Subleading power corrections for event shape variables in e^+e^- annihilation

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Based on arXiv:2311.12768 with L. Buonocore, M. Grazzini, L. Rottoli

Loops and Legs in Quantum Field Theory
Wittenberg, 18th April 2024

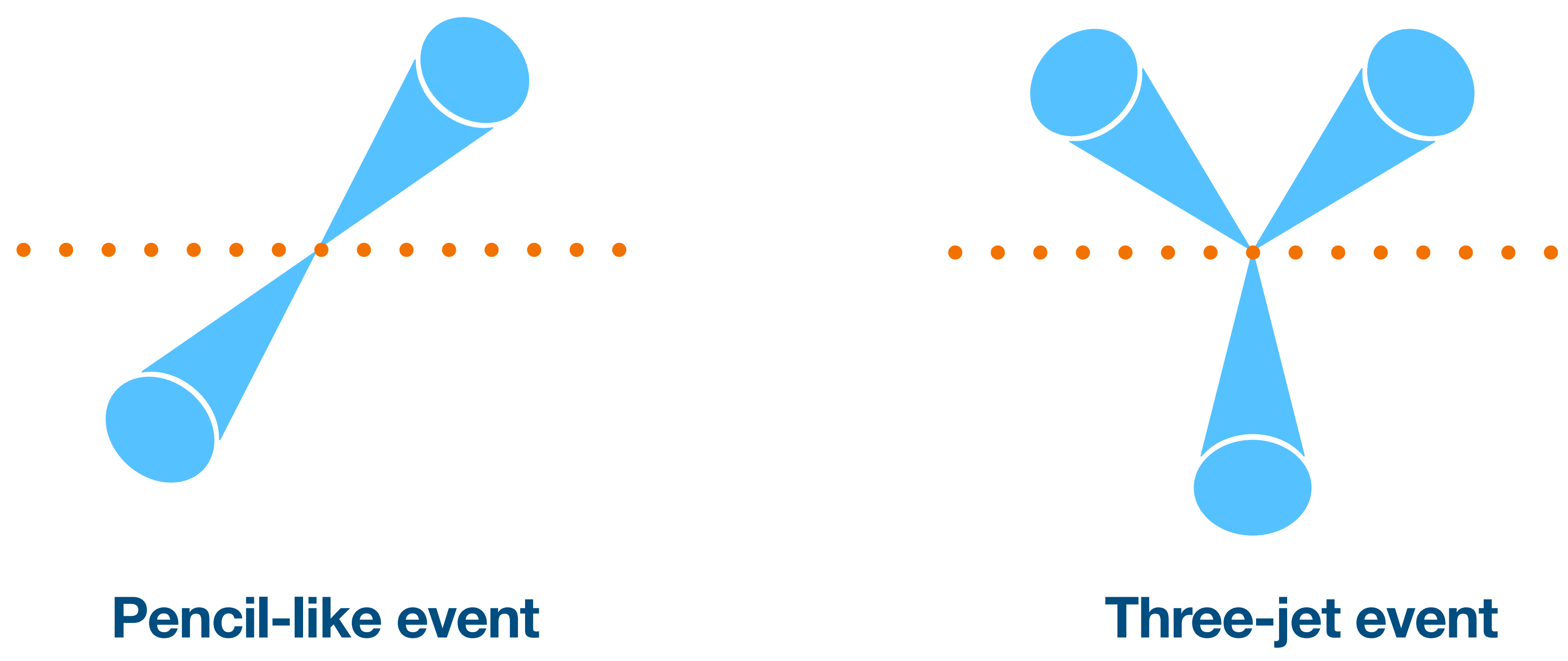
Universität Zürich[UZH]

Event shape variables

- **Event shape variables** play an important role in the study of QCD in e^+e^- annihilation. These variables characterize the **geometrical properties** of the hadronic final state. Some examples are Thrust and C-parameter.
- Event shape variables are defined to be **collinear and IR safe.**
- I will focus on 2-jet variables, i.e. variables that smoothly encode **transition between a pencil-like event and a three-jet event**.

	$1-T$	C-Parameter
Pencil-like event	0	0
Spherical event	1/2	1

$$T = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \hat{n}|}{\sum_i |\vec{p}_i|} \qquad C = 3 - \frac{3}{2}\sum_{i,j} \frac{(p_i \cdot p_j)^2}{(p_i \cdot q)(p_j \cdot q)}$$



Subleading power corrections

- In the limit in which a shape variable r is vanishing, the differential cross section in r **develops large logarithmic contributions** that have to be resummed. Leading power resummation for different shape variables have been extensively studied.

[Banfi, Becher, Bonciani, Catani, Dissertori, Gehrmann, Luisoni, Mangano, Marchesini, Monni, Nason, Rodrigo, Salam, Schmelling, Trentadue, Turnock, Webber, Zanderighi, …]

- However, only recently a systematic study of **subleading contributions in the $r \to 0$ limit** started. Subleading contributions can be used to improve the performances of slicing schemes when the shape variable is used as a slicing variable.

[Moult, Rothen, Stewart, Tackmann, Zhu (2017)] [Boughezal, Liu, Petriello (2017)] [Moult, Stewart, Vita, Zhu (2018)] [Moult, Stewart Vita (2019)] [Agarwal, van Beekveld, Laenen, Mishra, Mukhopadhyay, Tripathi (2023)][Beneke, Garny, Jaskiewicz, Strohm, Szafron, Vernazza, Wang (2022)][Vita (2024)]

- At NLO an event shape variable can be used to set up a slicing scheme by splitting the real contribution into a piece above a piece below a small cut v

$$\sigma_{\mathrm{NLO}} = \int d\sigma^R \theta(r - v) + \left(\int d\sigma^R \theta(v - r) + \int d\sigma^V + \int d\sigma^B \right)$$

- The term below the cut can be evaluated in the small v limit using suitable approximations for the phase space and the real matrix element. The IR poles from the real contributions will cancel against the ones in the virtual, and the below the cut part is:

$$\int d\sigma^R \theta(v - r) + \int d\sigma^V + \int d\sigma^B = \int d\sigma^B \left[1 + \frac{\alpha_s}{\pi} (A_r \log^2 v + B_r \log v + C_r + \mathcal{O}(v^p)) \right]$$

Subleading power corrections!

Observables

- We will now focus on the process $e^+e^- \to 2j$ at NLO. This simple case will allow us to perform fully analytical calculations

- In general, we will consider variables whose dependence of the momentum k of a single soft emission, collinear to the one of the hard legs of the Born event, can be parametrized as follows

$$r(\{p_i\}, k) = \left(\frac{k_t^{(\ell)}}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}}$$

- $\{p_i\}$ are the Born momenta and $k_t^{(\ell)}, \eta^{(\ell)} \geq 0$ denote the transverse momentum and the rapidity of k w.r.t. the leg ℓ.

- We will start our analysis by considering the variables 2-Jettiness τ_2, the jet rate y_{23} (based on the k_t clustering algorithm) and, limiting our self to NLO, the transverse momentum x_T^{FSR} of the emitted gluon w.r.t. the $q\bar{q}$ axis.

$$\tau_2 = \sum_{k=1}^{n} \min\left\{\frac{2p_k \cdot q_1}{Q^2}, \frac{2p_k \cdot q_2}{Q^2}\right\}$$

(q_1 and q_2 are jet axis obtained with the JADE algorithm) $a = 1, b = 1$

[Stewart, Tackmann, Waalewijn (2010)]

$$y_{23} = \min\{d_{12}, d_{13}, d_{23}\}$$

$$d_{ij} = \frac{2\min(E_i^2, E_j^2)(1 - \cos\theta_{ij})}{Q^2}$$

$a = 2, b = 0$

$$x_T^{\mathrm{FSR}} = \frac{1}{Q}\sqrt{\frac{2(p_1 \cdot p_3)(p_2 \cdot p_3)}{p_1 \cdot p_2}}$$

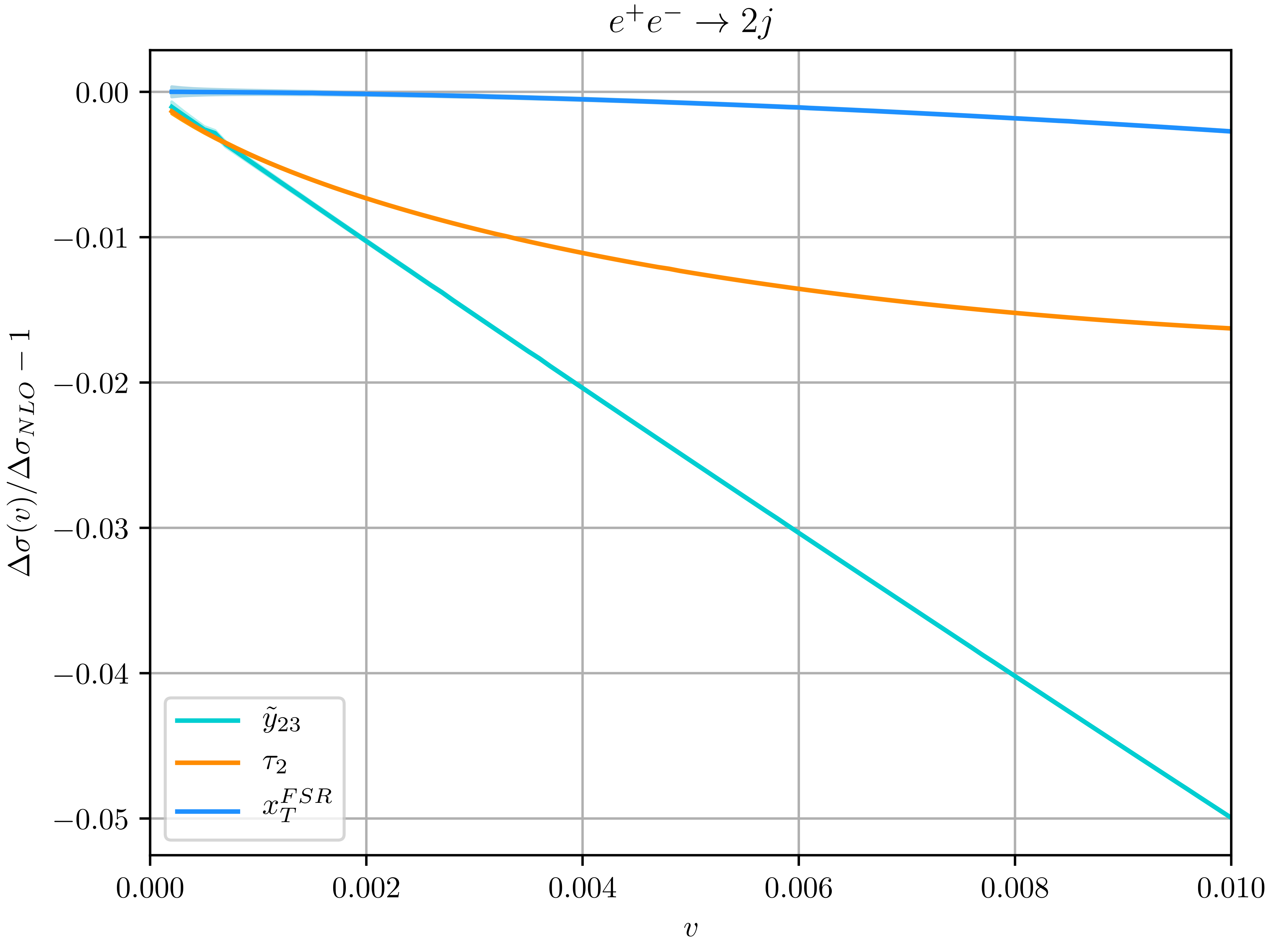
$a = 1, b = 0$

Observables

- Since the singularities of the matrix element are encoded in the limit $k_t^{(\ell)} \to 0$, we compare observables with an uniform scaling in the transverse momentum, so we will consider $\tilde{y}_{23} = \sqrt{y_{23}}$.

- We can test the size of the power corrections by plotting the relative deviation for the NLO correction $\Delta\sigma_{\text{NLO}}$ from its exact result as a function of the value of the cut v. The results for $\Delta\sigma_{\text{NLO}}$ are obtained numerically using a slicing method.

- The numerical plot suggests that the subleading power corrections are linear for $\tilde{y}_{23}$, linear logarithmically enhanced for τ_2 and quadratic for x_T^{FSR}. Our goal is to perform an analytical analysis of the observed different scaling. Furthermore, we would like to understand better the origin of the different scalings.



Setup of the calculation

- We now focus on the real contribution $d\sigma^R$. We can compute the complete tower of power corrections by integrating the real matrix element in the phase space region above the cut.

- The calculation is performed using a phase space parametrization in terms of the energy fractions x_i.

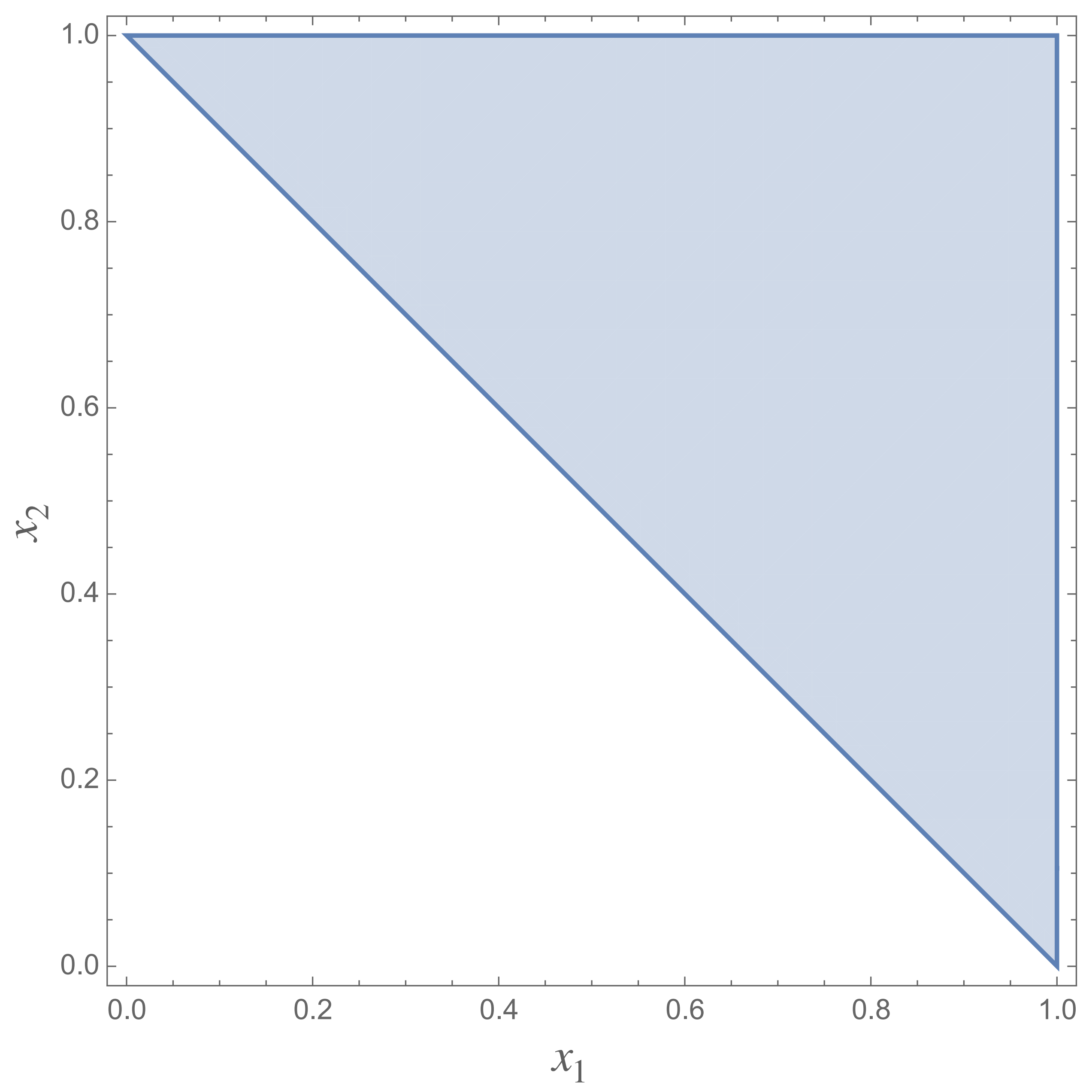
$$x_i = \frac{2p_i \cdot Q}{Q^2} \qquad x_1 + x_2 + x_3 = 2 \qquad \sigma_r^R(v) = \int d\sigma^R \theta(r - v) \equiv \sigma_0 \frac{\alpha_s}{2\pi} C_F R_r(v)$$

$$f(x_1, x_2) = \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \qquad R_r(v) = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, f(x_1, x_2)\theta(r(x_1, x_2) - v)$$

We want to obtain the subleading power corrections by computing this integral!

- The function $f(x_1, x_2)$ represents, up to a normalization factor, the squared matrix element for the process $e^+e^- \to \gamma^* \to q\bar{q}g$. The phase space in the (x_1, x_2) plane is the triangle $0 \le x_1 \le 1, 1 - x_1 \le x_2 \le 1$.

- The limits in which the emitted gluon is collinear to one of the quarks are reached for $x_1 \to 1, x_2 \to 1$. The soft limits occurs in the point $(x_1, x_1) = (1,1)$

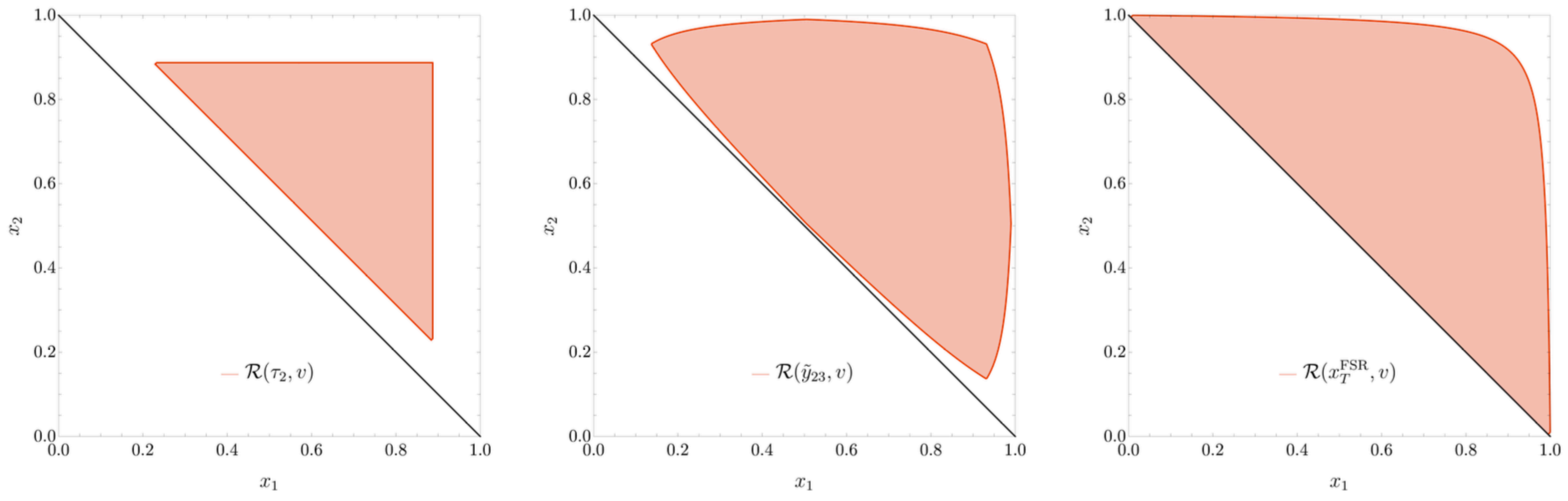


Observables in the x_1, x_2 plane

- The observables have the following expression in terms of the energy fractions

$$d_{ij} = \frac{\min\{x_i^2, x_j^2\}}{x_i x_j}(1 - x_k) \qquad \tau_2 = x_k(1 - x_k) \text{ when } s_{ij} < s_{ik}, s_{jk} \qquad x_T^{\mathrm{FSR}} = \frac{k_T^{\mathrm{FSR}}}{Q} = \sqrt{\frac{(1 - x_1)(1 - x_2)}{x_1 + x_2 - 1}}$$

- It is interesting to look at the phase space regions in which $r > v$, where v is a fixed value of the cut.



$\tau_2 > v$ $\qquad$ $\tilde{y}_{23} > v$ $\qquad$ $x_T^{\mathrm{FSR}} > v$

- We observe that the three variables cur in a different way the phase space region $x_2 \sim 1 - x_1$, that corresponds to the kinematical configuration in which an hard gluon is emitted. Since the matrix element is not singular in this limit, this region will give pure power corrections.

Results for $x_T^{\text{FSR}}, \tau_2, \tilde{y}_{23}$

- We analytically computed the full tower of power corrections for the variables $x_T^{\text{FST}}, \tau_2, \tilde{y}_{23}$

$$R_{x_T^{\text{FSR}}}(v) = \frac{7}{2} + v^2 + (3 + 4v^2 + v^4)\log\left(\frac{v^2}{1+v^2}\right) - 2\text{Li}_2\left(-\frac{1}{v^2}\right) = 4\log^2 v + 6\log v + \frac{7}{2} + \frac{\pi^2}{3} + \underline{4(2\log v - 1)v^2} + \mathcal{O}(v^4)$$

Quadratic subleading power corrections

$$R_{\tau_2}(v) = \frac{5}{2} - \frac{\pi^2}{3} + 2\log^2\left(\frac{1-u}{u}\right) + (6u-3)\log\left(\frac{1-2u}{u}\right) - 6u - \frac{9u^2}{2} + 4\text{Li}_2\left(\frac{u}{1-u}\right) = 2\log^2 v + 3\log v + \frac{5}{2} - \frac{\pi^2}{3} + \underline{v(7 + 2\log v)} + \mathcal{O}(v^2)$$

$$u = \frac{1}{2}(1 - \sqrt{1-4v})$$

Linear log-enhanced subleading power corrections

$$R_{\tilde{y}_{23}}(v) = 4\log^2 v + 6\log v + \frac{5}{2} - \frac{\pi^2}{6} + 6\log 2 + \underline{(4\log(1+\sqrt{2}) - 8\sqrt{2})v} + \mathcal{O}(v^2)$$

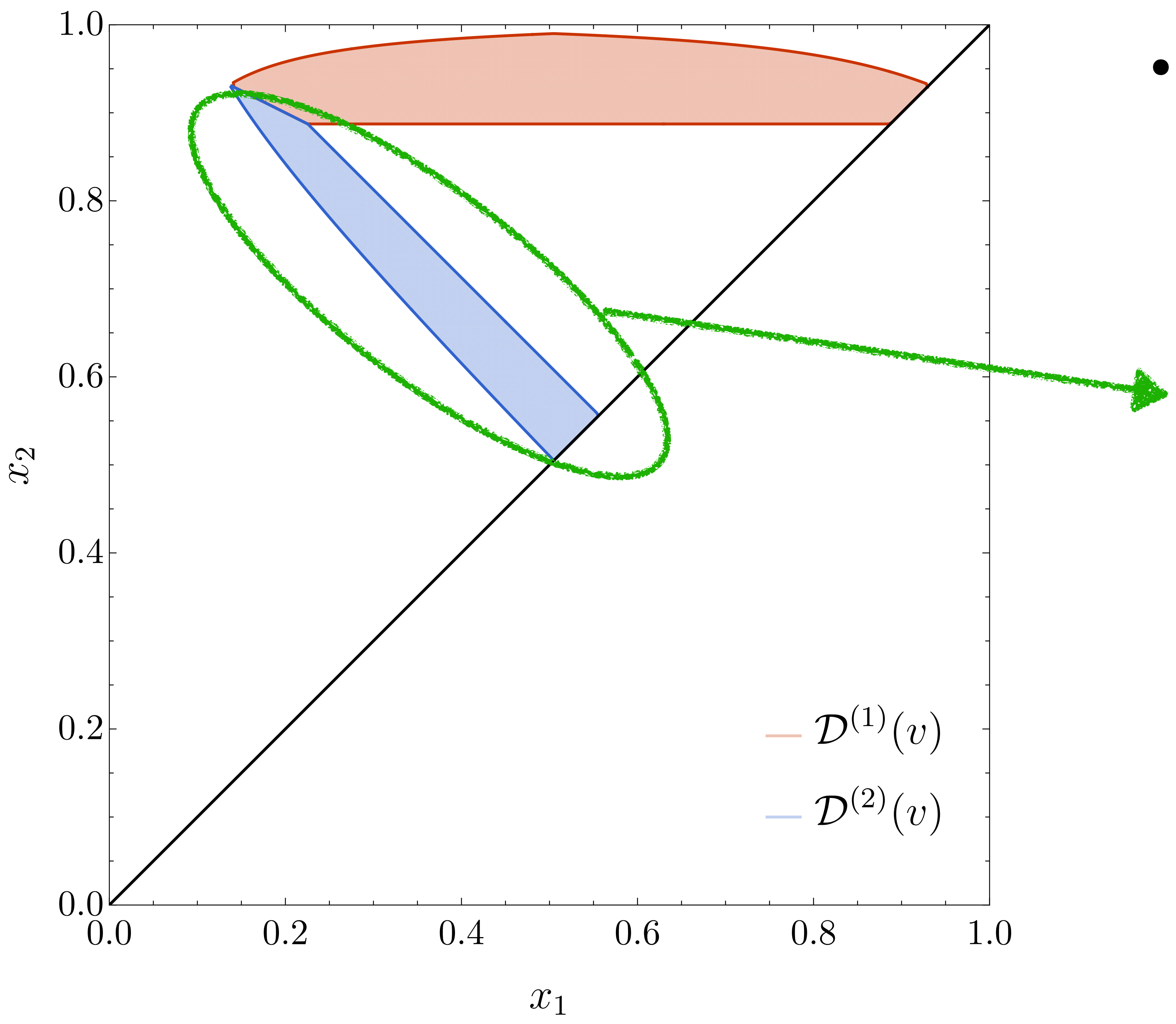
(The complete expression to all orders in v is reported in the paper)

Linear subleading power corrections

- This analysis **does not shed light on the physical origin** of the power corrections nor on the observed differences among the variables…

Comparison between τ_2 and $\tilde{y}_{23}$

- To gain further insight, we compare the regions covered by the variables τ_2 and $\tilde{y}_{23}$. The region responsible of the logarithmically enhanced term is given by the difference of these two regions.



- In particular, we see that the only the region close to the line $x_2 = 1 - x_1$ is responsible for the logarithmic power correction!

$$2\int_{D^{(2)}} dx_1 dx_2\, f(x_1, x_2) = -4v\log(1+\sqrt{2}) - 2v\log v + \mathcal{O}(v^2)$$

- This region corresponds to the physical configuration in which the **emitted gluon is hard** and recoils against the **collinear and/or soft quark-antiquark pair**.

Collinear expansion of the matrix element

- We now want to study the expansion of the matrix element in the singular limit $x_2 \to 1$, that corresponds to the configuration in which the momentum of the gluon becomes collinear to the one of the quark. This is the only singular limit that can be reached in the region $D^{(2)}$.

- The expansion of the matrix element in this limit is:

$$f(x_1, x_2) = \frac{1 + x_1^2}{(1 - x_1)(1 - x_2)} - \frac{2}{1 - x_1} + \mathcal{O}(1 - x_2) \equiv f_{\rm coll}^{(0)}(x_1, x_2) + f_{\rm coll}^{(1)}(x_1, x_2) + \mathcal{O}(1 - x_2)$$

$$2 \int_{D^{(2)}} dx_1 dx_2 \, f_{\rm coll}^{(0)}(x_1, x_2) = v \left(1 + 2 \log 2 - 4 \log(1 + \sqrt{2}) - 2 \log v \right) + \mathcal{O}(v^2)$$

- The **collinear approximation** of the matrix element is sufficient to capture the **logarithmically-enhanced linear power correction**.

- We associate this result to the fact that the phase space volume removed by a cut on $\tilde{y}_{23}$ scales quadratically with v, while it scales linearly for the case of τ_2.

- In conclusion, we shown that for τ_2 **the logarithmically enhanced power correction is a pure phase space effect!** This result is observable dependent.

Thrust

- The simple result obtained before for the logarithmic power correction of τ_2 does not hold for other variables, like thrust $1 - T$. In term of the energy fractions, thrust is given by:

$$1 - T = \min\{1 - x_1, 1 - x_2, 1 - x_3\}$$

- We report here the result for the cumulative cross-section up to $\mathcal{O}(v)$. **The result for $1 - T$ coincides with the one for τ_2 up to leading power**, including the constant term. This is due to the fact that τ_2 coincides with $1 - T$ for an appropriate choice of the jet axes.

$$R_{1-T}(v) = 2\log^2 v + 3\log v + \frac{5}{2} - \frac{\pi^2}{3} + 2v(2 - \log v) + \mathcal{O}(v^2)$$

- However, contrary to what happens for τ_2, the **subleading logarithmically-enhanced term** does not originate only from the hard gluon region, **but also from the region in which the gluon can be collinear to the quark**.

- In this case it is necessary to **include next-to-leading terms in the collinear expansion** of the matrix element to fully capture the logarithmically enhanced power correction.

- Our analysis is in agreement with the one performed in the SCET framework [Moult, Rothen, Stewart, Tackmann, Zhu (2017)]

C-parameter

- The same conclusions hold for the C-parameter. For massless final-state particles, C-parameter is given by:

$$C = 3 - \frac{3}{2} \sum_{i,j} \frac{(p_i \cdot p_j)^2}{(p_i \cdot q)(p_j \cdot q)}$$

- In the 2-jet limit $C \to 0$, thrust and C-parameter are related by $C = 6(1 - T)$, that is valid up to next-to-leading logarithmic accuracy. We can thus consider the variable $c = C/6$ that has the following expression in terms of energy fractions:

$$c = \frac{(1 - x_1)(1 - x_2)(1 - x_3)}{x_1 x_2 x_3}$$

- The evaluation of the cross-section in this case is more complicated and involves elliptic integrals [Gardi, Magnea (2003)][Agarwal, van Beekveld, Laenen, Mishra, Mukhopadhyay (2023)]. We find

$$R_c(v) = 2 \log^2 v + 3 \log v + \frac{5}{2} - \frac{2}{3}\pi^2 + v(7 - 4 \log v) + \mathcal{O}(v^2)$$

Summary for thrust and C-parameter

- Repeating the same region analysis, we have that also in this case the hard gluon and the collinear gluon regions contribute to the logarithmically enhanced power corrections.

- We report a summary of the result for the thrust and C-parameter. The symbol $\sim$ means that we are restricting the result to the logarithmically-enhanced correction

$$2\int_{D^{(2)}(v)} dx_1 dx_2\, f(x_1, x_2) \sim 2\int_{D^{(2)}(v)} dx_1 dx_2\, f^{(0)}_{\rm coll}(x_1, x_2) \sim -2v\log v$$

$$2\int_{D^{(1)}(v)} dx_1 dx_2 f(x_1, x_2) \sim \begin{cases} +4v\log v & \text{for } 1-T \\ +6v\log v & \text{for } c \end{cases}$$

The variable r_b

- We want to study a variable that depends on a parameter b that controls the dependence on the rapidity in the soft and collinear limits. Hence, we define:

$$r_b = (1-T)^b \, \tilde{y}_{23}^{1-b} \qquad\qquad r_b \simeq \left(\frac{k_t^{(\ell)}}{Q}\right) e^{-b\eta^{(\ell)}}$$

(Collinear and soft limits)

- Variables of this type have been studied in order to asses the logarithmic accuracy of Monte Carlo parton showers [Banfi, Salam, Zanderighi (2005)] [Dasgupta, Dreyer, Hamilton, Monni, Salam, Soyez (2020)].

- We find the following subleading power result for the cumulative cross-section for r_b

$$R_{r_b}(v) = \frac{2}{1+b}(2\log^2 v + 3\log v) + \frac{5}{2} - (1+b)\frac{\pi^2}{6} + 6\frac{1-b}{1+b}\log 2 + F_1(b)v + F_2(b)v^{\frac{2}{1+b}} + \mathcal{O}(v^2)$$

Subleading power correction of r_b

$$R_{r_b}(v) \sim F_1(b)v + F_2(b)v^{\frac{2}{1+b}} + \mathcal{O}(v^2)$$

$$F_1(b) = \frac{2^{\frac{5+b}{2}}b}{1+b} + 4B_{1/2}\left(-\frac{1+b}{2},0\right) - 2B_{1/2}\left(\frac{1-b}{2},0\right)$$

$$F_2(b) = 4B_{1/2}\left(\frac{b-1}{b+1},0\right) - 4B_{1/2}\left(\frac{2b}{1+b},0\right) + \frac{\Gamma\left(\frac{b-1}{b+1}\right)\left(4\left(b^4+3b^3+6b^2+b+1+\frac{b(b^3-7b^2+3b+3)}{b+1}B_{\frac{1}{2}}\left(\frac{b-1}{b+1},\frac{2}{b+1}\right)\right) - 4b^{\frac{2+b}{1+b}}(b+1)^2\right)}{(b+1)^3\,\Gamma\left(\frac{2b}{b+1}+1\right)} + \frac{5b^2+6b-3}{(1+b)^2}\left(\psi\left(\frac{b}{1+b}\right) - \psi\left(\frac{1+3b}{2(1+b)}\right)\right)$$

$$B_z(a,b) = \int_0^z dt\, t^{a-1}(1-t)^{b-1} \qquad\qquad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

- In this case we see that logarithmically enhanced terms are absent, but there are both linear and fractional subleading power corrections.

- For $b = 0$ and $b = 1$ we reproduce the results for $\tilde{y}_{23}$ and $1 - T$ respectively.

$$\lim_{b\to 0} R_{r_b}(v) = R_{\tilde{y}_{23}}(v) \qquad \lim_{b\to 1} R_{r_b}(v) = R_{1-T}(v)$$

Summary

- We studied the **subleading power corrections to event shape variables** in e^+e^-collisions, starting from τ_2 and y_{23}. Both variables have linear power corrections: for τ_2 the linear term is logarithmically enhanced, while for y_{23} is not.

- After computing the cumulative cross-section for these observables, **we discussed the origin of the different power corrections**. Our main observation is that these variables cover the phase space in different ways, and we **traced the origin of the logarithmically-enhanced term to specific kinematical configurations.**

- The **logarithmically-enhanced power correction for τ_2** can be obtained with a **collinear approximation** of the matrix element.

- For Thrust and C-parameter, the logarithmically-enhanced power corrections do not originate only form the hard gluon region. In addition, it cannot be reconstructed using a leading power collinear expansion of the real matrix element.

- We finally considered a class of observables r_b**, depending on a parameter b that gives different weights to central and forward emissions**. These variables have an highly non-trivial structure of the subleading power corrections.

- Recent studies of subleading power corrections were mostly carried out in the SCET framework, for Thrust and Jettiness. **Our result extend these findings to other observables, offering a different perspective on the structure of power corrections**.