\mu\nu}\right)/2 \\
\varepsilon^{*\,\mu}(k,\lambda_R)\, \varepsilon^\nu(k,\lambda_R) &= \left(\mathcal{P}_{\mathrm{tot}}^{\mu\nu} - \mathcal{P}_{\mathrm{L}}^{\mu\nu} - \mathcal{P}_{\mathrm{A_{FB}}}^{\mu\nu}\right)/2
\end{aligned}$$

where $\mathcal{P}_{\mathrm{A_{FB}}}^{\mu\nu} \equiv \frac{-i\epsilon^{\mu\nu}_{p\,k}}{m_t |\vec{k}|}$ (known as the forward-backward-asymmetry projector [Altarelli, Lampe 93])

Loop and Phase-space Integration

- Loop integrals are reduced using IBP [Chetyrkin 81] relations done with `Blade` [Guan, Liu, Ma 20] , and the resulting *masters* are calculated using DE method [Kotikov 90; Remiddi 97] with `AMFlow` [Liu, Ma 22]

- The phase-space integrals, except for W-momentum k, are treated in the same manner as loop integrals by means of the reverse unitarity [Anastasiou, Melnikov 02]

- The IR-divergent phase-space integration of $\mathcal{W}_{tb}^{\mu\nu}$ over k are done "*manually*" using its power-log series representation (PSE) with ϵ assigned with non-zero numbers.

- **Level of Complexity**:
  2988 master integrals, for which PSE about 200 orders in k_0 are derived with the above method. (c.f. only 185 MIs in the color-leading part [→ See talk by J. Wang])

Generalized Power-Log Series and the Integration Formula

According to Expansion-by-Region as well as Frobenius series solution for DE of **dimensionally-regularized loop integrals**:

$$f(\epsilon, x) = \sum_{a,b\in S} x^a \ln^b(x)\, T_{ab}(\epsilon, x) = \sum_{a,b\in S} x^a \ln^b(x) \left(\sum_{n=0}^{\infty} C_{abn}(\epsilon)\, x^n \right)$$

where $a = a_0 + a_1\epsilon$ with rational a_0, a_1 and non-negative integer b, belonging to a finite set S.

- Termwise integration formula we used:

$$\int_0^u x^a \ln^b(x)\, \mathrm{d}x = \begin{cases} \frac{\ln^{1+b}(u)}{1+b} & \text{if } a = -1 \\ u^{1+a} \frac{1}{(1+a)^{1+b}} \sum_{i=0}^{b} (-1)^{b-i} \frac{b!}{i!} (1+a)^i \ln^i(u) & \text{if } a \neq -1 \end{cases}$$

- The list of ϵ sampled: $10^{-3} + n \times 10^{-4}$ for $n = 0,\ 1,\ \cdots, 15$.
- **Fit in ϵ is done only at the very end for the final finite (physical) objects of interest**.

- **Consistency Check**:
  A perfect agreement in the result for the **inclusive** Γ_t with $\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1} 2E}$ calculated in this way and by directly applying the reverse unitarity [Anastasiou, Melnikov 02]

Results for the Inclusive Γ_t

The QCD effects on Γ_t in SM can be parameterized as

$$\Gamma_t = \Gamma_0 \left[\mathbf{c}_0 + \frac{\alpha_s}{\pi} \mathbf{c}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathbf{c}_2 + \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{c}_3 + \mathcal{O}(\alpha_s^4) \right],$$

with $\Gamma_0 \equiv \frac{G_F\ m_W^2\, m_t\, |V_{tb}|^2}{12\sqrt{2}}$.

We choose $\mu = m_t/2$, motivated by the kinetic energy $m_t - m_W - m_b$ of the QCD radiations, at which our N3LO result reads: [LC, Chen, Guan, Ma 23]

$$\begin{aligned} \Gamma_t &= 1.48642 - 0.140877 - 0.023306 - 0.007240\ \text{GeV} \\ &= \mathbf{1.31500}\ \text{GeV} \end{aligned}$$

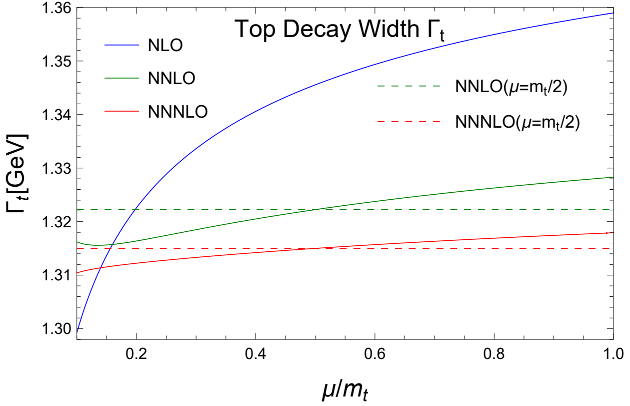
SM Inputs:

$$G_F = 1.166379 \times 10^{-5}\text{GeV}^{-2}, \quad m_t = 172.69\text{GeV},$$
$$m_W = 80.377\text{GeV}, \quad \alpha_s(m_t/2) \approx 0.1189.$$

The leading-color part of Γ_t agrees with a parallel computation [Chen, Li, Li, Wang, Wang, Wu 23]

The QCD Scale Uncertainty of Γ_t

The scale dependence of the fixed-order results for Γ_t in $\mu/m_t \in [0.1, 1]$



- NNLO scale-variation **never** cover the NNNLO result at any scales less than $\mu/m_t = 0.6$.
- Pure $\mathcal{O}(\alpha_s^3)$ correction decreases Γ_t by $\sim 0.8\%$ of the NNLO result at $\mu = m_t$ roughly 10 MeV(exceeding NNLO scale-hand)

The Offshell W and Finite m_b Effects

- With $\frac{1}{k^2-m_W^2+i\epsilon} \to \frac{1}{k^2-m_W^2+im_W\Gamma_W}$,

$$\Gamma_t(m_W) \to \tilde{\Gamma}_t = \int_0^{m_t^2} \frac{\mathrm{d}\,k^2}{2\pi} \frac{2m_W\Gamma_W}{(k^2-m_W^2)^2+(m_W\Gamma_W)^2} \Gamma_t(m_W^2 \to k^2)$$
$$= \tilde{\Gamma}_0 \left[\tilde{\mathbf{c}}_0 + \frac{\alpha_s}{\pi}\mathbf{c}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathbf{c}_2 + \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{c}_3 + \mathcal{O}(\alpha_s^4)\right],$$

  we find: $\frac{\tilde{\mathbf{c}}_i - \mathbf{c}_i}{\mathbf{c}_i}$ takes $-1.54\%, -1.53\%, -1.39\%, -1.23\%$ for $i = 0, 1, 2, 3$.

- Similarly, keeping $m_b = 4.78\,\mathrm{GeV}$, we find: $\frac{\mathbf{c}_1^{m_b} - \mathbf{c}_1}{\mathbf{c}_1} \approx \frac{\mathbf{c}_2^{m_b} - \mathbf{c}_2}{\mathbf{c}_2} \approx -1.47\%$.

- The NLO electroweak K-factor is re-evaluated to be $K_{EW}^{NLO} = 1.0168$.

Taking these misc-effects into account, we finally obtain the **to-date most-precise high-precision theoretical prediction**:

$$\boxed{\Gamma_t = 1.3148^{+0.003}_{-0.005} \times |V_{tb}|^2 + 0.027\,(m_t - 172.69)\,\mathrm{GeV}}$$

the error of which meets the request by future colliders.

Results for W-helicity Fractions

Top decay width with polarized W:

$$\Gamma_{\lambda} \;=\; \frac{1}{2\,m_t} \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1} 2E}\, \mathcal{W}_{tb}^{\mu\nu}\, \varepsilon^{*}_{\mu}(k,\lambda)\, \varepsilon_{\nu}(k,\lambda)$$

The W-**helicity fractions** $f_{\lambda}^{[n]} = \frac{\sum_{i=0}^{n} \Gamma_{\lambda}^{[n]}}{\sum_{i=0}^{n} \Gamma_{t}^{[n]}}$ truncated to $\mathcal{O}(\alpha_s^3)$ in massless QCD:

$$\begin{aligned}
{f_0}^{[3]} &= 0.697706 - 0.008401 - 0.001954 - 0.000613\,,\\
&= 0.686737\,,\\
{f_{\mathrm{L}}}^{[3]} &= 0.302294 + 0.007254 + 0.001799 + 0.000586\,,\\
&= 0.311933\,,\\
{f_{\mathrm{R}}}^{[3]} &= 0. + 0.001147 + 0.000155 + 0.000027\,,\\
&= 0.001330\,.
\end{aligned}$$

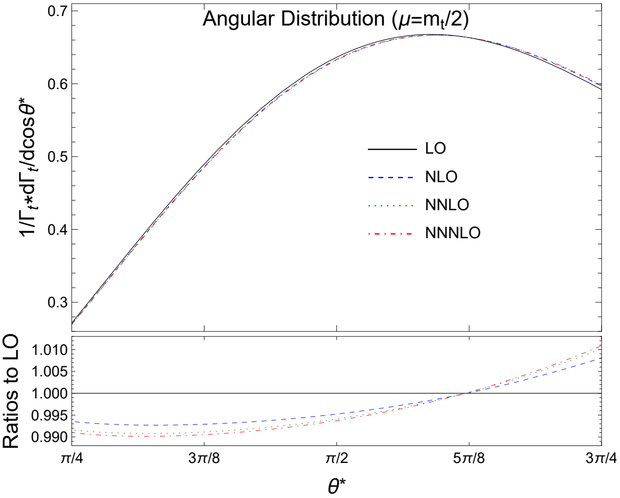
(The above results evaluated at $\mu = m_t$ agree with [Czarnecki, Korner, Piclum 10] up to NNLO)

With NLO EW-correction and m_b effects included, our **final results** read:

$$\boxed{{f_0}^{[3]} = 0.686^{+0.002}_{-0.003}\,, \quad {f_{\mathrm{L}}}^{[3]} = 0.312^{+0.001}_{-0.002}\,, \quad {f_{\mathrm{R}}}^{[3]} = 0.00157^{+0.00002}_{-0.00002}\,.}$$

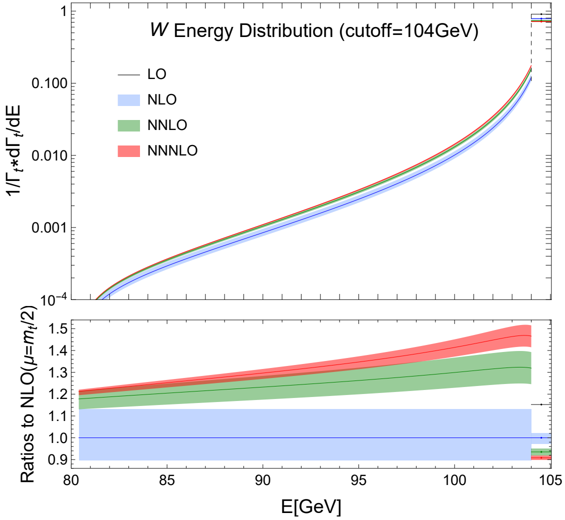
Results for $\cos\theta^*$ Angular Distribution

$$\frac{1}{\Gamma_t}\frac{\mathrm{d}\Gamma_t}{\mathrm{d}\cos\theta^*} = \frac{3}{4}(\sin^2\theta^*)\, f_0 + \frac{3}{8}(1-\cos\theta^*)^2 f_{\mathrm{L}} + \frac{3}{8}(1+\cos\theta^*)^2 f_{\mathrm{R}}$$



where θ^* is the angle between the charged-lepton momentum from the W-decay in W-rest frame and the W-momentum in t-rest frame.

Results for W-energy Distribution

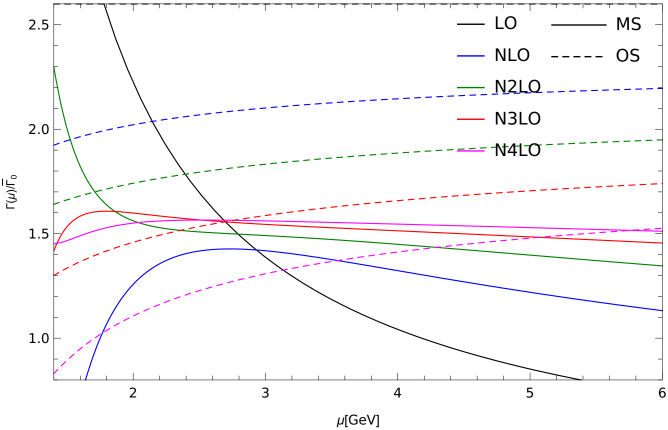


- **In the bulk**: QCD corrections are positive and quite sizable: pure $\mathcal{O}(\alpha_s^3)$ correction modifies the lowest order by $7 \sim 14\%$ for $E \in [94, 104]$ GeV.
- **In the rightmost 1 GeV-bin**: QCD corrections up to $\mathcal{O}(\alpha_s^3)$ decrease the Born-level result.

Bonus: The $b \to u\, e^+ \nu_e$ decay at higher orders in QCD

$$\Gamma(b \to u l \bar{\nu}_l)|_{\mu=\bar{m}_b} = \overline{\Gamma}_0 \left(1 + 4.2536 \left(\frac{\alpha_s}{\pi}\right) + 26.809 \left(\frac{\alpha_s}{\pi}\right)^2 + 188.15 \left(\frac{\alpha_s}{\pi}\right)^3 + 1317.0 \left(\frac{\alpha_s}{\pi}\right)^4\right)$$

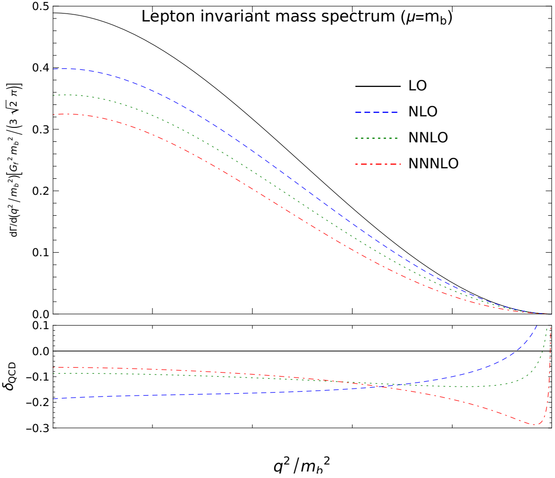
$$= \overline{\Gamma}_0 \left(1 + 0.3036075 + 0.1365820 + 0.06841766 + 0.034184\right)$$



Every one more perturbative order higher in the $\overline{\text{MS}}$ result, the term is reduced roughly by 1/2.

Bonus: lepton-pair invariant-mass spectrum in $b \to u\, e^+ \nu_e$

$$\Gamma(b \to u l \bar{\nu}_l) = \Gamma_0 \left(1 - 2.4131 \left(\frac{\alpha_s}{\pi}\right) - 21.27 \left(\frac{\alpha_s}{\pi}\right)^2 - 270.7 \left(\frac{\alpha_s}{\pi}\right)^3\right).$$



- $\mathcal{O}(\alpha_s^3)$ Leading-color inclusive part agrees with [Chen, Li, Li, Wang, Wang, Wu 23] . [→ See also talk by J. Wang]
- $\mathcal{O}(\alpha_s^3)$ corrections agrees with a recent approximation [Fael, Usovitsch 23] . [→ See also talk by M. Fael]

Summary and Outlook

- ☑ We have provided the **to-date most-precise high-precision theoretical prediction** for top-quark decay width:

$$\boxed{\Gamma_t = 1.3148^{+0.003}_{-0.005} \times |V_{tb}|^2 + 0.027\,(m_t - 172.69)\,\text{GeV}}$$

the error of which meets the request by future colliders.

- ☑ By a novel approach to complete phase-space integration over W momentum with IR-divergent integrand, we determined, in addition, W-helicity fractions, $\cos\theta^*$ distribution and W-energy distribution at α_s^3 for the first time.

- ☑ Furthermore, the lepton invariant-mass distribution in $b \to ul\bar{\nu}_l$ is derived up to α_s^3, and an estimation of $\mathcal{O}(\alpha_s^4)$ correction for $\Gamma_{b\to ul\bar{\nu}_l}$ based on geometric series behavior is provided.

- ☒ The approach can be readily applied to the decay of polarized t-quarks at α_s^3, as well as the mixed QCD-electroweak corrections.

Thank you!

Summary and Outlook

- ☑ We have provided the **to-date most-precise high-precision theoretical prediction** for top-quark decay width:

$$\boxed{\Gamma_t = 1.3148^{+0.003}_{-0.005} \times |V_{tb}|^2 + 0.027\,(m_t - 172.69)\,\text{GeV}}$$

  the error of which meets the request by future colliders.

- ☑ By a novel approach to complete phase-space integration over W momentum with IR-divergent integrand, we determined, in addition, W-helicity fractions, $\cos\theta^*$ distribution and W-energy distribution at α_s^3 for the first time.

- ☑ Furthermore, the lepton invariant-mass distribution in $b \to u l \bar{\nu}_l$ is derived up to α_s^3, and an estimation of $\mathcal{O}(\alpha_s^4)$ correction for $\Gamma_{b\to u l \bar{\nu}_l}$ based on geometric series behavior is provided.

- ☒ The approach can be readily applied to the decay of polarized t-quarks at α_s^3, as well as the mixed QCD-electroweak corrections.

Thank you!