Worldline integration of photon amplitudes

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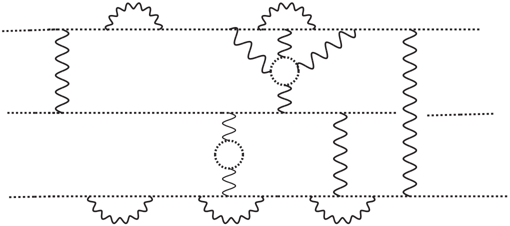
Loops and Legs in Quantum Field Theory
Wittenberg, April 15, 2024

Content

- Worldline representation of the QED S-matrix.
- One-loop photon amplitudes in vacuum.
- One-loop photon amplitudes in external fields:
  - Constant fields.
  - Plane-wave fields.
  - Constant plus plane-wave fields.
- Multi-loop photon amplitudes.
- Some remarks on gluon amplitudes.

Worldline representation of the QED S-matrix

Feynman 1948, 1950, 1951: Representation of the perturbative QED S-matrix in terms of particle path integrals interconnected by photons in all possible ways:



Equivalent to Feynman diagrams, but

1. Avoids the break-up of the scalar/spinor lines or loops into individual propagators.
2. A priori does not require ordering of the photon legs along a line or loop.

One-loop N-photon amplitudes in scalar QED

$$\Gamma[\{k_i,\varepsilon_i\}] = (-ie)^N \int \frac{dT}{T}\, \mathrm{e}^{-m^2 T} \int \mathcal{D}x\, V^A_{\mathrm{scal}}[k_1,\varepsilon_1] \ldots V^A_{\mathrm{scal}}[k_N,\varepsilon_N] \mathrm{e}^{-\int_0^T d\tau \frac{\dot{x}^2}{4}}$$

V^A_{scal} denotes the same photon vertex operator as is used in string perturbation theory,

$$V^A_{\mathrm{scal}}[k,\varepsilon] = \int_0^T d\tau\, \varepsilon \cdot \dot{x}(\tau)\, \mathrm{e}^{ikx(\tau)}$$

The zero mode $x_0 = \frac{1}{T}\int_0^T d\tau x(\tau)$ factors out and produces the momentum conservation factor $(2\pi)^D \delta(\sum k_i)$.

Master formula for the N-photon amplitudes in scalar QED

Polyakov 1987, Bern-Kosower 1991, Strassler 1992:

$$\Gamma[\{k_i,\varepsilon_i\}] = (-ie)^N \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times \exp\Big\{ \sum_{i,j=1}^N \big[\frac{1}{2} G_{ij} k_i \cdot k_j + i\dot{G}_{ij} k_i \cdot \varepsilon_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i \cdot \varepsilon_j\big]\Big\} \big|_{\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)}$$

$$G(\tau_1,\tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1-\tau_2)^2}{T}$$
$$\dot{G}(\tau_1,\tau_2) = \mathrm{sgn}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}$$
$$\ddot{G}(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2) - \frac{2}{T}$$

T is the proper-time of the scalar particle in the loop, τ_i parametrizes the position of photon i along the loop.

Master formula for the N-photon amplitudes in spinor QED

Spin can be incorporated by an additional Grassmann path integral (Fradkin 1966).
In practice it is usually preferable to use a certain integration-by-parts algorithm, removing the $\ddot{G}_{ij}$, and the Bern-Kosower replacement rule

$$\dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_n i_1} \rightarrow \dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_n i_1} - G_{Fi_1 i_2}G_{Fi_2 i_3}\cdots G_{Fi_n i_1} \; ,$$

involving the "τ-cycles" $\dot{G}_{i_1 i_2}\dot{G}_{i_2 i_3}\cdots\dot{G}_{i_n i_1}$ and the fermionic worldline Green's function G_F,

$$G_F(\tau,\tau') \equiv \mathrm{sgn}(\tau-\tau')$$

Bern and Kosower 1991 (on-shell gluon case), Strassler 1992 (off-shell photon case), C S 1997 (maintaining full permutation invariance)

Worldline representation of the four-photon amplitude

After the IBP, the four-photon amplitude in spinor QED appears naturally decomposed as follows:

$$\Gamma_{\rm spin}(k_1,\varepsilon_1,\dots,k_4,\varepsilon_4) \quad = \quad -\frac{e^4}{8\pi^2}\Big(\Gamma^{(1)}+\Gamma^{(2)}+\Gamma^{(3)}+\Gamma^{(4)}+\Gamma^{(5)}\Big)\,,$$

$$\begin{aligned}
\Gamma^{(1)} &= \Gamma^{(1)}_{(1234)}T^{(1)}_{(1234)}+\Gamma^{(1)}_{(1243)}T^{(1)}_{(1243)}+\Gamma^{(1)}_{(1324)}T^{(1)}_{(1324)}\,,\\
\Gamma^{(2)} &= \Gamma^{(2)}_{(12)(34)}T^{(2)}_{(12)(34)}+\Gamma^{(2)}_{(13)(24)}T^{(2)}_{(13)(24)}+\Gamma^{(2)}_{(14)(23)}T^{(2)}_{(14)(23)}\,,\\
\Gamma^{(3)} &= \sum_{i=1,2,3}\Gamma^{(3)}_{(123)i}T^{(3)r_4}_{(123)i}+\sum_{i=2,3,4}\Gamma^{(3)}_{(234)i}T^{(3)r_1}_{(234)i}+\sum_{i=3,4,1}\Gamma^{(3)}_{(341)i}T^{(3)r_2}_{(341)i}+\sum_{i=4,1,2}\Gamma^{(3)}_{(412)i}T^{(3)r_3}_{(412)i}\,,\\
\Gamma^{(4)} &= \sum_{i<j}\Gamma^{(4)}_{(ij)ii}T^{(4)}_{(ij)ii}+\sum_{i<j}\Gamma^{(4)}_{(ij)jj}T^{(4)}_{(ij)jj}\,,\\
\Gamma^{(5)} &= \sum_{i<j}\Gamma^{(5)}_{(ij)ij}T^{(5)}_{(ij)ij}+\sum_{i<j}\Gamma^{(5)}_{(ij)ji}T^{(5)}_{(ij)ji}\,.
\end{aligned}$$

Tensor basis for the four-photon amplitude

$$\begin{aligned}
T^{(1)}_{(1234)} &\equiv Z_4(1234)\,, \\
T^{(2)}_{(12)(34)} &\equiv Z_2(12)Z_2(34)\,, \\
T^{(3)r_4}_{(123)i} &\equiv Z_3(123)\,\frac{r_4\cdot f_4\cdot k_i}{r_4\cdot k_4}\,, \quad i=1,2,3\,, \\
T^{(4)}_{(12)ii} &\equiv Z_2(12)\,\frac{k_i\cdot f_3\cdot f_4\cdot k_i}{k_3\cdot k_4}\,, \quad i=1,2, \\
T^{(5)}_{(12)ij} &\equiv Z_2(12)\,\frac{k_i\cdot f_3\cdot f_4\cdot k_j}{k_3\cdot k_4}\,, \quad (i,j)=(1,2),(2,1).
\end{aligned}$$

$$\begin{aligned}
f_i^{\mu\nu} &\equiv k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu\,, \text{(photon field strength tensor)} \\
Z_2(ij) &\equiv \frac{1}{2}\mathrm{tr}\left(f_i f_j\right) = \varepsilon_i\cdot k_j \varepsilon_j\cdot k_i - \varepsilon_i\cdot\varepsilon_j k_i\cdot k_j\,, \\
Z_n(i_1 i_2 \ldots i_n) &\equiv \mathrm{tr}\Big(\prod_{j=1}^{n} f_{i_j}\Big)\,, \quad (n\geq 3)\,. \quad \text{(``Lorentz cycle'')}
\end{aligned}$$

Coefficient functions for the four-photon amplitude

$$\Gamma^{(k)}_{\dots} = \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \, \mathrm{e}^{-m^2 T} \int_0^1 \prod_{i=1}^4 du_i \, \Gamma^{(k)}_{\dots}(\dot{G}_{ij}) \, \mathrm{e}^{\frac{1}{2} T \sum_{i,j=1}^4 G_{ij} k_i \cdot k_j}$$

$$\begin{aligned}
\Gamma^{(1)}_{(1234)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41} \,, \\
\Gamma^{(2)}_{(12)(34)} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\left(\dot{G}_{34}\dot{G}_{43} - G_{F34}G_{F43}\right) , \\
\Gamma^{(3)}_{(123)1} &= \left(\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31}\right)\dot{G}_{41} \,, \\
\Gamma^{(4)}_{(12)11} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\dot{G}_{13}\dot{G}_{41} \,, \\
\Gamma^{(5)}_{(12)12} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\dot{G}_{13}\dot{G}_{42}
\end{aligned}$$

$$\begin{aligned}
G_{ij} &= |u_i - u_j| - (u_i - u_j)^2 \\
G_{Fij} &= \mathrm{sgn}(u_i - u_j)
\end{aligned}$$

Results on four-photon amplitudes in the worldline approach

In previous work, the coefficient functions have been evaluated for the off-shell case, but with two legs taken in the low-energy limit: N. Ahmadiniaz, C. Lopez-Arcos, M. A. Lopez-Lopez, C. S., Nucl. Phys. B **991** (2023) 116216, Nucl. Phys. B **991** (2023) 116217.

In preparation: Victor M. Banda, James P. Edwards, C. Moctezuma Mata, Luis A. Rodriguez Chacón and C.S.:

1. On-shell coefficient functions for both scalar and spinor QED.
2. Tables of worldline integrals that allow one to do all integrals without fixing the ordering of the photons.

Worldline integrals with four on-shell legs (1)

$$\Lambda \equiv -\frac{T}{2}\Big[(G_{12}+G_{34})s+(G_{13}+G_{24})t+(G_{14}+G_{23})u\Big]$$

$$\begin{aligned}\int_0^1 du_4\ \mathrm{e}^{\Lambda} \quad = \quad & \frac{1}{T}\left[\frac{2}{u+\dot G_{12}t+\dot G_{13}s}+\frac{2}{u-\dot G_{12}t-\dot G_{13}s}\right]\ \mathrm{e}^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT} \\ & +\frac{1}{T}\left[\frac{2}{t+\dot G_{23}s+\dot G_{21}u}+\frac{2}{t-\dot G_{23}s-\dot G_{21}u}\right]\ \mathrm{e}^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT} \\ & +\frac{1}{T}\left[\frac{2}{s+\dot G_{31}u+\dot G_{32}t}+\frac{2}{s-\dot G_{31}u-\dot G_{32}t}\right]\ \mathrm{e}^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT}\ .\end{aligned} \tag{1}$$

Worldline integrals with four on-shell legs (2)

$$\Lambda \equiv -\frac{T}{2}\left[(G_{12}+G_{34})s+(G_{13}+G_{24})t+(G_{14}+G_{23})u\right]$$

$$\begin{aligned}
\int_0^1 du_4\,\dot G_{41}\,\mathrm{e}^{\Lambda} \quad = \quad & \Biggl[-\frac{8}{T^2(u+\dot G_{12}t+\dot G_{13}s)^2}+\frac{8}{T^2(u-\dot G_{12}t-\dot G_{13}s)^2}\\
&+\frac{2}{T(u+\dot G_{12}t+\dot G_{13}s)}-\frac{2}{T(u-\dot G_{12}t-\dot G_{13}s)}\Biggr]\,\mathrm{e}^{\frac{1}{2}(G_{12}+G_{13}-G_{23})uT}\\
&+\Biggl[-\frac{8}{T^2(t+\dot G_{23}s+\dot G_{21}u)^2}+\frac{8}{T^2(t-\dot G_{23}s-\dot G_{21}u)^2}\\
&-\frac{2\dot G_{12}}{T(t+\dot G_{23}s+\dot G_{21}u)}-\frac{2\dot G_{12}}{T(t-\dot G_{23}s-\dot G_{21}u)}\Biggr]\,\mathrm{e}^{\frac{1}{2}(G_{12}+G_{23}-G_{13})tT}\\
&+\Biggl[-\frac{8}{T^2(s+\dot G_{31}u+\dot G_{32}t)^2}+\frac{8}{T^2(s-\dot G_{31}u-\dot G_{32}t)^2}\\
&-\frac{2\dot G_{13}}{T(s+\dot G_{31}u+\dot G_{32}t)}-\frac{2\dot G_{13}}{T(s-\dot G_{31}u-\dot G_{32}t)}\Biggr]\,\mathrm{e}^{\frac{1}{2}(G_{13}+G_{23}-G_{12})sT}\,.
\end{aligned} \tag{2}$$

Example of a coefficient function (1)

$$\int_0^\infty \frac{dT}{T}\, T^{4-\frac{D}{2}}\, e^{-m^2 T} \int_{(4)} \mathrm{e}^{T\,\Lambda}\, \dot{G}_{12}\dot{G}_{21}\dot{G}_{13}\dot{G}_{42} \quad = \quad r^{(1)}_{(12)12} + r^{(2)}_{(12)12} \ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right) + r^{(3)}_{(12)12} \ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)$$

$$+r^{(4)}_{(12)12} \ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right) + r^{(5)}_{(12)12}\left[\ln\left(\frac{\beta_{\hat{s}}-1}{\beta_{\hat{s}}+1}\right)\right]^2 + r^{(6)}_{(12)12}\left[\ln\left(\frac{\beta_{\hat{t}}-1}{\beta_{\hat{t}}+1}\right)\right]^2 + r^{(7)}_{(12)12}\left[\ln\left(\frac{\beta_{\hat{u}}-1}{\beta_{\hat{u}}+1}\right)\right]^2$$

$$+r^{(8)}_{(12)12}\bar{B}(s,t,u) + r^{(9)}_{(12)12}\bar{B}(s,u,t) + r^{(10)}_{(12)12}\bar{B}(t,u,s),$$

Here we have introduced

$$\beta_{\hat{s}} \equiv \sqrt{1-\frac{1}{\hat{s}}}, \quad \beta_{\hat{t}\hat{u}} \equiv \sqrt{1+\frac{\hat{s}}{\hat{t}\hat{u}}} = \sqrt{1-\frac{1}{\hat{t}}-\frac{1}{\hat{u}}}.$$

with $\hat{s} \equiv \frac{s}{4m^2}$, $\hat{t} \equiv \frac{t}{4m^2}$, $\hat{u} \equiv \frac{u}{4m^2}$.

Example of a coefficient function (2)

The functions $r^{(1)}_{(12)12}, \ldots, r^{(10)}_{(12)12}$ are rational, and all dilogs are contained in a single integral $\bar{B}(s,t,u)$,

$$\bar{B}(s,t,u) = \frac{1}{s-t}\int_0^1 dx \left[\frac{s(2t+u)(1-2x)}{m^2 - s(1-x)x} - \frac{t(2s+u)(1-2x)}{m^2 - t(1-x)x}\right] \ln\left(\frac{x - \frac{1+\beta_{\hat{s}\hat{t}}}{2}}{x - \frac{1-\beta_{\hat{s}\hat{t}}}{2}}\right).$$

Incorporating a constant external field

To generalize all the previous formulas from vacuum to a constant external field, just

1. Change the worldline Green's functions G_B, G_F to field-dependent ones $\mathcal{G}_B, \mathcal{G}_F$,

$$G_B(\tau_1,\tau_2) \quad \to \quad \mathcal{G}_B(\tau_1,\tau_2) = \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\,\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}} + i\mathcal{Z}\dot{G}_{B12} - 1\right)$$

$$G_F(\tau_1,\tau_2) \quad \to \quad \mathcal{G}_F(\tau_1,\tau_2) = G_{F12}\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}}{\cos\mathcal{Z}}$$

   where $\mathcal{Z}_{\mu\nu} \equiv eF_{\mu\nu}T$.

2. Add global determinant factors

$$\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \quad (\textit{Scalar QED}), \qquad \det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right] \quad (\textit{Spinor QED})$$

N-photon amplitudes in a constant field

Master formula for the scalar QED N-photon amplitudes in a constant field (Shaisultanov 1995, Reuter, Schmidt and C.S. 1996)

$$\Gamma_{\rm scal}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N|F) = (-ie)^N$$
$$\times \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} e^{-m^2T} {\rm det}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times \exp\biggl\{\sum_{i,j=1}^N \Bigl[\frac{1}{2} k_i\cdot\mathcal{G}_{Bij}\cdot k_j - i\varepsilon_i\cdot\dot{\mathcal{G}}_{Bij}\cdot k_j + \frac{1}{2}\varepsilon_i\cdot\ddot{\mathcal{G}}_{Bij}\cdot\varepsilon_j\Bigr]\biggr\}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}\,.$$

Applications and generalizations

1. $N = 2$ (vacuum polarisation in the field) becomes very simple (C.S. 2000).
2. Magnetic photon splitting (Adler and C.S. 1996).
3. Two-loop Euler-Heisenberg Lagrangians (Reuter, Schmidt and C.S. 1996, G.V. Dunne and C.S. 2000).
4. One-loop photon-graviton conversion in a constant field (Bastianelli and C.S. 2005, Bastianelli, Nucamendi, C. S. and Villanueva 2007, Ahmadiniaz, Bastianelli, Karbstein and C.S. 2021).
5. N-photon amplitudes in a constant field in the low-energy limit (N. Ahmadiniaz, M. A. Lopez-Lopez and C.S., PLB 852 (2024) 138619).

Low-energy limit of N-photon amplitudes in vacuum (1)

After the removal of the $\ddot{G}'_{ij}s$ the low-energy limit can simply be taken by replacing the universal exponential factor $\mathrm{e}^{\frac{1}{2}T\sum_{i,j=1}^{4}G_{ij}k_i\cdot k_j}$ by unity. E.g. for $N=4$:

$$\Gamma^{(k)}_{\dots} = \int_0^\infty \frac{dT}{T}T^{4-\frac{D}{2}}\,\mathrm{e}^{-m^2T}\int_0^1\prod_{i=1}^{4}du_i\,\Gamma^{(k)}_{\dots}(\dot{G}_{ij})\,\cancel{\mathrm{e}^{\frac{1}{2}T\sum_{i,j=1}^{4}G_{ij}k_i\cdot k_j}}$$

Then all terms in the integrand that are not just products of cycles turn into total derivatives and integrate to zero:

$$\begin{aligned}
\Gamma^{(1)}_{(1234)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41}\,,\\
\Gamma^{(2)}_{(12)(34)} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\left(\dot{G}_{34}\dot{G}_{43} - G_{F34}G_{F43}\right)\,,\\
\cancel{\Gamma^{(3)}_{(123)1}} &= \left(\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31}\right)\dot{G}_{41}\,,\\
\cancel{\Gamma^{(4)}_{(12)11}} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\dot{G}_{13}\dot{G}_{41}\,,\\
\cancel{\Gamma^{(5)}_{(12)12}} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\dot{G}_{13}\dot{G}_{42}\,.
\end{aligned}$$

Low-energy limit of N-photon amplitudes in vacuum (2)

The cycle-integrals can be done in closed form, leading to Bernoulli numbers B_n:

$$\int_0^1 du_1 du_2 \ldots du_n \left(\dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{n1} - G_{F12}G_{F23}\cdots G_{Fn1}\right) = (2-2^n)\, b_n$$

$$b_n = \begin{cases} -2^n \frac{\mathcal{B}_n}{n!} & n \quad \text{even} \\ 0 & n \quad \text{odd} \end{cases}$$

This leads to the closed-form expression

$$\Gamma^{(\mathrm{LE})}_{\mathrm{spin}}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) = (-2)\frac{e^N\Gamma(N-\frac{D}{2})}{(4\pi)^2 m^{2N-D}} \exp\left\{\sum_{m=1}^{\infty}(1-2^{2m-1})\frac{b_{2m}}{2m}\,\mathrm{tr}\,(f_{\mathrm{tot}}^{2m})\right\}\Big|_{f_1\ldots f_N}$$

where $f_{\mathrm{tot}} \equiv \sum_{i=1}^N f_i$.

Low-energy limit of photon amplitudes in a constant field

In the constant-field background, it is still true that the N-photon amplitudes can, in the weak-field limit, be reduced to terms that factorize into cycles. However, since the generalized worldline Green's functions are non-trivial Lorentz matrices, these do not any more factorize into "τ-cycles" and "Lorentz-cycles", instead they combine like

$$\dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_n i_1} Z_n(i_1 i_2 \ldots i_n) \rightarrow \mathrm{tr}\left(f_{i_1} \cdot \dot{\mathcal{G}}_{B i_1 i_2} \cdot f_{i_2} \cdot \dot{\mathcal{G}}_{B i_2 i_3} \cdots f_{i_n} \cdot \dot{\mathcal{G}}_{B i_n i_1} \right)$$

Thus the basic mathematical problem becomes the computation of the "open-index cycle integral"

$$\int_0^1 du_1 \cdots \int_0^1 du_n \, \dot{\mathcal{G}}_{B12} \otimes \dot{\mathcal{G}}_{B23} \otimes \cdots \otimes \dot{\mathcal{G}}_{Bn1}$$

which at first seemed to generate a large number of component integrals. However, it turns out that there is a nice way of avoiding this. Let us show it for the purely magnetic-field case.

Magnetic worldline Green's function

The magnetic worldline Green's function has the matrix decomposition

$$\dot{\mathcal{G}}_B(\tau_1,\tau_2) = \dot{G}_{12}\, g_- + S_{B12}(z) g_+ - A_{B12}(z) i r_+$$

where $z = eBT$, $\dot{G}_{12} = \mathrm{sgn}(\tau_1 - \tau_2)$,

$$g_+ \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad g_- \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$r_+ \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r_- \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \qquad (3)$$

$$S_{B12}(z) = \frac{\sinh(z\,\dot{G}_{12})}{\sinh z}$$
$$A_{B12}(z) = \frac{\cosh(z\,\dot{G}_{12})}{\sinh z} - \frac{1}{z}.$$

The magic magnetic master integral

Introducing the function

$$H_{ij}(z) \equiv \frac{e^{z\dot{G}_{ij}}}{\sinh z} - \frac{1}{z}$$

the three component functions of the Green's function can be written as

$$\begin{aligned}
\dot{G}_{ij} &= H_{ij}(0) \\
S_{Bij}(z) &= \frac{1}{2}\Big[H_{ij}(z) + H_{ij}(-z)\Big] \\
A_{Bij}(z) &= \frac{1}{2}\Big[H_{ij}(z) - H_{ij}(-z)\Big]
\end{aligned}$$

The nice thing about this function is that it is self-reproducing under folding:

$$\begin{aligned}
H^{(2)}_{ik}(z,z') &\equiv \int_0^T d\tau_j H_{ij}(z)H_{jk}(z') = \frac{H_{ik}(z)}{z'-z} + \frac{H_{ik}(z')}{z-z'} \\
H^{(3)}_{il}(z,z',z'') &\equiv \int_0^T d\tau_j \int_0^T d\tau_k H_{ij}(z)H_{jk}(z')H_{kl}(z'') \\
&= \frac{H_{il}(z)}{(z'-z)(z''-z)} + \frac{H_{il}(z')}{(z-z')(z''-z')} + \frac{H_{il}(z'')}{(z-z'')(z'-z'')} \\
H^{(n)}_{i_1 i_{n+1}}(z_1,\dots,z_n) &= \sum_{k=1}^{n} \frac{H_{i_1 i_{n+1}}(z_k)}{\prod_{l\neq k}(z_l - z_k)}
\end{aligned}$$

Application to the low-energy photon amplitudes

Defining

$$z_0 \equiv 0\,, \quad z_+ \equiv z\,, \quad z_0 \equiv -z$$

and

$$\mathfrak{g}_0 \equiv g_-\,, \quad \mathfrak{g}_+ \equiv \frac{1}{2}(g_+ - ir_+)\,, \quad \mathfrak{g}_- \equiv \frac{1}{2}(g_+ + ir_+)$$

we can now write

$$\int_0^1 du_2 \cdots \int_0^1 du_n \, \dot{\mathcal{G}}_{B12} \otimes \dot{\mathcal{G}}_{B23} \otimes \cdots \otimes \dot{\mathcal{G}}_{Bn(n+1)} = \sum_{\alpha_1,\dots,\alpha_n} H^{(n)}_{1(n+1)}(z_{\alpha_1},\dots,z_{\alpha_n})\mathfrak{g}_{\alpha_1} \otimes \cdots \otimes \mathfrak{g}_{\alpha_n}$$

where each index α_i runs over $0, +, -$. This reduces the calculation of the low-energy limit of the magnetic N-photon amplitudes to simple algebra and a single global proper-time integral with trigonometric integrand.

N-photon amplitudes in a plane-wave background

The plane-wave background can be defined by a vector potential $A(x)$ of the form

$$eA_\mu(x) = a_\mu(n \cdot x)$$

where n^μ is a null vector, $n^2 = 0$, and as is usual we will further impose the *light-front gauge condition* $n \cdot a = 0$. Until recently it seemed intractable in the worldline formalism, since it leads to path integrals that are far from gaussian. Only in 2019 a way was found to rewrite the path integrals for the N-photon amplitudes in scalar and spinor QED in terms of gaussian ones.

N-photon amplitude in a plane-wave background

Master formula for the scalar QED N-photon amplitude in a plane-wave background
James P. Edwards and C.S. 2019, Phys. Lett. B **822** (2021) 136696.

$$\Gamma_{\rm scal}(\{k_i,\varepsilon_i\};a) \quad = \quad (-ie)^N(2\pi)^3\delta(\sum_{i=1}^N k_i^1)\delta(\sum_{i=1}^N k_i^2)\delta(\sum_{i=1}^N k_i^+)\int_{-\infty}^{\infty}dx_0^+\,{\rm e}^{-ix_0^+\sum_{i=1}^N k_i^-}$$
$$\times\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\prod_{i=1}^N\int_0^T d\tau_i\,{\rm e}^{\sum_{i,j=1}^N\left[\frac{1}{2}G_{ij}k_i\cdot k_j-i\dot G_{ij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot G_{ij}\varepsilon_i\cdot\varepsilon_j\right]}$$
$$\times\,{\rm e}^{-\left(m^2+\langle\langle a^2\rangle\rangle-\langle\langle a\rangle\rangle^2\right)T+2\sum_{i=1}^N k_i\cdot\left(I(\tau_i)-\langle\langle I\rangle\rangle\right)-2i\sum_{i=1}^N\left(a(\tau_i)-\langle\langle a\rangle\rangle\right)\cdot\varepsilon_i}\Big|_{\varepsilon_1\cdots\varepsilon_N}$$

where we have introduced light-cone coordinates and

$$\langle\langle f\rangle\rangle\equiv\frac{1}{T}\int_0^T d\tau f(\tau)$$

The fermionic worldline Green's function for spinor QED

Contrary to the constant-field case, the generalization to spinor QED requires the explicit evaluation of the Grassmann path integral (no replacement rule). The appropriate generalization of the vacuum correlator $\langle\psi^\mu(\tau)\psi^\nu(\tau')\rangle = \frac{1}{2}\mathrm{sgn}(\tau-\tau')$ is (James P. Edwards and C.S. 2019)

$$\langle\psi^\mu(\tau)\psi^\nu(\tau')\rangle = \frac{1}{2}\mathfrak{G}_F^{\mu\nu}(\tau,\tau'),$$

where

$$\mathfrak{G}_F^{\mu\nu}(\tau,\tau') \equiv \Big\{\delta^{\mu\nu} + 2in^\mu\mathcal{J}^\nu(\tau,\tau') + 2i\mathcal{J}^\mu(\tau',\tau)n^\nu + 2\Big[\mathcal{J}^2(\tau,\tau') - \frac{T^2}{4}\langle\langle a'\rangle\rangle^2\Big]n^\mu n^\nu\Big\}G_F(\tau,\tau')$$

and we have further defined

$$\begin{aligned} J_\mu(\tau) &\equiv \int_0^\tau d\tau'\Big(a'_\mu(\tau') - \langle\langle a'_\mu\rangle\rangle\Big)\,, \\ \mathcal{J}_\mu(\tau,\tau') &\equiv J_\mu(\tau) - J_\mu(\tau') - \frac{T}{2}\dot{G}(\tau,\tau')\langle\langle a'_\mu\rangle\rangle\,. \end{aligned}$$

The combined constant and plane-wave field

The arguably most complex background field for which the Klein-Gordon and Dirac field can be solved in closed form is the combination of a constant and a plane-wave field where the directions of the magnetic and of the electric field coincide with each other and the direction of the wave propagation (Redmond 1965, Batalin and Fradkin 1970). Very recently, we have obtained for this background the following master formula (C.S. and R. Shaisultanov, PLB 843 (2023) 137969)

$$\Gamma_{\rm scal}(\{k_i,\varepsilon_i\};a,F) = (-ie)^N(2\pi)^3\delta(\sum_{i=1}^N k_i^1)\,\delta(\sum_{i=1}^N k_i^2)\,\delta(\sum_{i=1}^N k_i^+)\int_{-\infty}^{\infty}dx_0^+\,{\rm e}^{-ix_0^+\sum_{i=1}^N k_i^-}$$

$$\times\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}{\rm det}^{\frac12}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\prod_{i=1}^N\int_0^T d\tau_i\,{\rm e}^{\sum_{i,j=1}^N\left[\frac12 k_i\cdot\mathcal{G}_{Bij}\cdot k_j-i\varepsilon_i\cdot\dot{\mathcal{G}}_{Bij}\cdot k_j+\frac12\varepsilon_i\cdot\ddot{\mathcal{G}}_{Bij}\cdot\varepsilon_j\right]}$$

$$\times\,{\rm e}^{-\left[m^2+\frac12\int_0^T d\tau\int_0^T d\tau'\tilde{a}(\tau)\cdot\ddot{\mathcal{G}}_B(\tau,\tau')\cdot\tilde{a}(\tau')\right]T-\sum_{i=1}^N\int_0^T d\tau\left[\tilde{a}(\tau)\cdot\dot{\mathcal{G}}_B(\tau,\tau_i)\cdot k_i+i\tilde{a}(\tau)\cdot\ddot{\mathcal{G}}_B(\tau,\tau_i)\cdot\varepsilon_i\right]}\Big|_{\varepsilon_1\cdots\varepsilon_N}$$

where now

$$\tilde{a}_\mu(\tau)\equiv a_\mu\Big(x_0^++n\cdot\sum_{i=1}^N\big[-i\mathcal{G}_B(\tau,\tau_i)\cdot k_i+\dot{\mathcal{G}}_B(\tau,\tau_i)\cdot\varepsilon_i\big]\Big)\,.$$

The fermionic worldline Green's function for the mixed field

$$\begin{aligned}
\widetilde{\mathfrak{G}}_F(\tau,\tau') \;=\;& \mathcal{G}_F(\tau,\tau') + 2i\Big[n\otimes \tilde{J}(\tau)\cdot \mathcal{G}_F(\tau,\tau') - \mathcal{G}_F(\tau,\tau')\cdot n\otimes \tilde{J}(\tau')\Big] \\
&+ 2i\Big[\mathcal{G}_F(\tau,\tau')\cdot \tilde{J}(\tau')\otimes n - \tilde{J}(\tau)\otimes n\cdot \mathcal{G}_F(\tau,\tau')\Big] \\
&+2\tilde{J}^2(\tau) n\otimes n\cdot \mathcal{G}_F(\tau,\tau') + 2\mathcal{G}_F(\tau,\tau')\cdot n\otimes n\tilde{J}^2(\tau') \\
&-4\tilde{J}(\tau)\cdot \mathcal{G}_F(\tau,\tau')\cdot \tilde{J}(\tau') n\otimes n \\
&-\frac{iT}{z_\parallel + \lambda z_\perp}\Big[\mathcal{G}_F(\tau,\tau')\cdot\Big(n\otimes\langle\langle \tilde{a}'_\lambda\rangle\rangle_F m_\lambda - \langle\langle \tilde{a}'_\lambda\rangle\rangle_F m_\lambda\otimes n\Big) \\
&\qquad -\Big(n\otimes\langle\langle \tilde{a}'_\lambda\rangle\rangle_F m_\lambda - \langle\langle \tilde{a}'_\lambda\rangle\rangle_F m_\lambda\otimes n\Big)\cdot \mathcal{G}_F(\tau,\tau')\Big] \\
&+2\frac{T}{z_\parallel + \lambda z_\perp}\Big[\langle\langle \tilde{a}'_\lambda\rangle\rangle_F m_\lambda\cdot \tilde{J}(\tau')\mathcal{G}_F(\tau,\tau')\cdot n\otimes n + \langle\langle \tilde{a}'_\lambda\rangle\rangle_F m_\lambda\cdot \tilde{J}(\tau) n\otimes n\cdot \mathcal{G}_F(\tau,\tau') \\
&\qquad -(\tilde{J}(\tau)\cdot \mathcal{G}_F(\tau,\tau')\cdot m_\lambda\langle\langle \tilde{a}'_\lambda\rangle\rangle_F + \langle\langle \tilde{a}'_\lambda\rangle\rangle_F m_\lambda\cdot \mathcal{G}_F(\tau,\tau')\cdot \tilde{J}(\tau'))n\otimes n\Big] \\
&+\frac{T^2}{z_\parallel^2 - z_\perp^2}\Big[\frac{\langle\langle \tilde{a}'\rangle\rangle_F^2}{2}\mathcal{G}_F(\tau,\tau')\cdot n\otimes n + \frac{\langle\langle \tilde{a}'\rangle\rangle_F^2}{2} n\otimes n\cdot \mathcal{G}_F(\tau,\tau') \\
&\qquad -\langle\langle \tilde{a}'\rangle\rangle_F\cdot \mathcal{G}_F(\tau,\tau')\cdot \langle\langle \tilde{a}'\rangle\rangle_F n\otimes n\Big] .
\end{aligned}$$

The fermionic worldline Green's function for the mixed field (2)

Here $m_\pm \equiv \frac{1}{\sqrt{2}}(1, \pm i, 0, 0)$

$$\tilde{j}^\mu(\tau) \equiv \sum_{\lambda=\pm} m_\lambda^\mu \, \mathrm{e}^{-2\frac{\tau}{T}(z_\parallel + \lambda z_\perp)} \int_0^\tau d\bar{\tau} \Big(\tilde{a}'_\lambda(\bar{\tau}) - \langle\langle \tilde{a}'_\lambda \rangle\rangle_F \Big) \, \mathrm{e}^{2\frac{\bar{\tau}}{T}(z_\parallel + \lambda z_\perp)}$$

where $z_\perp = eBT$, $z_\parallel = ieET$

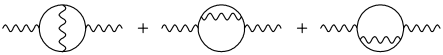
$$\langle\langle \tilde{a}'_\lambda \rangle\rangle_F \equiv \frac{2(z_\parallel + \lambda z_\perp)}{1 - e^{-2(z_\parallel + \lambda z_\perp)}} \frac{1}{T} \int_0^T d\tau \, \tilde{a}'_\lambda(\tau) \, \mathrm{e}^{-2(z_\parallel + \lambda z_\perp)\frac{T-\tau}{T}} \, .$$

In progress (C.S. and R. Shaisultanov): application to the vacuum polarization amplitude in the mixed field.

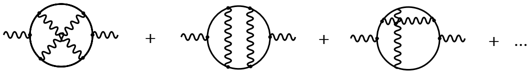
Quenched multi-loop photon amplitudes

Dealing with the amplitude as a whole becomes important when one uses the one-loop amplitudes to construct higher-loop amplitudes by sewing:

From the four-photon amplitude we can construct the two-loop quenched photon propagator,



From the one-loop six-photon amplitude we get the three-loop quenched propagator



etcetera

This type of sums of diagrams is known to suffer from extensive cancellations...

Multi-loop worldline Green's functions

M. G. Schmidt and C.S. 1994: More efficient than sewing is the use of multi-loop wordline Green's functions that hold the information on photon insertions. For a single insertion,

$$G_B^{(1)}(\tau_1,\tau_2) = G_B(\tau_1,\tau_2) + \frac{1}{2}\frac{[G(\tau_1,\tau_a)-G(\tau_1,\tau_b)][G(\tau_a,\tau_2)-G(\tau_b,\tau_2)]}{\bar{T}+G(\tau_a,\tau_b)}\,.$$

where $\bar{T}$ is the proper-time length of the inserted propagator, and τ_a, τ_b the points on the loop between which the propagator is inserted.
It leads to integral representations for the l-loop photon propagator naturally written in the variables

$$G_{a_1 b_1}, G_{a_2 b_2}, \ldots, G_{a_l b_l}, C_{a_1 b_1 a_2 b_2}, \ldots, C_{a_{l-1} b_{l-1} a_l b_l}$$

where the $G_{a_i b_i}$ depend only on a single propagator, and the $C_{a_i b_i a_j b_j}$ on pairs of propagators.

Two-loop vacuum polarisation in scalar QED

V.M. Banda and C.S. (in preparation): compact integral representation for the two-loop photon polarization function in scalar QED,

$$\Pi^{(2)}_{\rm scal}(k^2) = -\frac{e^6}{2(4\pi)^D}\int_0^\infty \frac{dT}{T^{D+1}} e^{-m^2T}\int_0^\infty d\bar T\int_0^T d\tau_a\int_0^T d\tau_b \gamma_{ab}^{D/2} \times \int_0^T d\tau_1\int_0^T d\tau_2\, \mathrm{e}^{-(G_{12}-\frac{\gamma_{ab}}{4}C^2)k^2} I$$

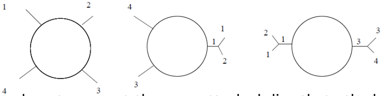
$$\begin{aligned} I = {} & D(-\ddot G_{ab}+\frac{\gamma_{ab}}{2}\dot G_{ab}^2)(-\ddot G_{12}-\frac{\gamma_{ab}}{2}\partial_1 C\partial_2 C)\frac{1}{k^2} \\ & +\Big[(-\ddot G_{a1}-\frac{\gamma_{ab}}{2}\dot G_{ab}\partial_1 C)(-\ddot G_{b2}+\frac{\gamma_{ab}}{2}\dot G_{ab}\partial_2 C)+(1\leftrightarrow 2)\Big]\frac{1}{k^2} \\ & -[\partial_a C-\frac{\gamma_{ab}}{2}\dot G_{ab}C][\partial_b C+\frac{\gamma_{ab}}{2}\dot G_{ab}C]\Big(\ddot G_{12}+\frac{\gamma_{ab}}{2}\partial_1 C\partial_2 C\Big) \end{aligned}$$

$(\gamma_{ab} = (\bar T + G_{ab})^{-1})$

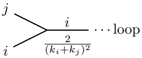
On-shell N-gluon matrix elements

When computing the on-shell N-gluon matrix elements, we have to use the full connected amplitude, not just the irreducible one. Following Bern and Kosower 1991, the additional one-particle-reducible terms can be obtained from Q_N by the following procedure:

(i) Draw all possible ϕ^3 1-loop diagrams D_i with N legs, labelled $1, \ldots, N$ (following the ordering of the color trace). Diagrams where the loop is a tadpole or isolated on an external leg can be omitted. E.g. at the four-point level there are single and double pinches only:



(ii) A diagram will contribute if each vertex except the ones attached directly to the loop corresponds to a possible pinch. A vertex with labels $i < j$ can be pinched if Q_N is linear in $\dot{G}_{ij}$. The pinching replaces this $\dot{G}_{ij}$ by a factor of $2/(k_i + k_j)^2$, removes the vertex and transfers the label i to the ingoing leg.



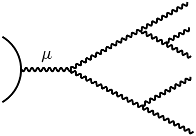
The τ_j - integration is omitted and the index j replaced by i in all $G_{kl}, \dot{G}_{kl}$. The pinching can thus be represented by a pinch operator $\mathcal{D}_{ij}$,

$$\mathcal{D}_{ij} f(\dot{G}) \equiv \frac{\partial}{\partial \dot{G}_{ij}} f(\dot{G}) \Big|_{\substack{\dot{G}_{ij}=0 \\ \dot{G}_{jk} \to \dot{G}_{ik}}} .$$

(N. Ahmadiniaz, F.M. Balli, C. Lopez-Arcos, A. Quintero Velez and C. S., PRD 104 (2021) L941702).
The trees are to be "pruned" recursively starting with the outermost vertices.

Berends-Giele Currents

Returning to the Bern-Kosower formalism, without the pinch rules we would have to construct the reducible contributions attaching off-shell currents to the loop:



Such currents were recognized as central objects in Yang-Mills theory since the eighties:

- They are naturally written in terms of multi-particle polarizations (F.A. Berends and W.T. Giele, NPB 306 (1988) 759) and then are called Berends-Giele currents.
- They are instrumental in the perturbiner approach where tree-level amplitudes are constructed directly from the field equations (A.A. Rosly and K.G. Selivanov, PLB 399 (1997) 135, S. Mizera and B. Skrzypek, JHEP 10 (2018) 018).
- They are important building blocks for amplitudes obeying color-kinematics duality (Z. Bern, J.J.M. Carrasco and H. Johansson, PRD 78, 085011 (2008)). This requires a specific gauge, BCJ gauge.

Multiparticle polarizations and field strength tensors

Multi-particle polarization tensors:

$$\begin{aligned} \varepsilon_{12}^{\mu} &= \tfrac{1}{2}\left[\varepsilon_2 \cdot k_1 \varepsilon_1^{\mu} - \varepsilon_{1\rho} f_2^{\rho\mu} - (1 \leftrightarrow 2)\right] \\ \varepsilon_{123}^{\mu} &= \tfrac{1}{2}\Big[(k_3 \cdot \varepsilon_{12})\, \varepsilon_3^{\mu} - (k_{12} \cdot \varepsilon_3)\, \varepsilon_{12}^{\mu} + \varepsilon_{12\nu} f_3^{\nu\mu} \\ &\quad - \varepsilon_{3\nu} f_{12}^{\nu\mu}\Big] - k_{123}^{\mu} \tfrac{1}{4} \varepsilon_1 \cdot \varepsilon_2\, \varepsilon_3 \cdot (k_2 - k_1) \end{aligned}$$

etc.

Multi-particle field-strength tensors:

$$\begin{aligned} f_{12}^{\mu\nu} &= \varepsilon_2 \cdot k_1 f_1^{\mu\nu} - (f_1 f_2)^{\mu\nu} - (1 \leftrightarrow 2) \\ f_{123}^{\mu\nu} &= k_{123}^{\mu} \varepsilon_{123}^{\nu} - k_{12} \cdot k_3 \varepsilon_{12}^{\mu} \varepsilon_3^{\nu} \\ &\quad - k_1 \cdot k_2 (\varepsilon_1^{\mu} \varepsilon_{23}^{\nu} + \varepsilon_{13}^{\mu} \varepsilon_2^{\nu}) - (\mu \leftrightarrow \nu) \end{aligned}$$

etc.

BCJ gauge and generalized Jacobi identities

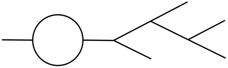
The multi-particle polarizations are subject to generalized gauge transformations. To construct currents in BCJ gauge, they must obey the generalized Jacobi identities

$$\varepsilon^\mu_{123} + \varepsilon^\mu_{213} = 0\,, \quad \varepsilon^\mu_{123} + \varepsilon^\mu_{312} + \varepsilon^\mu_{231} = 0\,, \quad etc.$$

(C.R. Mafra and O. Schlotterer, JHEP 03, 090 (2016)).

Multi-particle polarizations from pinching

Clearly the Bern-Kosower pinching procedure must hold the information on the Berends-Giele currents. It turns out that to obtain the currents, it is sufficient to look at the maximal pinch of the N-gluon amplitude, defined by the consecutive pinching of $N-2$ adjacent legs. It corresponds to the Bern-Kosower diagram



(which in the original Bern-Kosower rules was actually discarded, since it is absorbed by the gluon wave-function renormalization).

Only single-cycle terms contribute to it, thus in its calculation we can replace Q_N by $\tilde{Q}_N \equiv Q_N^2 + Q_N^3 + \ldots Q_N^N$. It turns out that the $(N-1)$ - field-strength tensor $f^{\mu\nu}_{12\cdots(N-1)}$ can be harvested through

$$\mathcal{D}_{1(N-1)} \cdots \mathcal{D}_{13}\mathcal{D}_{12}\tilde{Q}_N = \frac{1}{2} f^{\mu\nu}_{12\cdots(N-1)} f_{N\nu\mu} \dot{G}^2_{1N}$$

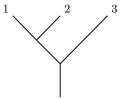
and (less obviously) the $(N-2)$ - polarization tensor $\varepsilon_{12\cdots(N-2)}$ directly from the $(N-2)$ - tail:

$$\mathcal{D}_{1(N-2)} \cdots \mathcal{D}_{13}\mathcal{D}_{12} T(1, 2, \ldots, N-2) = \varepsilon_{12\cdots(N-2)} \cdot k_{N-1} \dot{G}_{1(N-1)} + \varepsilon_{12\cdots(N-2)} \cdot k_N \dot{G}_{1N}$$

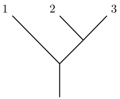
(N. Ahmadiniaz, F.M. Balli, C. Lopez-Arcos, A. Quintero Velez and C. S., PRD 104 (2021) L941702)

BCJ gauge comes for free

It turns out that these polarization and field strength tensors automatically fulfill the generalized Jacobi identities. This can be shown using the natural mapping between the Bern-Kosower pinch diagrams and the Lie-bracketing algebra for N ordered legs,



$[[1,2],3]$



$[1,[2,3]]$

etc.

The proof does not involve any specific properties of the integrand, i.e. it would work with any symmetric polynomial in the $\dot{G}_{ij}$.

Constructing the tree-level N-gluon amplitude (1)

N. Ahmadiniaz, F.M. Balli, O. Corradini, C. Lopez-Arcos, A. Quintero Velez and C. S., NPB 975 (2022) 115690

To compute the N-gluon tree-level amplitude:

1. Use the above to calculate the generalized polarization tensor ε_{N-1} in BCJ gauge. (in the above paper we calculate them up to multiplicity five).
2. Sum over all pinch diagrams to this order to construct the color-stripped Berends-Giele currents $A^{\mu}_{12\cdots(N-1)}$:

$$
\begin{aligned}
A^{\mu}_{1} &= \varepsilon^{\mu}_{1}, \\
A^{\mu}_{12} &= \frac{\varepsilon^{\mu}_{[1,2]}}{s_{12}}, \\
A^{\mu}_{123} &= \frac{\varepsilon^{\mu}_{[[1,2],3]}}{s_{12}s_{123}} + \frac{\varepsilon^{\mu}_{[1,[2,3]]}}{s_{23}s_{123}}, \\
A^{\mu}_{1234} &= \frac{\varepsilon^{\mu}_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{\varepsilon^{\mu}_{[[1,[2,3]],4]}}{s_{123}s_{1234}s_{23}} + \frac{\varepsilon^{\mu}_{[[1,2],[3,4]]}}{s_{12}s_{1234}s_{34}} + \frac{\varepsilon^{\mu}_{[1,[[2,3],4]]}}{s_{1234}s_{23}s_{234}} + \frac{\varepsilon^{\mu}_{[1,[2,[3,4]]]}}{s_{1234}s_{234}s_{34}}, \\
&\vdots
\end{aligned}
$$

The denominators can be read off from the pinch diagram.

Constructing the tree-level N-gluon amplitude (2)

3 From this we can get the colour-ordered partial amplitude of N gluons through the Berends-Giele formula

$$A^{\rm tree}(1,2,\ldots,N) = s_{12\cdots(N-1)} A^{\mu}_{12\cdots(N-1)} A_{N\mu}\,.$$

The factor $s_{12\cdots(N-1)}$ is inserted to cancel the final off-shell propagator, and the factor $A_{N\mu} = \varepsilon_{N\mu}$ puts the final gluon on-shell.

4 The color-dressed Berends-Giele currents $\mathcal{A}^{\mu}_{12\cdots(N-1)}$ are obtained from the color-stripped ones $A^{\mu}_{12\cdots(N-1)}$ by summing over all inequivalent orderings ($(2N-5)!!$ terms in total), and supplying color factors that (by color-kinematics duality) have the same Lie bracketing structure in color space. E. g.

$$\begin{aligned} \varepsilon^{\mu}_{[[1,2],3]} &\longrightarrow \varepsilon^{\mu}_{[[1,2],3]} c^{a}_{[[1,2],3]}\,, \\ c^{a}_{[[1,2],3]} &= \tilde{f}_{a_1 a_2}{}^{b} \tilde{f}_{b a_3}{}^{a} \end{aligned}$$

$(\tilde{f}_{ab}{}^{c} \equiv i\sqrt{2}\, f_{ab}{}^{c})$.

Constructing the tree-level N-gluon amplitude (3)

E.g. For $N = 5$:

$$\mathcal{A}^{a\mu}_{1234} = \frac{c^a_{[[[1,2],3],4]}\varepsilon^\mu_{[[[1,2],3],4]}}{s_{12}s_{123}s_{1234}} + \frac{c^a_{[[[1,2],4],3]}\varepsilon^\mu_{[[[1,2],4],3]}}{s_{12}s_{124}s_{1234}} + \frac{c^a_{[[[1,3],4],2]}\varepsilon^\mu_{[[[1,3],4],2]}}{s_{13}s_{134}s_{1234}} + \frac{c^a_{[[[2,3],4],1]}\varepsilon^\mu_{[[[2,3],4],1]}}{s_{23}s_{234}s_{1234}}$$
$$+ \frac{c^a_{[[[1,3],2],4]}\varepsilon^\mu_{[[[1,3],2],4]}}{s_{13}s_{123}s_{1234}} + \frac{c^a_{[[[1,4],2],3]}\varepsilon^\mu_{[[[1,4],2],3]}}{s_{14}s_{124}s_{1234}} + \frac{c^a_{[[[1,4],3],2]}\varepsilon^\mu_{[[[1,4],3],2]}}{s_{14}s_{134}s_{1234}} + \frac{c^a_{[[[2,3],1],4]}\varepsilon^\mu_{[[[2,3],1],4]}}{s_{23}s_{123}s_{1234}}$$
$$+ \frac{c^a_{[[[2,4],1],3]}\varepsilon^\mu_{[[[2,4],1],3]}}{s_{24}s_{124}s_{1234}} + \frac{c^a_{[[[2,4],3],1]}\varepsilon^\mu_{[[[2,4],3],1]}}{s_{24}s_{234}s_{1234}} + \frac{c^a_{[[[3,4],1],2]}\varepsilon^\mu_{[[[3,4],1],2]}}{s_{34}s_{134}s_{1234}} + \frac{c^a_{[[[3,4],2],1]}\varepsilon^\mu_{[[[3,4],2],1]}}{s_{34}s_{234}s_{1234}}$$
$$+ \frac{c^a_{[[1,2],[3,4]]}\varepsilon^\mu_{[[1,2],[3,4]]}}{s_{12}s_{34}s_{1234}} + \frac{c^a_{[[1,3],[2,4]]}\varepsilon^\mu_{[[1,3],[2,4]]}}{s_{13}s_{24}s_{1234}} + \frac{c^a_{[[1,4],[2,3]]}\varepsilon^\mu_{[[1,4],[2,3]]}}{s_{14}s_{23}s_{1234}}.$$

5 From this we get the total tree-level N - gluon amplitude,

$$A^{\text{tree}}_N = s_{12\cdots(N-1)}\mathcal{A}^\mu_{12\cdots(N-1)}\mathcal{A}_{N\mu}.$$

Things that had to be left out

1. Photon- dressed electron propagator in vacuum and in a constant crossed field
Feynman 1951; Fradkin 1966; Fradkin and Gitman 1991; M. Reuter, M.G. Schmidt and C.S. 1996, N. Ahmadiniaz, V.M. Banda Guzmán, F. Bastianelli, O. Corradini, J.P. Edwards and C. S. 2020:
Worldline master formulas for the dressed electron propagator, part 1: Off-shell amplitudes, JHEP 08 (2020) 018
Worldline master formulas for the dressed electron propagator, part 2: On-shell amplitudes, JHEP 01 (2022) 050
2. Berends-Giele currents for Gravity (N. Ahmadiniaz, F.M. Balli, O. Corradini, C. Lopez-Arcos, A. Quintero Velez and C. S., PRD 104 (2021) L941702)). Nucl. Phys. B **975** 115690 (2022),
3. Photonic processes in Coulomb and Sauter backgrounds.
4. etc.