Analytic Evaluation of Multiple Mellin-Barnes Integrals

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Based on: arXiv:2309.00409 & PhysRevLett.127.151601

Loops & Legs in QFT

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Outline

- Multiple Mellin-Barnes Representation
- Analytical Evaluation
  - Conic Hull Approach
  - Triangulation Approach
- Applications
- Conclusion & Outlook

Multiple MB Representation

Motivation

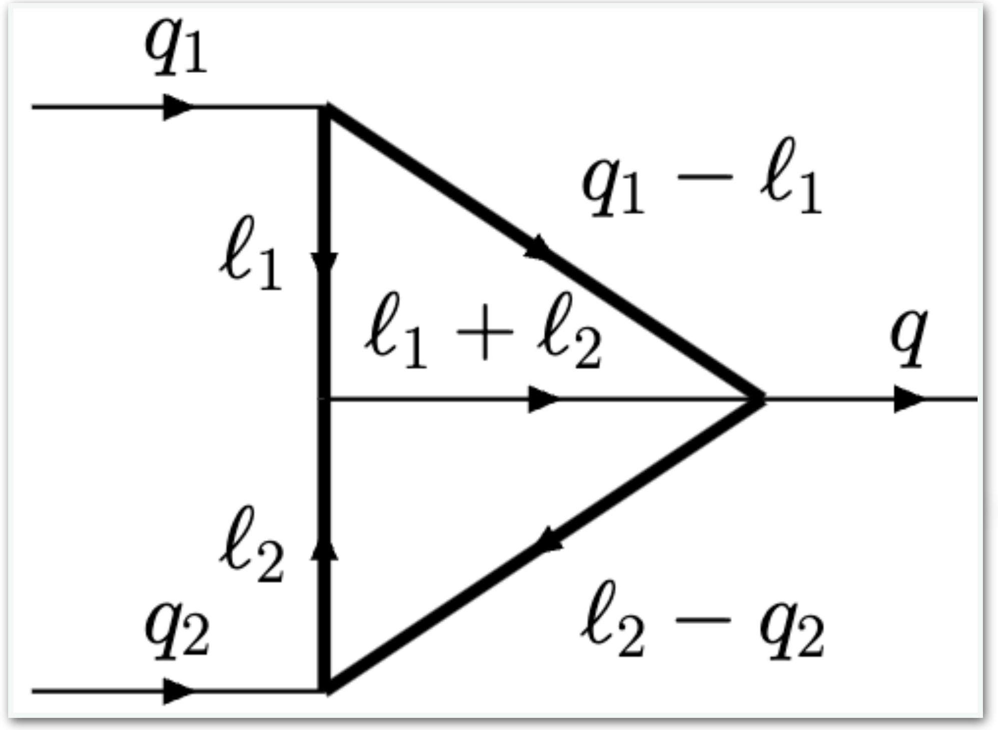
Overview: [arXiv: 2211.13733]

- Feynman Integrals can be evaluated using Mellin-Barnes (MB) Representation

$$\frac{1}{(A+B)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \Gamma(-z)\Gamma(\alpha+z) A^{-\alpha-z} B^{z}$$

Separate poles of $\Gamma(-z)$ and $\Gamma(\alpha+z)$

- Example: Two-Loop Triangle



[arXiv: 0704.2423]

AMBRE

Fold Scale

$$\int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi \mathrm{i}} \left(\frac{s}{4m^2}\right)^{z} \frac{\Gamma(-z)\Gamma^3(1+z)\Gamma(1+z+\epsilon)\Gamma(1+z+2\epsilon)}{\Gamma^2(2+z)\Gamma(2+z-\epsilon)\Gamma\left(\frac{3}{2}+z+\epsilon\right)}$$

- Useful for computing boundary conditions

Talk by H. Zhang

Multiple MB Representation

Motivation

- Hypergeometric Functions have MB representation

Useful for deriving analytic continuations

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{\Gamma(-z_1)\Gamma(a+z_1)\Gamma(b+z_1)}{\Gamma(c+z_1)} (-x)^{z_1} $$

- Multiple Polylogs special class of MB

# of numerator $\Gamma(\cdots)$ $\propto$ weights

$$ \mathrm{Li}_{m_1,m_2}(x_1,x_2) = x_1 x_2^2 \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_3}{2\pi i} \Gamma(-z_1)\Gamma(-z_2)\Gamma(1+z_1)\Gamma(1+z_2) \frac{\Gamma^{m_1}(1+z_1)\Gamma^{m_2}(2+z_{12})}{\Gamma^{m_1}(2+z_1)\Gamma^{m_2}(3+z_{12})} (-x_1x_2)^{z_1}(-x_2)^{z_2} $$

Multiple MB Representation

N-Fold Case

- N-fold MB Representation

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} \frac{\prod_{i=1}^{k} \Gamma^{a_i}(\mathbf{e}_i \cdot \mathbf{z} + g_i)}{\prod_{j=1}^{l} \Gamma^{b_j}(\mathbf{f}_j \cdot \mathbf{z} + h_j)} x_1^{z_1} \cdots x_N^{z_N}$$

$\mathbf{e}_i$ & $\mathbf{f}_j$

N-dimensional

Degenerate:

$$\sum_i \mathbf{e}_i - \sum_j \mathbf{f}_j = \mathbf{0}$$

$$\mathbf{z} = \{z_1, \cdots z_N\}$$

Analytic Evaluation

- MBConichulls.wl
- MBsums.m [arXiv: 1511.01323]

This talk

Numerical Evaluation

- MB.m
- MBresolve.m
- MBnumerics.m

[arXiv: 2211.00009]

Analytic Evaluation

Conic Hull Approach

- Find All Possible N-Combinations of Numerator Gamma functions
- Associate Series (Building Block) with each N-Combination
- Associate Conic Hull with each N-Combination
- Find Largest Subsets of Intersecting Conic Hulls
- Final Solution = Sum of Building Blocks associated with each Largest Subset

Intersecting Region (Master Conic Hull)

Several Solutions possible for a given MB

Analytic Evaluation

Evaluating Appell F_1 using Conic Hulls

$^5C_2 = 10$ possible 2-combinations

- Appell F_1 MB Representation:

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}(-u_1)^{z_1}(-u_2)^{z_2}\frac{\overset{1}{\Gamma}(-z_1)\overset{2}{\Gamma}(-z_2)\overset{3}{\Gamma}\left(a+z_1+z_2\right)\overset{4}{\Gamma}\left(b_1+z_1\right)\overset{5}{\Gamma}\left(b_2+z_2\right)}{\Gamma\left(c+z_1+z_2\right)}$$

- 2-Combinations of Numerator Gamma Functions

1. $\{\overset{1}{\Gamma}(-z_1), \overset{2}{\Gamma}(-z_2)\}$
2. $\{\overset{1}{\Gamma}(-z_1), \overset{3}{\Gamma}(a+z_1+z_2)\}$
3. $\{\overset{3}{\Gamma}(a+z_1+z_2), \overset{5}{\Gamma}(b_2+z_2)\}$
4. $\{\overset{4}{\Gamma}(b_1+z_1), \overset{5}{\Gamma}(b_2+z_2)\}$
5. $\{\overset{2}{\Gamma}(-z_2), \overset{4}{\Gamma}(b_1+z_1)\}$
6. $\{\overset{3}{\Gamma}(a+z_1+z_2), \overset{4}{\Gamma}(b_1+z_1)\}$
7. $\{\overset{2}{\Gamma}(-z_2), \overset{3}{\Gamma}(a+z_1+z_2)\}$
8. $\{\overset{1}{\Gamma}(-z_1), \overset{5}{\Gamma}(b_2+z_2)\}$

- Singular 2-Combinations Omitted: ✗ $\{\Gamma(-z_1), \Gamma(b_1+z_1)\}$ ✗ $\{\Gamma(-z_2), \Gamma(b_2+z_2)\}$

Analytic Evaluation

Evaluating Appell F_1 using Conic Hulls

Residues of poles of $\{\Gamma(-z_1), \Gamma(-z_2)\}$ at $(z_1, z_2) = (n_1, n_2)$

- 8 Associated Building Blocks

1

$$\{\overset{1}{\Gamma}(-z_1), \overset{2}{\Gamma}(-z_2)\}$$

$B_{1,2}$

$$\sum_{n_1,n_2=0}^{\infty} \frac{\Gamma(a+n_1+n_2)\Gamma(b_1+n_1)\Gamma(b_2+n_2)}{\Gamma(c+n_1+n_2)} \frac{u_1^{n_1} v_2^{n_2}}{n_1! n_2!}$$

- 8 Associated Conic Hulls

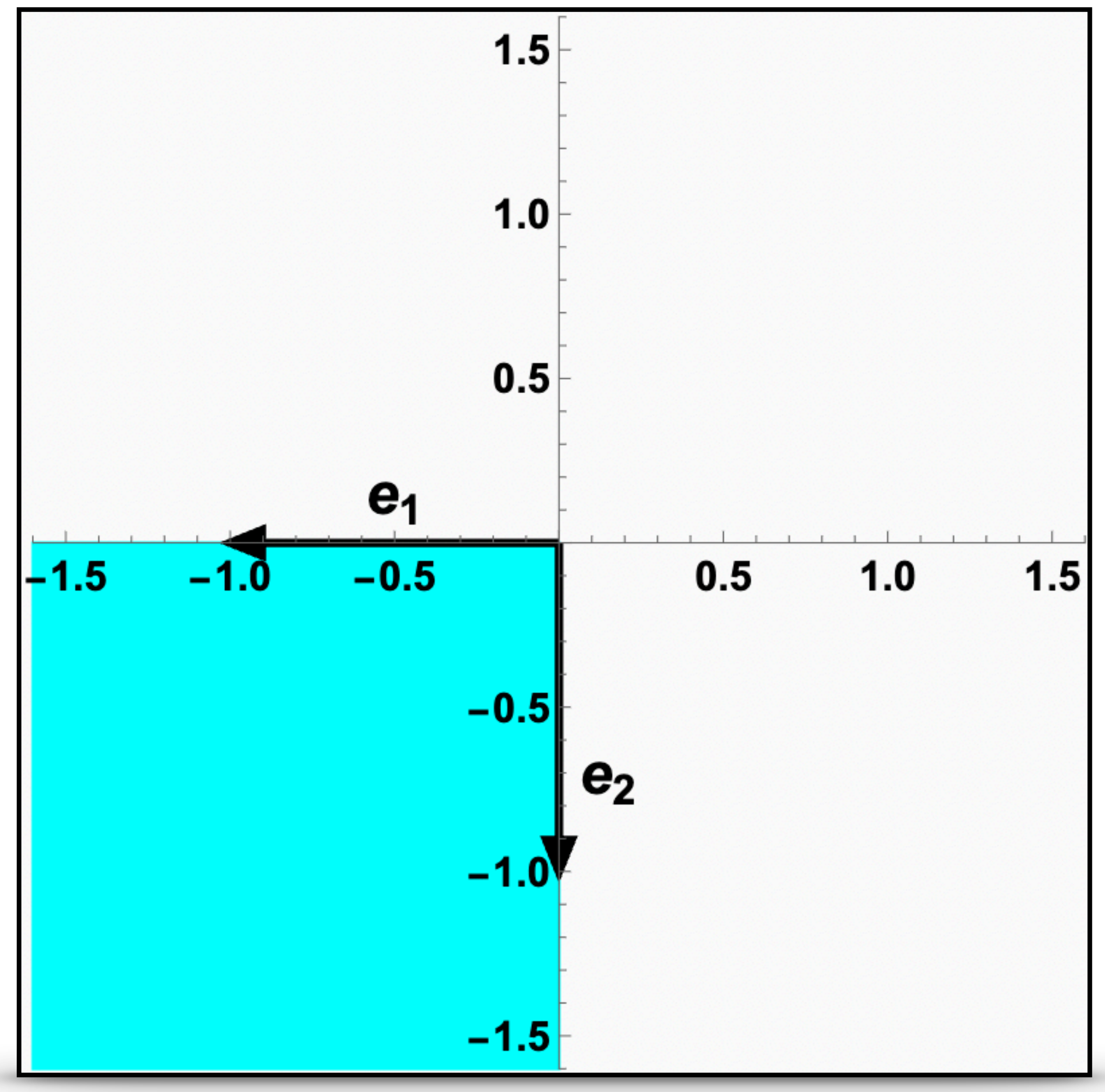
1

$$\{\overset{1}{\Gamma}(-z_1), \overset{2}{\Gamma}(-z_2)\}$$

$\vec{e}_1 = (-1,0)$

$\vec{e}_2 = (0,-1)$

$C_{1,2}$



Analytic Evaluation

Appell F_1 Solutions

- 5 Largest Subsets → 5 Series Solutions

Largest Intersecting Subset

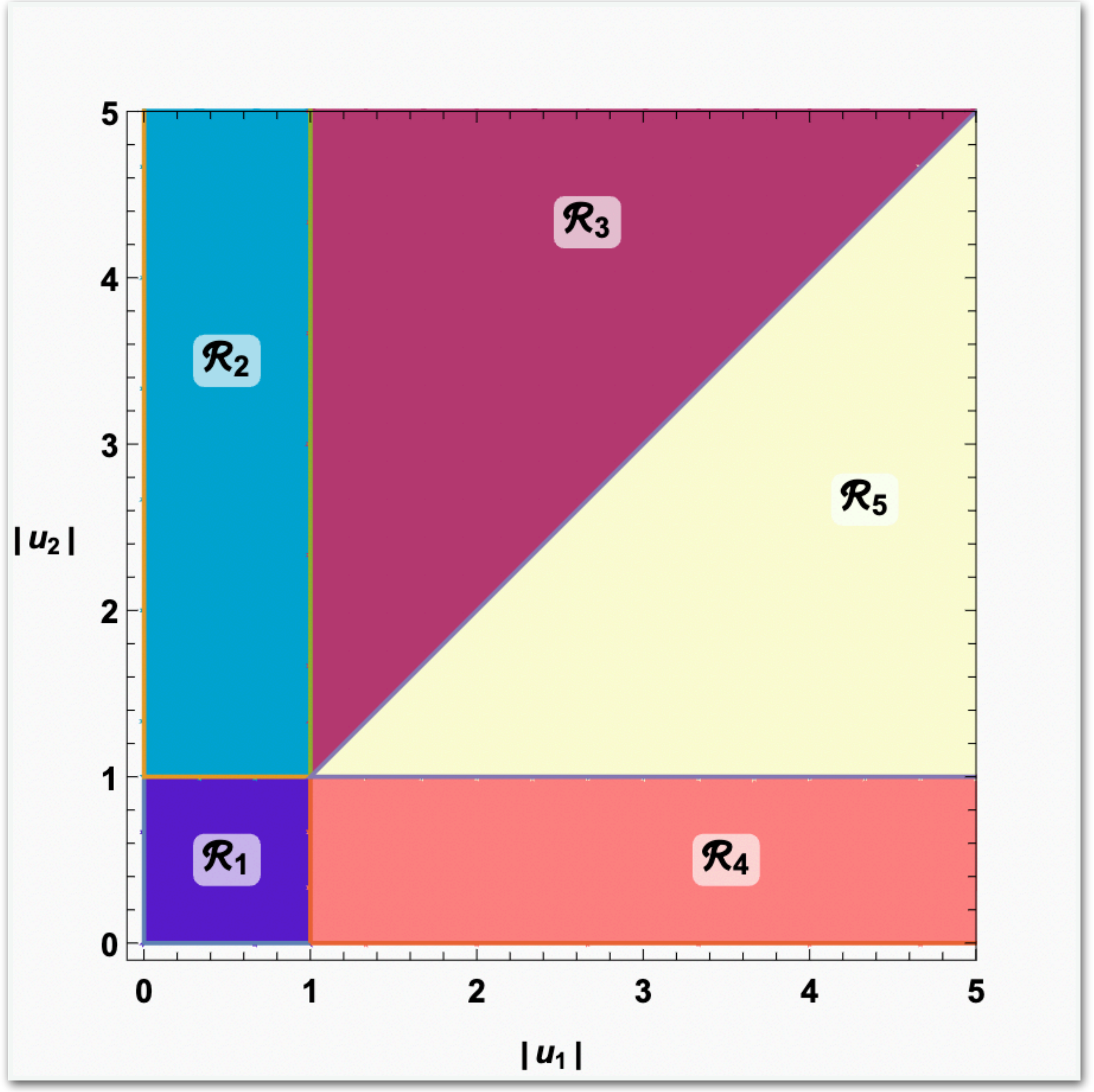
- Solutions:

1. $B_{1,2}$
2. $B_{1,3} + B_{3,5} + B_{4,5}$
3. $B_{1,3} + B_{1,5}$
4. $B_{2,3} + B_{2,4}$
5. $B_{2,3} + B_{3,4} + B_{4,5}$

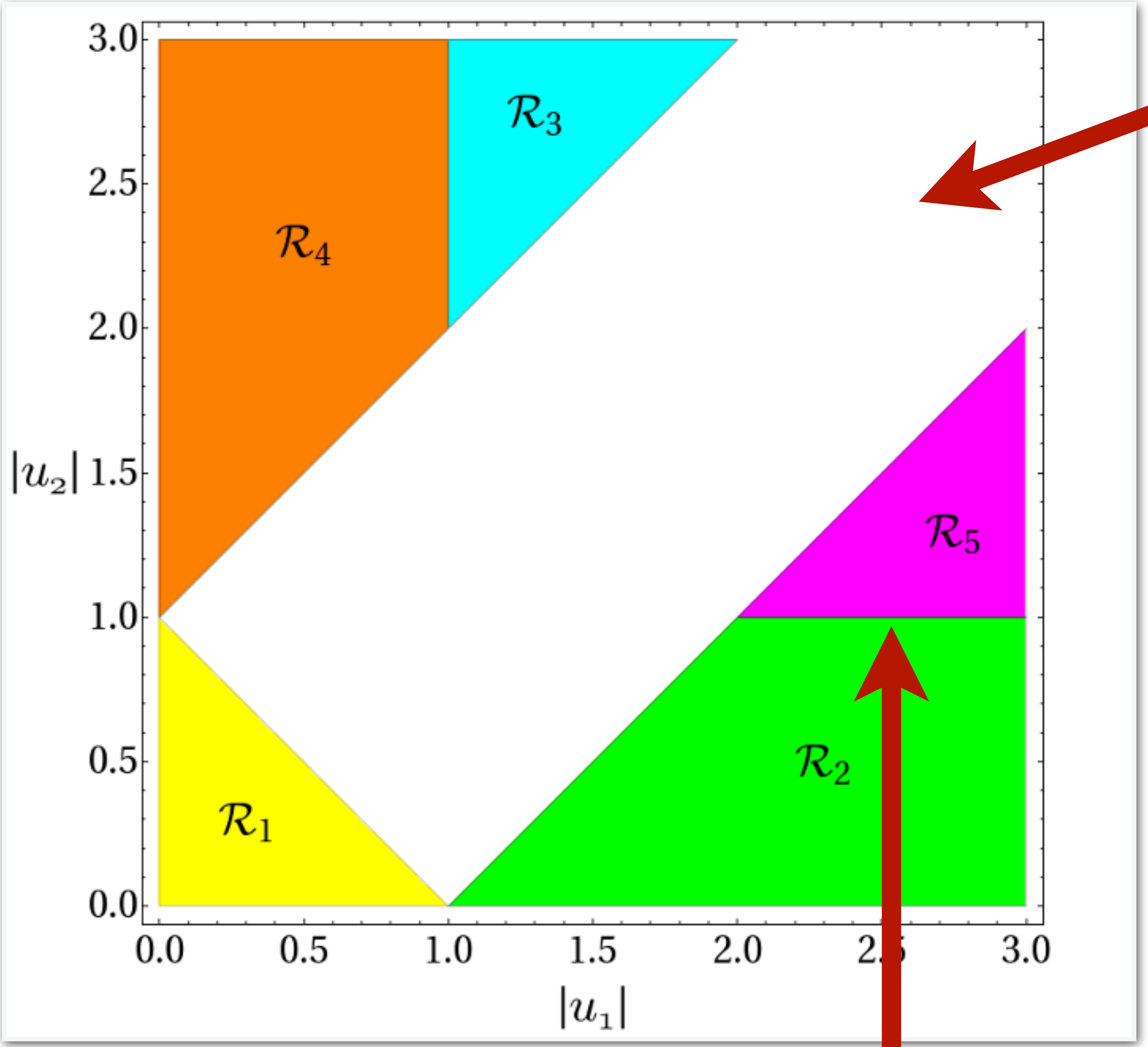
Analytic Evaluation

Challenges in Conic Hull Approach

- Convergent Solutions if # of Scales= # of Folds
- Full set of solutions may not always converge for all values (White Zone)



Appell F_1



$R_{-1}(u_1, u_2)$

- Final solutions may converge slowly near boundaries
- Slow for high-fold MB

Analytic Evaluation

Triangulation Approach

- Find All Possible N-Combinations of Numerator Gamma functions
- Associate Series (Building Block) with each N-Combination
- Associate Point Configuration
- Find all Possible Regular Triangulations
- Final Solution = Sum of Building Blocks associated with each Triangulation

Triangles in each triangulation mapped to building blocks

# of solutions = # of triangulations

Analytic Evaluation

Evaluating Appell F_1 using Triangulation

- Appell F_1 MB Representation:

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}(-u_1)^{z_1}(-u_2)^{z_2}\frac{\overset{1}{\Gamma}(-z_1)\overset{2}{\Gamma}(-z_2)\overset{3}{\Gamma}(a+z_1+z_2)\overset{4}{\Gamma}(b_1+z_1)\overset{5}{\Gamma}(b_2+z_2)}{\Gamma(c+z_1+z_2)}$$

- Point Configuration: # of points = # of numerator $\Gamma(\cdots)$

$$P_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad P_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad P_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad P_4 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad P_5 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

z_1 coefficients of non-trivial gamma

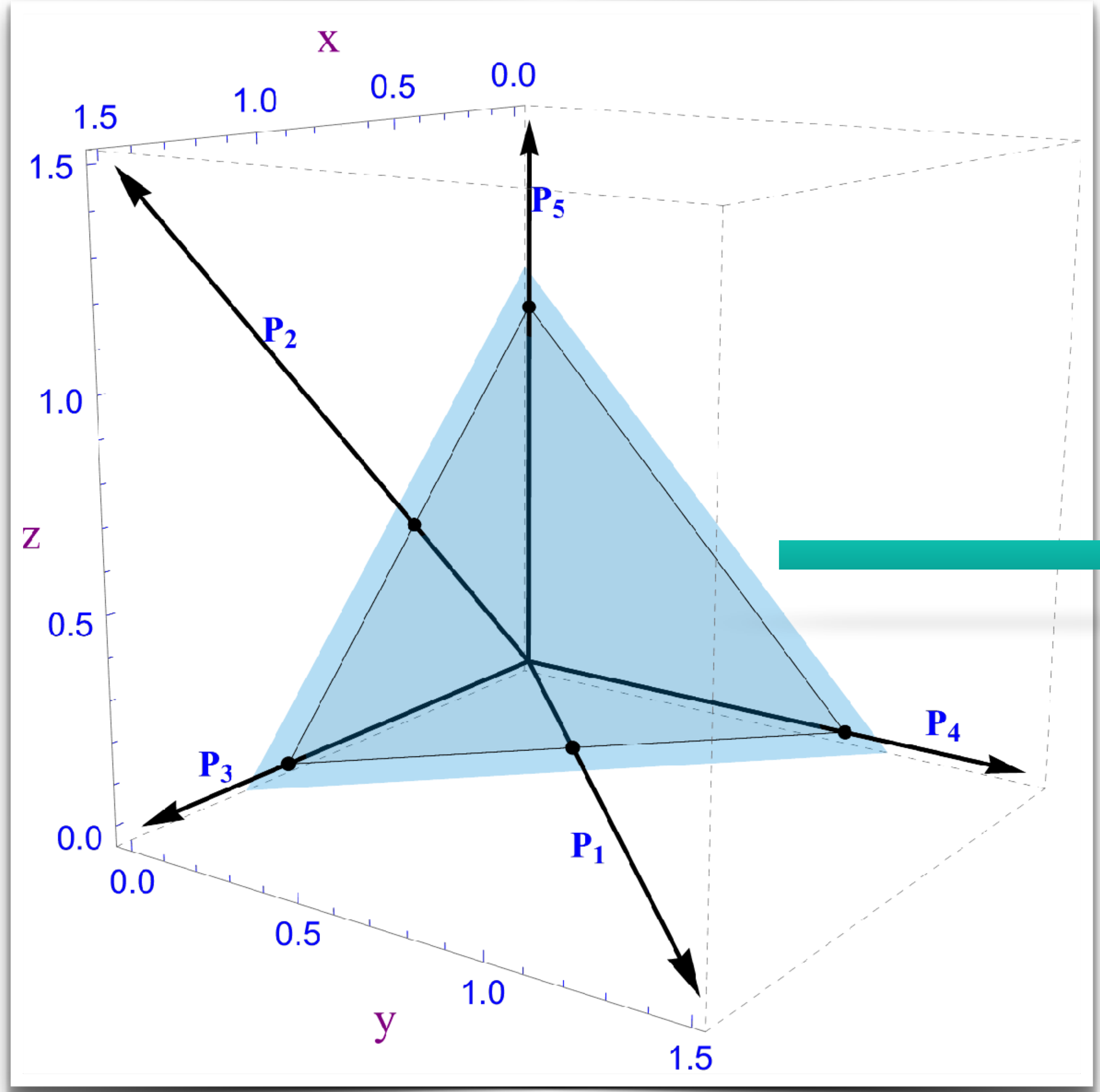
z_2 coefficients of non-trivial gamma

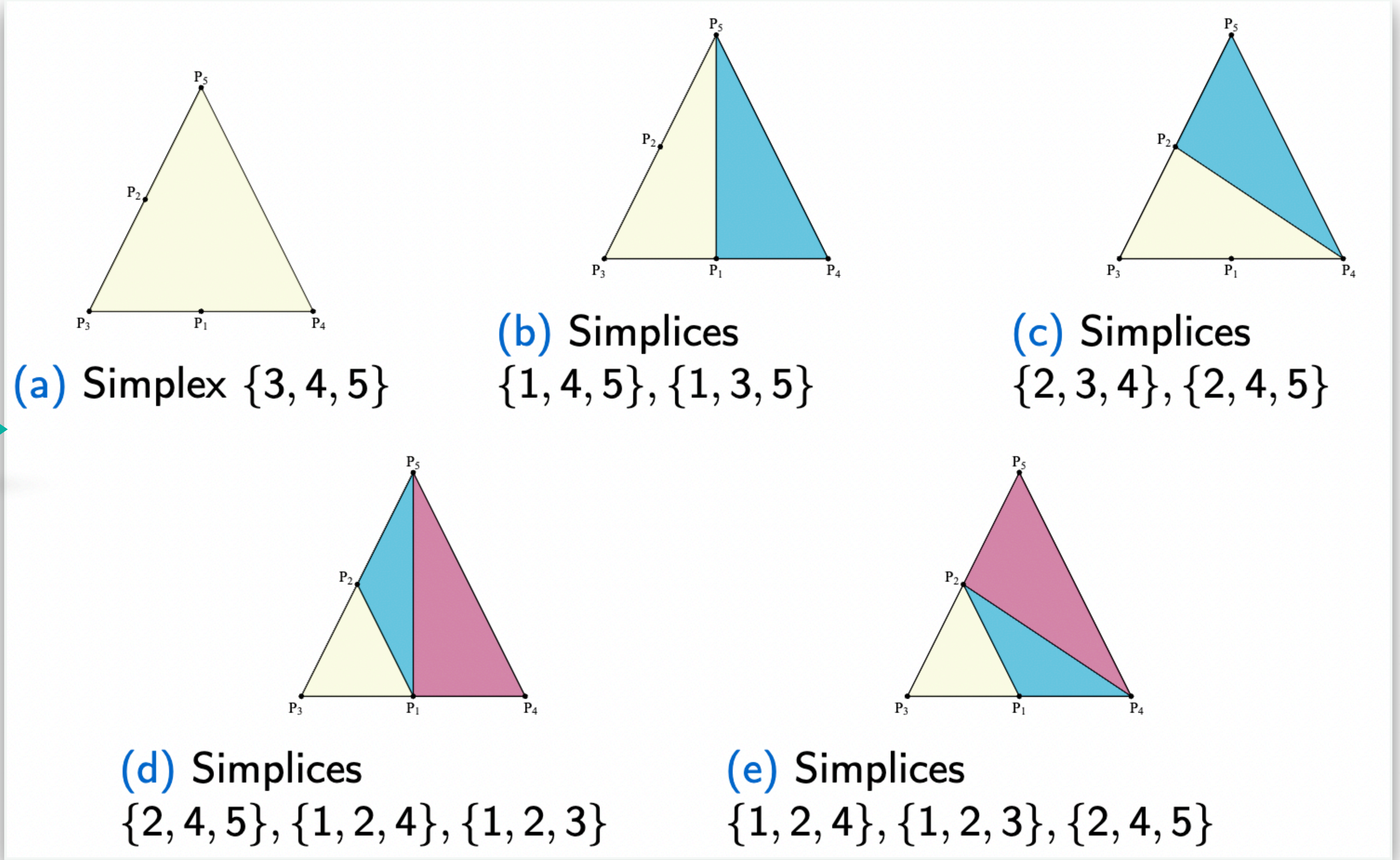
Unit Vectors

Analytic Evaluation

Evaluating Appell F_1 using Triangulation

- Point Configuration: $P = \{P_1, P_2, P_3, P_4, P_5\}$

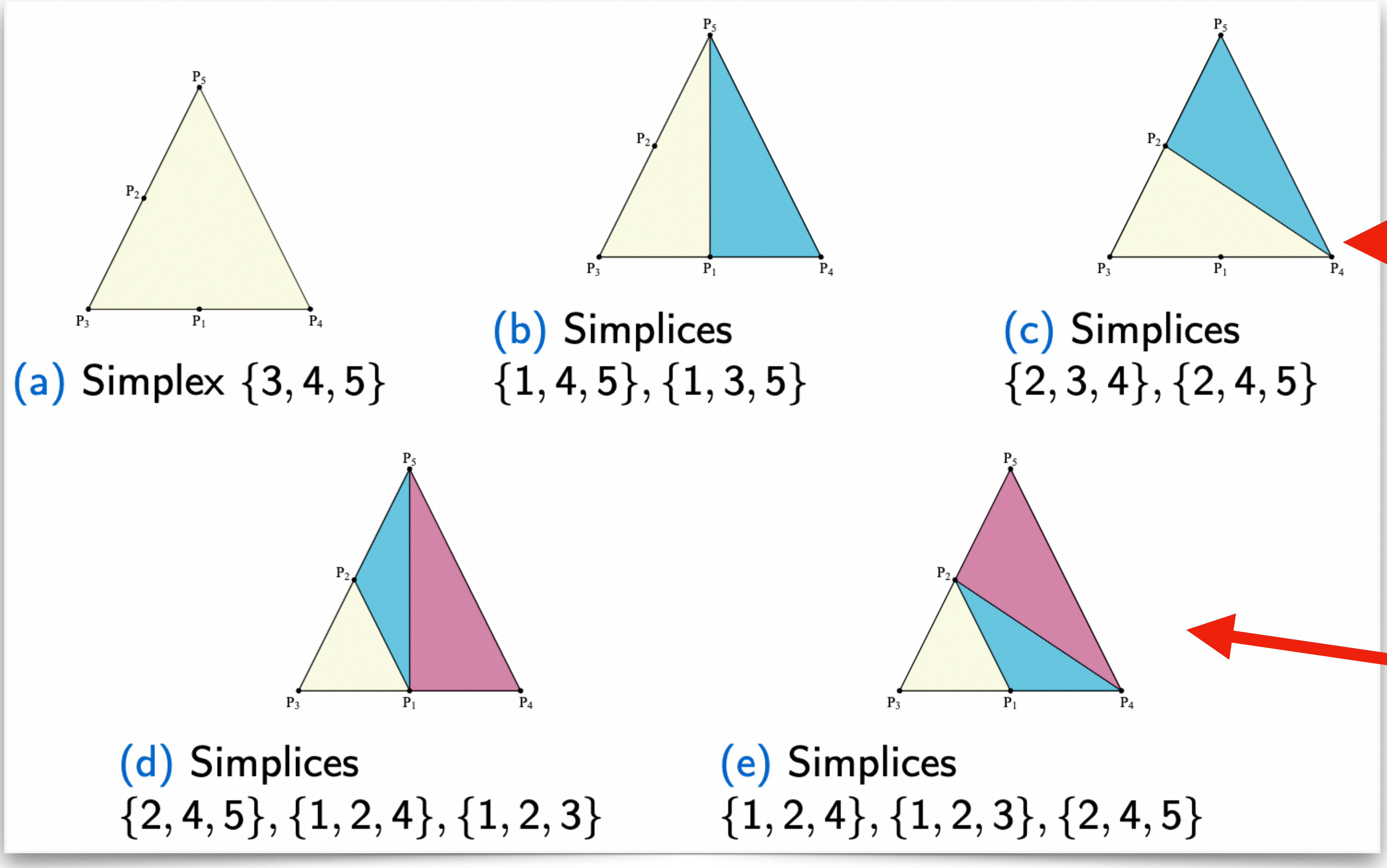




Analytic Evaluation

Evaluating Appell F_1 using Triangulation

- **Five** Possible Triangulations



Same solution as conic hull approach

1. $\{C_{1,2}\}$
2. $\{C_{1,3}, C_{1,5}\}$
3. $\{C_{2,3}, C_{2,4}\}$
4. $\{C_{1,3}, C_{3,5}, C_{4,5}\}$
5. $\{C_{2,3}, C_{3,4}, C_{4,5}\}$

- Take complement of $\{1,2,\cdots,5\}$ with each simplex in the triangulation

Analytic Evaluation

Speed Comparison

- MBConicHulls.wl for automated evaluation

TOPCOM interfaced with Mathematica

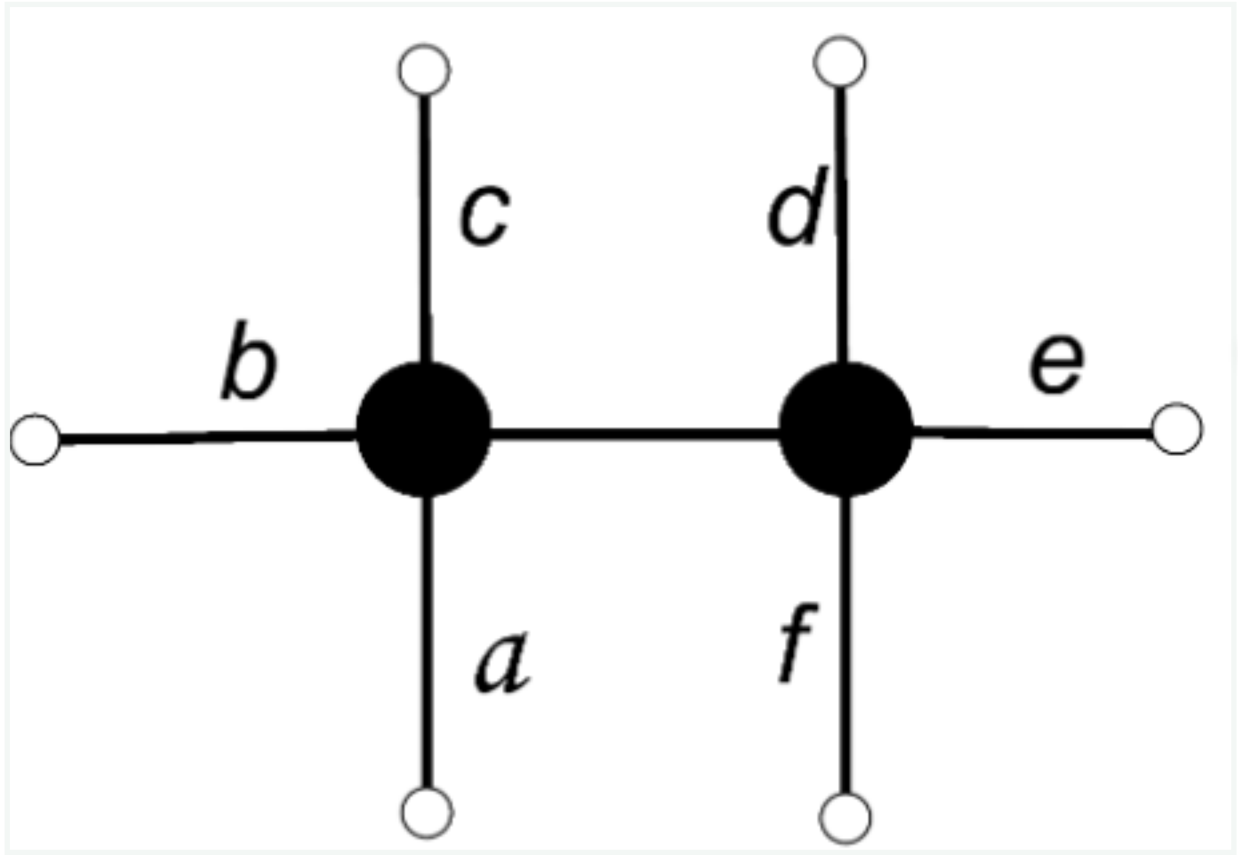
- Triangulation approach much faster than the conic hull approach

Feynman integral	MB folds	Total solution number	Conic hulls method		Triangulation method	
			One solution	All solutions	One solution	All solutions
Conformal triangle	3	14	0.186 sec.	1.44 sec.	0.543 sec.	0.483 sec.
Massless pentagon	5	70	1.276 sec.	1.25 h.	0.318 sec.	2.78 sec.
Conformal hexagon	9	194160	1 min.	-	0.489 sec.	40 min.
Conformal double-box	9	243186	1.9 min.	-	0.635 sec.	1.8 h.
Hard diagram	8	1471926	6 min.	-	1.4 sec.	-

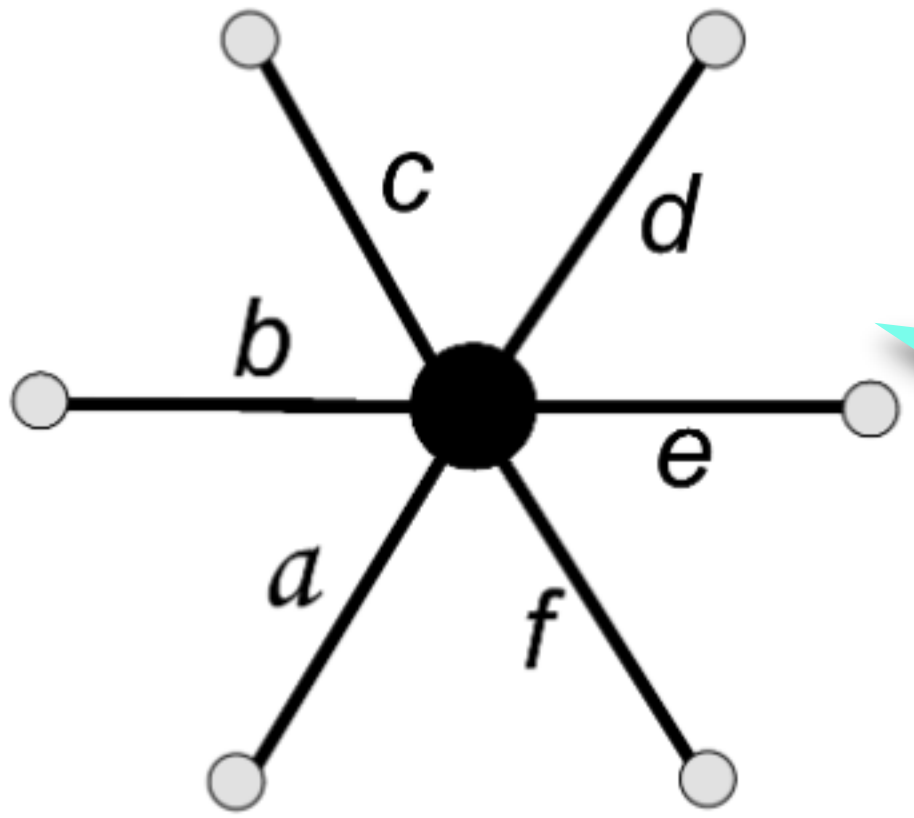
Double Box and Hexagon

New Solutions

- Recomputed conformal off-shell double box and hexagon
- 9-fold MB representation



4834 Building Blocks
Old Solution Length: 44



2530 Building Blocks
Old Solution Length: 26

- Simpler solution of length 25 found using triangulation approach

Multiple Polylogarithms

Analytic Continutions

- MPLs have a MB representation

$$\text{Li}_m(x) = x \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \Gamma(-z_1)\Gamma(1+z_1) \frac{\Gamma^m(1+z_1)}{\Gamma^m(2+z_1)} (-x)^{z_1}$$

- General Expressions:

[arXiv: 1311.1425]

$$\text{Li}_{m_1,\ldots,m_N}(x_1,\cdots,x_N) = \left(x_1 x_2^2 \cdots x_N^N\right) \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} \Gamma(-z_1)\cdots\Gamma(-z_N)\Gamma(1+z_1)\cdots\Gamma(1+z_N)$$

$$\times \frac{\Gamma^{m_1}(1+z_1)\Gamma^{m_2}(2+z_{12})}{\Gamma^{m_1}(2+z_1)\Gamma^{m_2}(3+z_{12})} \cdots \frac{\Gamma^{m_N}(N+z_{1\ldots N})}{\Gamma^{m_N}(N+1+z_{1\ldots N})} (-x_1 x_2 \cdots x_N)^{z_1} (-x_2 \cdots x_N)^{z_2} (-x_N)^{z_N}$$

- Degenerate MB and no white zones

Multiple Polylogarithms

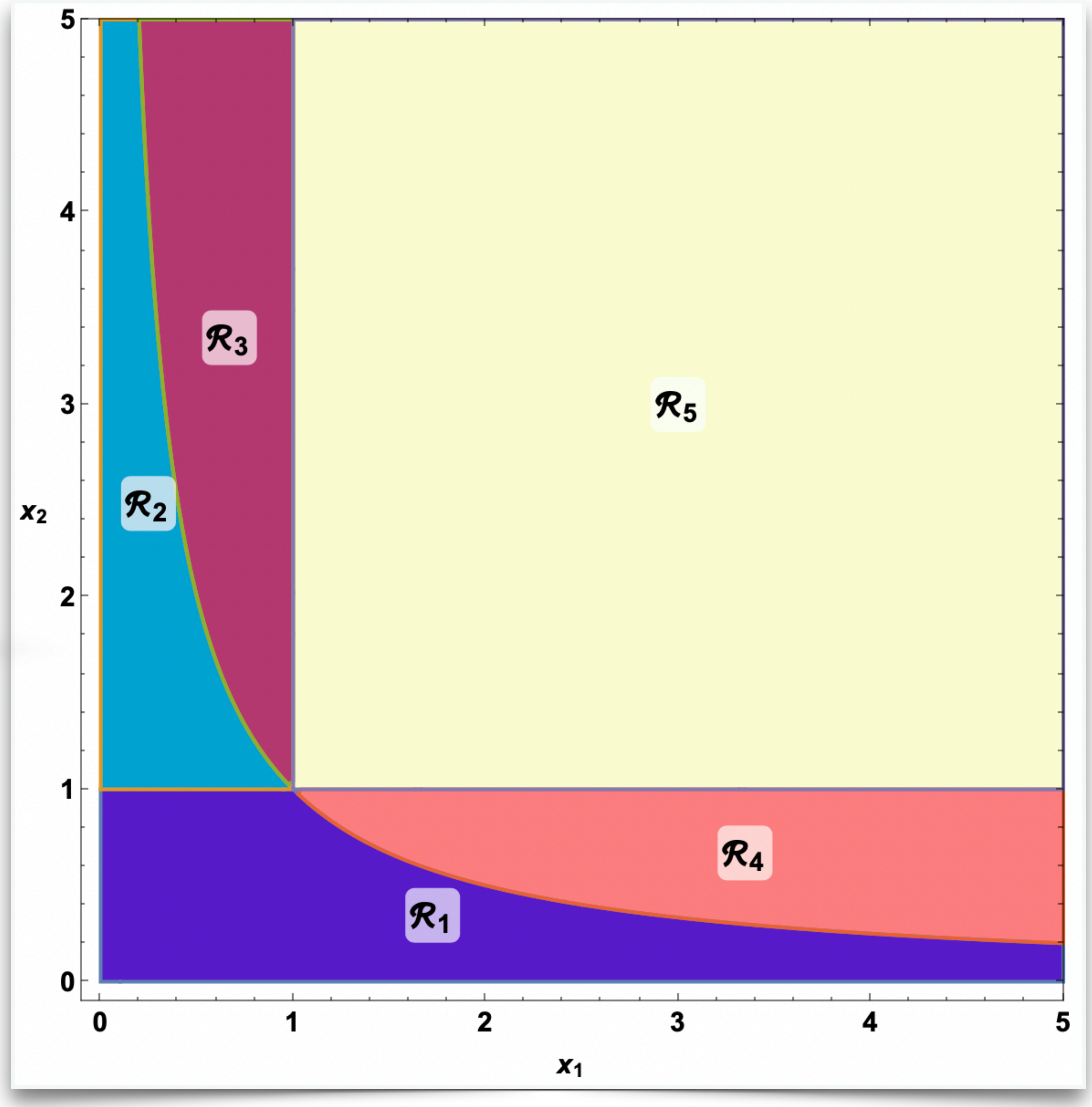
Analytic Continutions

- Example: $\mathrm{Li}_{1,1}(x_1, x_2)$

$$\mathrm{Li}_{1,1}(x_1,x_2) = x_1 x_2^2 \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_3}{2\pi i} \Gamma(-z_1)\Gamma(-z_2)\Gamma(1+z_1)\Gamma(1+z_2) \frac{\Gamma(1+z_1)\Gamma(2+z_{12})}{\Gamma(2+z_1)\Gamma(3+z_{12})} (-x_1 x_2)^{z_1} (-x_2)^{z_2}$$

- 11 conic hulls associated; 6 points for triangulations

- 5 series solutions with no white zones



Conclusion & Summary

- MB integrals can be solved using conic hulls and triangulations
- MBConicHulls.wl for automated evaluation
- Simpler solution of conformal double box and hexagon diagram
- MPLs special class of MB with no white zones

  For eMPLs?
- Reducing fold of MB integrals a key challenge

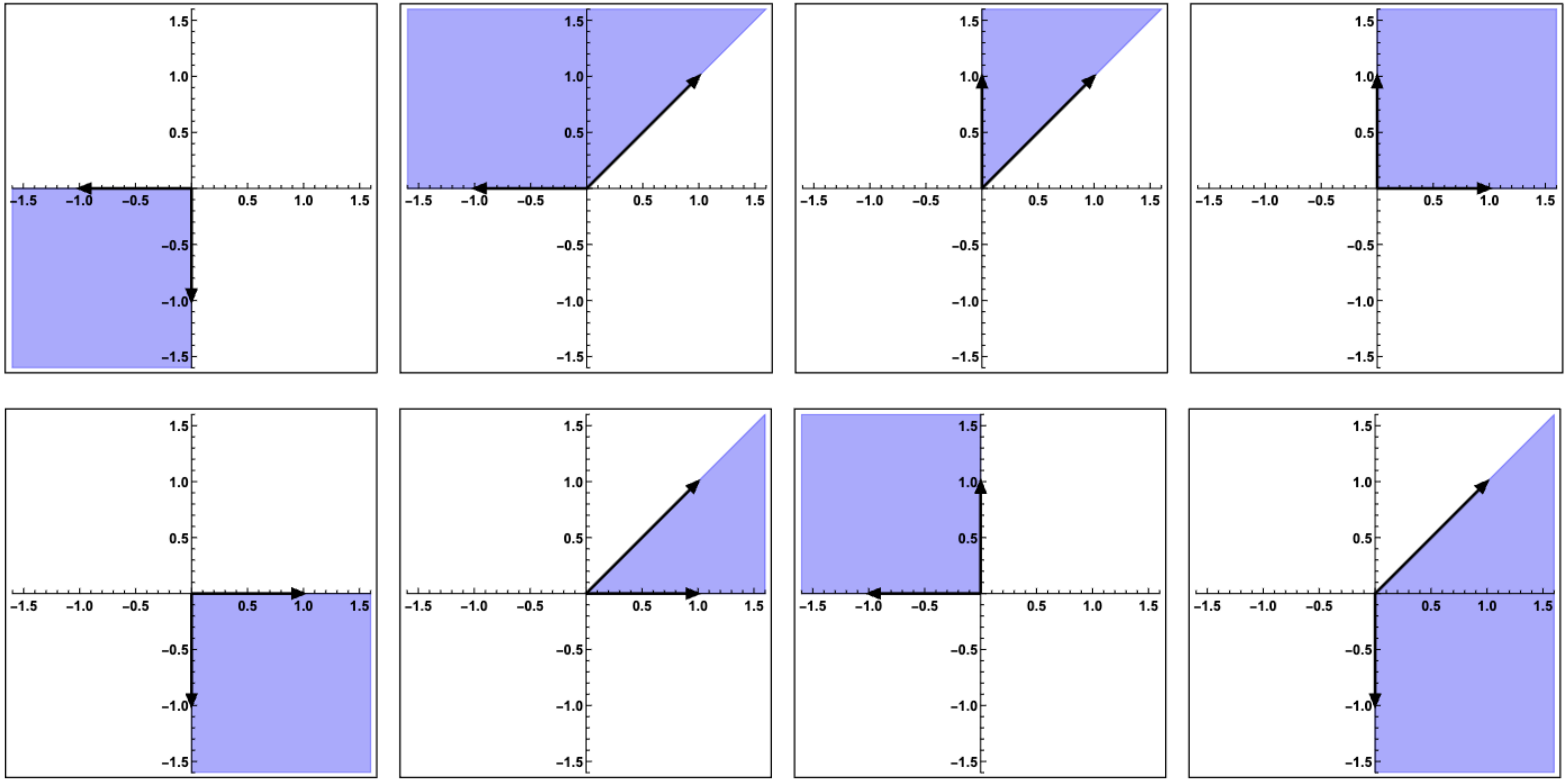
Thank you for your attention!

Backup

Analytic Evaluation

Appell F_1 Solutions

- All 8 Conic Hulls



Double Box and Hexagon

Numerical Comparison

Numerical Comparison for Hexagon		
Upper Sum Limit	Series Representation (Time)	Feynman Parametrization (Time)
2	636.76884 (14 sec)	636.76882 (9 Hours)

Table: Computed for $u_1 = 1, u_2 = 10^{12}, u_3 = 1/10^{12}, u_4 = 1, u_5 = 1, u_6 = 100, u_7 = 1/100, u_8 = 10000, u_9 = 1/10^8$ for propagator powers $a = 42/100, b = 11/100, c = 15/100, d = 32/100, e = 59/100, f = 55/100$

Multiple Polylogarithms

Analytic Continutions

- Numerical comparison with GiNaC for $\mathrm{Li}_{1,1,1}(x_1, x_2, x_3)$ [Link]

In[87]:=
```
Sub = {x1 → 0.03, x2 → 0.02, x3 → 10};
Lim = 80;
SumAllSeries[PolyLogWMBSeriesOut, Sub, Lim, RunInParallel → True];
```

```
Numerical Result: -0.000409347 - 0.0000191085 i
Time Taken 2.63637 seconds
```

In[90]:=
```
Sub = {x1 → 0.03, x2 → 0.02, x3 → 10};
SubVal = XVars /. Sub;
Ginsh[Li[WeightList, SubVal], {}]
```

Out[92]=
```
-0.0004093467863786605081355196475494932002 - 0.000019108543116203301158737782799149320410 i
```