NO-π SCHEMES FOR MULTI-COUPLING THEORIES

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Introduction

- Noticed that ζ_4 cancelled in the QCD Adler function up to $O(\alpha_s^3)$ Gorishnii, Kataev and Larin (1991) and $O(\alpha_s^4)$ Baikov, Chetyrkin, Kühn (2008)
- More recent evidence from other physical quantities Chetyrkin et al (2017), Ruijl et al (2018), Davies and Vogt (2018)
- Explained at 3 loops Broadhurst (1999) and 4 loops Baikov, Chetyrkin (2010), Baikov, Chetyrkin, Kühn (2017) in terms of dependence of Feynman integrals on certain combinations of ζ-functions
- "C-scheme" introduced Boito, Jamin, Miravitllas (2016) - additional RG functions free of even ζs (i.e. powers of π^2) up to certain orders Jamin, Miravitllas (2018)
- Proposed that "no-π" property holds to all orders in $\hat{G}$ scheme Baikov and Chetyrkin (2018, 2019)

Introduction

- Meanwhile work on MOM schemes has also demonstrated evidence for no-π property Chetyrkin, Retey (2000), von Smekal, Maltman, Sternbeck (2009), Gracey (2013), Gracey, Mason (2022), Gracey (2024)
- All this mostly for single-coupling theories except for Wess-Zumino model - no-π property demonstrated up five loops for general tensor coupling Gracey (2022).

Introduction

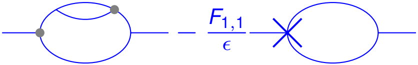
Proposal:

- For any (multi-coupling) theory there are (a range of?) renormalisation schemes in which even-ζs are absent. These can all be specified by redefinition of the couplings
- Some of these schemes have simple physical definitions
- One of them is a minimal scheme where we absorb even-ζ-dependent finite parts of RG functions
- Another is a variant of "MOM", i.e. it is specified by absorbing all finite parts of RG functions in some well-defined way
- Easiest to demonstrate for Wess-Zumino model where β function defined by two-point function
- Three-point superpotential $g^{ijk}\Phi_i\Phi_j\Phi_k + \text{c.c.}$

Basic Ideas

Define counterterms recursively by standard R-operation.

$$F = \sum F(G), \quad F(G) = -\overline{R}(G)$$

- Sum is over all the relevant graphs G
- $\overline{R} \Rightarrow$ subtractions of diagrams with counterterm insertions corresponding to divergent subgraphs.
- $\overline{R}(G_2) =$ 

- The final minus sign implements the subtraction of the overall pole.
- Standard minimal subtraction involves subtraction only of pole term

Basic Ideas

We may write the bare coupling g_B (in minimal subtraction) as

$$g_B = \mu^{\frac{1}{2}\epsilon}\left(g + \frac{1}{\epsilon}\left\{F_{1,1} + F_{2,1} + F^{\zeta_3}_{3,1}\zeta_3 + F^{\zeta_4}_{4,1}\zeta_4\right\} + \frac{1}{\epsilon^2}\left\{F_{2,2} + F^{\zeta_3}_{4,2}\zeta_3\right\}\right) + \ldots$$

- $F^{\zeta_n}_{L,m}$: L-loop, order ϵ^{-m} ζ_n-dependent contribution to F
- $F_{L,m}$: L-loop, order ϵ^{-m} non-ζ-dependent contribution to F
- $d = 4 - \epsilon$
- μ: usual dimensional regularisation mass parameter

Basic Ideas

We have

$$\mu \frac{d}{d\mu} g_B = 0.$$

β-function defined by

$$\hat{\beta}(g) = \mu \frac{d}{d\mu} g = -\epsilon g + \beta(g)$$
$$\Rightarrow \beta_L^{\zeta_n} = L F_{L,1}^{\zeta_n}, \quad \beta_L = L F_{L,1}$$

where $\beta_L^{\zeta_n}/\beta_L$ are the ζ_n-dependent/purely rational L-loop contributions to $\beta(g)$. Also have

$$F_{4,2}^{\zeta_3} = \frac{1}{4}\left(\beta_3^{\zeta_3} \cdot F_{1,1} + \beta_1 \cdot F_{3,1}^{\zeta_3}\right) = \frac{1}{4}\left(3F_{3,1}^{\zeta_3} \cdot F_{1,1} + F_{1,1} \cdot F_{3,1}^{\zeta_3}\right),$$
$$f \cdot \equiv f \frac{\partial}{\partial g} \quad \text{or} \quad f \cdot \equiv f^{ijk} \frac{\partial}{\partial g^{ijk}}$$

Basic Ideas

- Now want to find renormalisation scheme where $\beta^{\zeta_4} = 0$.
- Change of scheme $\Rightarrow$ redefinition $g \to g'(g)$ $\Rightarrow$ variation $\beta(g) \to \beta'(g')$ given by

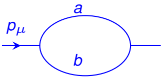
$$\beta'(g') = \mu \frac{d}{d\mu} g' = (\beta(g))^{klm} \frac{\partial}{\partial g^{klm}} g'(g) = \beta(g) \cdot g'(g).$$

- Infinitesimal variation $g' = g + \delta g$ then gives $\beta' = \beta + \delta\beta$ with

$$\begin{aligned} \delta\beta =& [\beta, \delta g] + \dots \\ [X, Y] =& X \cdot Y - Y \cdot X. \end{aligned}$$

Graphical properties

ζ_4 terms in β-function are not independent! Back to Feynman diagrams: for a generic loop



we have the well-known result

$$\frac{1}{(p^2)^{a+b-\frac{d}{2}}} L(a,b), \quad L(a,b) = (4\pi)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{d}{2}-a)\Gamma(\frac{d}{2}-b)\Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)}$$

A large class of Feynman integrals may be written as a product of $L(a,b)$ for various a, b. There is an expansion

$$L(1+\alpha\epsilon, 1+\beta\epsilon) = \frac{2}{\epsilon(2\alpha+2\beta+1)} \exp\left\{\frac{\epsilon}{2}\left[\ln 4\pi - \gamma - \frac{\epsilon}{4}\zeta_2\right]\right\}$$
$$\times \exp\left[\sum_{j=1}^{\infty}(\alpha+\beta+1)^j \frac{\epsilon^j}{j} + \sum_{j=3}^{\infty} h_j(\alpha,\beta)\zeta_j \frac{\epsilon^j}{j}\right]$$

Graphical properties

where

$$h_j(\alpha,\beta) = (\alpha+\tfrac{1}{2})^j+(\beta+\tfrac{1}{2})^j+(-\alpha-\beta-\tfrac{1}{2})^j-(-\alpha)^j-(-\beta)^j-(\alpha+\beta+1)^j.$$

Easy to check:

$$h_4 = \frac{3}{4}h_3 + \frac{1}{64}$$

irrespective of α, β, and similarly for higher even values of j.

- We would like a relation between $F^{\zeta_4}_{L,m}$ and $F^{\zeta_3}_{L,m+1}$
- But these quantities may include ζ-dependent counterterms which do not obey relation between F_3 and F_4
- Define $G^{\zeta_n}_{L,m}$ to be value of $F^{\zeta_n}_{L,m}$ after omitting ζ_n-dependent counterterms
- Then $G^{\zeta_4}_{L,m} = \frac{3}{4}G^{\zeta_3}_{L,m+1} \quad (m \geq 0)$

We have

$$G^{\zeta_3}_{3,1} = F^{\zeta_3}_{3,1}, \quad G^{\zeta_4}_{3,0} = F^{\zeta_4}_{3,0}, \quad G^{\zeta_4}_{4,1} = F^{\zeta_4}_{4,1},$$
$$G^{\zeta_3}_{4,2} = F^{\zeta_3}_{4,2} - F^{\zeta_3}_{3,1} \cdot F_{1,1}$$

Recall

$$F^{\zeta_3}_{4,2} = \frac{1}{4}\left(3F^{\zeta_3}_{3,1} \cdot F_{1,1} + F_{1,1} \cdot F^{\zeta_3}_{3,1}\right)$$

and so

$$G^{\zeta_3}_{4,2} = \frac{1}{4}[F_{1,1}, F^{\zeta_3}_{3,1}] = \frac{1}{4}[\beta_1, G^{\zeta_3}_{3,1}].$$

We then find using

$$G^{\zeta_4}_{4,1} = \frac{3}{4} G^{\zeta_3}_{4,2},$$
$$G^{\zeta_4}_{3,0} = \frac{3}{4} G^{\zeta_3}_{3,1}.$$

$$\beta^{\zeta_4}_4 = 4F^{\zeta_4}_{4,1} = 4G^{\zeta_4}_{4,1} = [\beta_1, G^{\zeta_4}_{3,0}] = [\beta_1, F^{\zeta_4}_{3,0}]$$

Application to β-functions

This can be removed using $\delta\beta = [\beta, \delta g]$ by a redefinition with

$$\delta g = -F_{3,0}^{\zeta_4}\zeta_4$$

i.e. removing the ζ_4-dependent finite part.

We further obtain

$$G^{\zeta_6}_{L,m} = \frac{5}{4}\left(G^{\zeta_5}_{L,m+1} - \frac{1}{3}G^{\zeta_4}_{L,m+2}\right),$$
$$G^{\zeta_8}_{L,m} = \frac{7}{4}\left(G^{\zeta_7}_{L,m+1} - \frac{1}{2}G^{\zeta_6}_{L,m+2} + \frac{1}{24}G^{\zeta_4}_{L,m+4}\right),$$

- Relations like this can be extended to higher even ζ and higher loops (in fact for all known "p-integrals" - two-point integrals with a single momentum dependence).
- Proposal: all even ζ may be removed by

$$\delta g = -\left(\left[F^{\zeta_4}_{3,0} + F^{\zeta_4}_{4,0}\right]\zeta_4 + \left[F^{\zeta_6}_{4,0} + F^{\zeta_6}_{5,0}\right]\zeta_6 + \left[F^{\zeta_8}_{5,0} + F^{\zeta_8}_{6,0}\right]\zeta_8\right) + \ldots$$

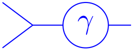
- This is a scheme where we subtract all even-ζ-dependent finite parts - we call it MOM'
- We have shown cancellation in MOM' one loop order beyond first appearance for ζ_4 (i.e. 5 loops) and ζ_6 (i.e. 6 loops).

Application to β-functions

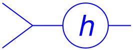
- Consider the MOM scheme (subtract all finite parts)
- Can show (at least up to five loops for ζ_4) that even ζs also cancel in MOM

Example: Wess-Zumino model

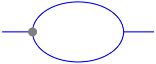
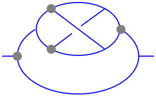
For the Wess-Zumino model

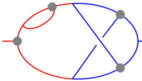
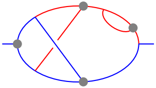
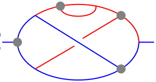
$$\beta = \mathcal{S}_3$$ 

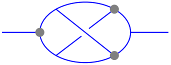
where $\mathcal{S}_3$ denotes the sum over the three terms where γ is attached to each external line. Likewise

$$\delta g = \mathcal{S}_3$$ 

Example: Wess-Zumino model

$$\gamma = \frac{1}{2}$$  $$+ \ldots - \frac{3}{4}(2\zeta_3 + \zeta_4)$$ 

$$-\frac{3}{4}(2\zeta_3 - \zeta_4)\Bigg($$  $$+$$  $$+\,2$$  $$\Bigg) + \ldots$$

$$h^{(3)} = -\frac{1}{4}\gamma_3^{\zeta_3}\zeta_4 = F_{3,0}^{\zeta_4}\zeta_4 = -\frac{3}{8}\zeta_4$$  ;

$$\delta\beta = [\delta g + \delta g^*, \beta] \Rightarrow \delta\gamma = (\delta g + \delta g^*)\cdot\gamma - (\beta + \beta^*)\cdot h$$

$$= (\delta g)^{(3)klm}\frac{\partial}{\partial g^{klm}}\gamma^{(1)} - \beta^{(1)klm}\frac{\partial}{\partial g^{klm}}h^{(3)} + * \text{ terms}$$

$\Rightarrow \zeta_4$ terms cancel.

Conclusions

- Higher loops and higher even-ζs require more work
- So does general all orders relation of $\hat{G}$, MOM and MOM'
- Theoretical underpinning based on p-integrals, i.e. integrals with a single momentum dependence. Extension to 3-point and 4-point graphs based on systematic nullification of external momenta?
- Well-motivated nullification procedure for QCD, see talk by Gracey