A+AMFlow

for master integrals in singular kinematics

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Loops&Legs in QFT

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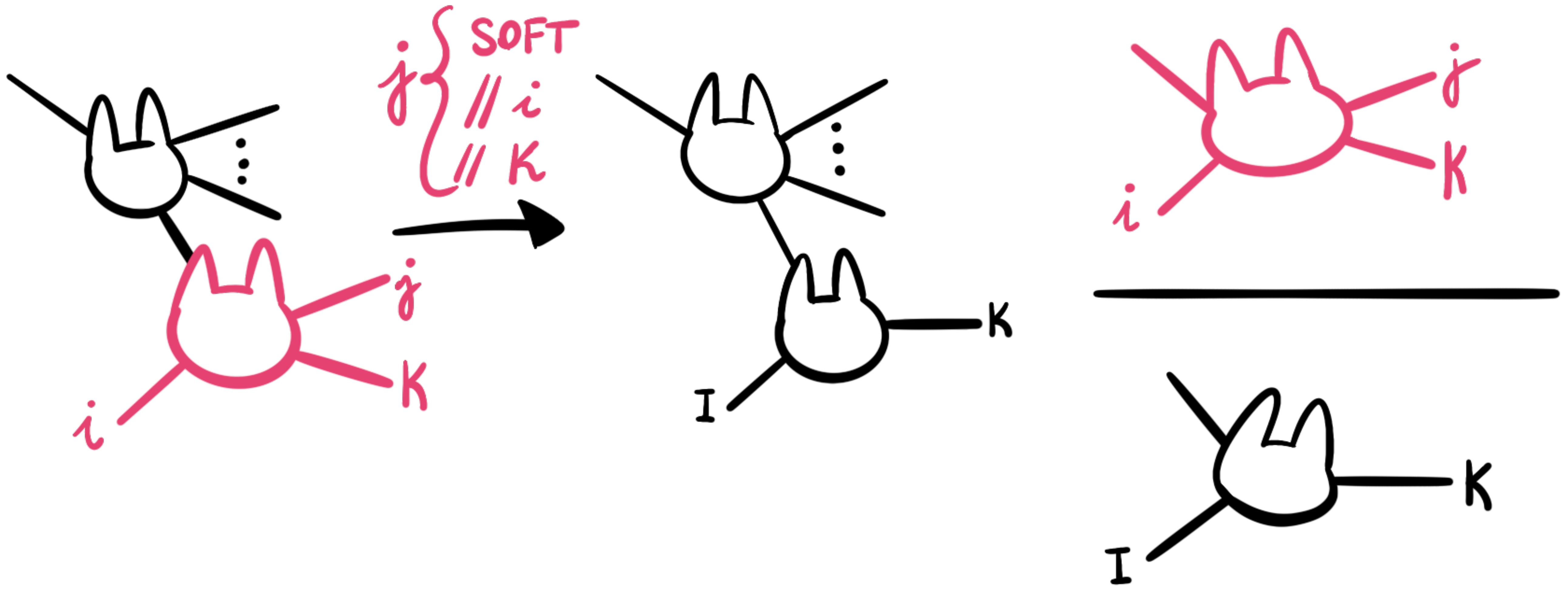
Context: antenna subtraction scheme

- **Antenna functions**
  - Built from simple matrix elements
  - Mimic the divergent behaviour in singular limits
  - Can be easily integrated over phase space

$$d\sigma^{S}_{NLO} \sim X^{\ell}_{2+\,extra\ radiation}\ \tilde{M}^{\ell}_{hard\ partons}\ J_m$$

- **Initial-final antennae:**

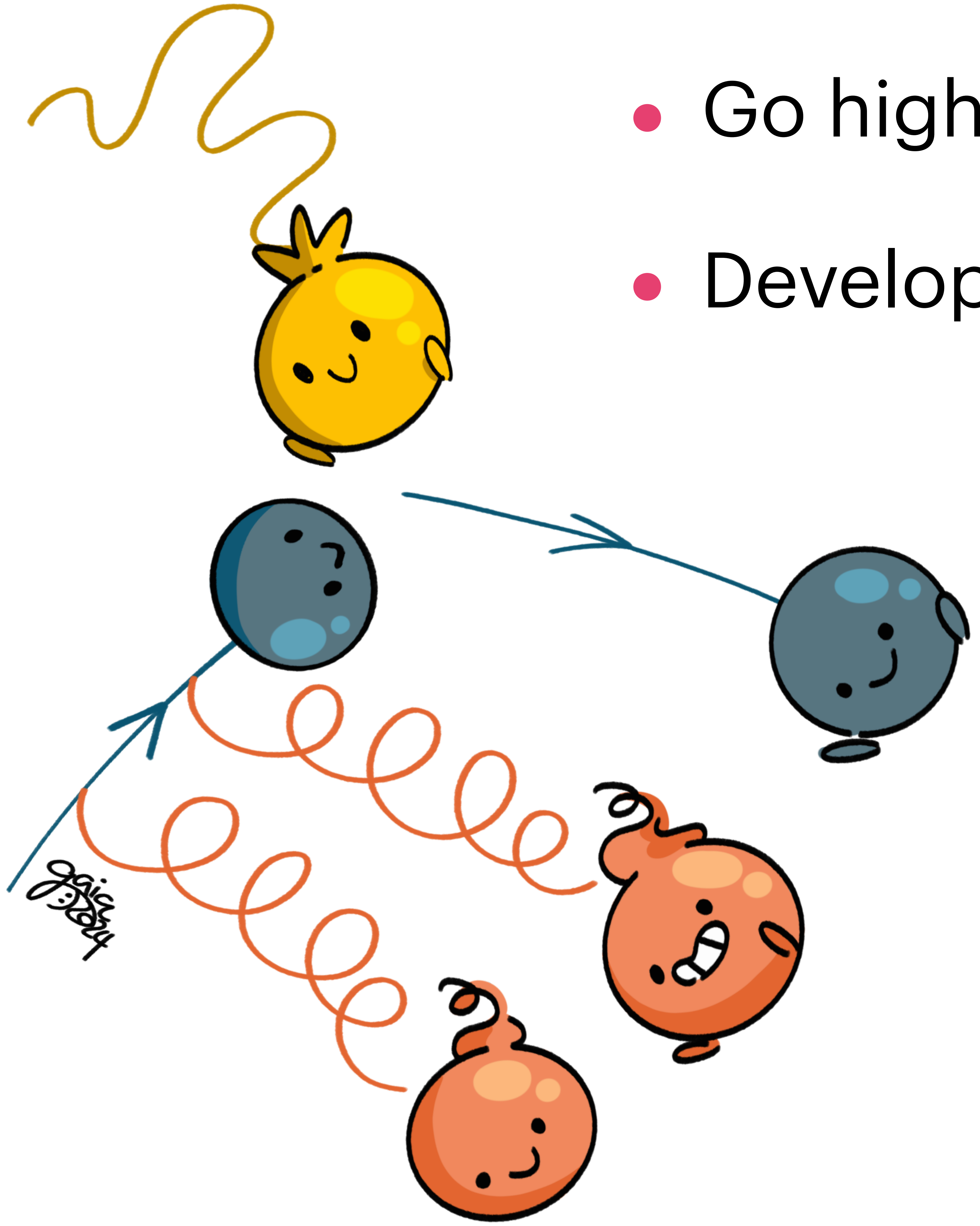
  Initial and final states hard radiators



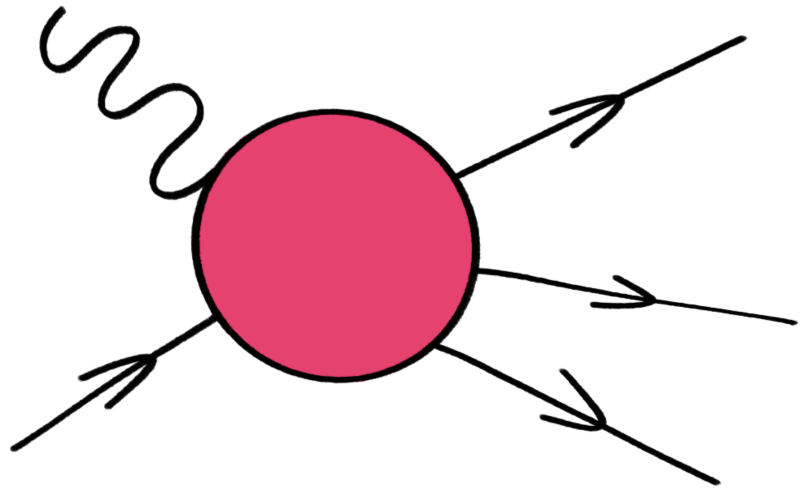
This work:

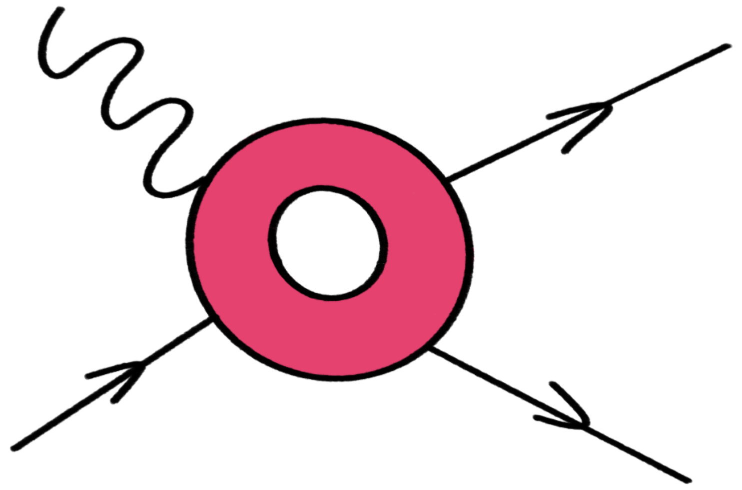
Rederivation of NNLO $2 \to 3, 2 \to 2$ IF antennae

- Known but required a lot of hands-on labour — Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (2009)
- Go higher in the $\epsilon-$expansion [N3LO]
- Develop a more **automated workflow**



Building blocks:
NNLO phase-space integrals for DIS





NNLO DIS

kinematics

- $q_2^2 = -Q^2 < 0$
- $q_1^2 = 0$
- $p_i^2 = 0, \quad i = 1,2,3$

$$q_1 + q_2 \to p_1 + p_2 + (p_3)$$

invariants

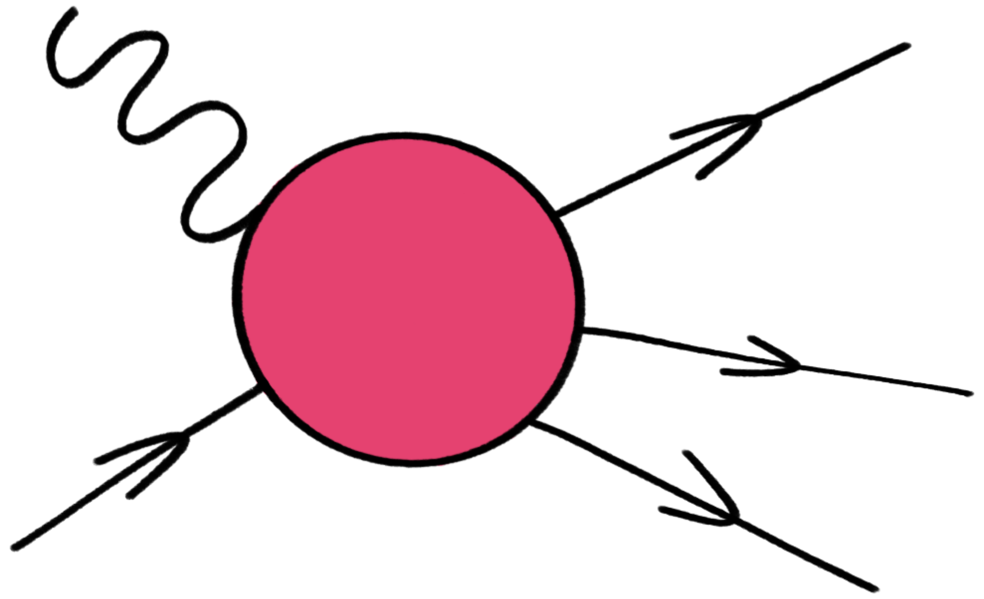
- $s = (q_1 + q_2)^2$
- $z = \dfrac{1}{2q_1 q_2} \longrightarrow s = \dfrac{(1-z)}{z}$

Reverse Unitarity

Anastasiou, Melnikov (2002)

RR

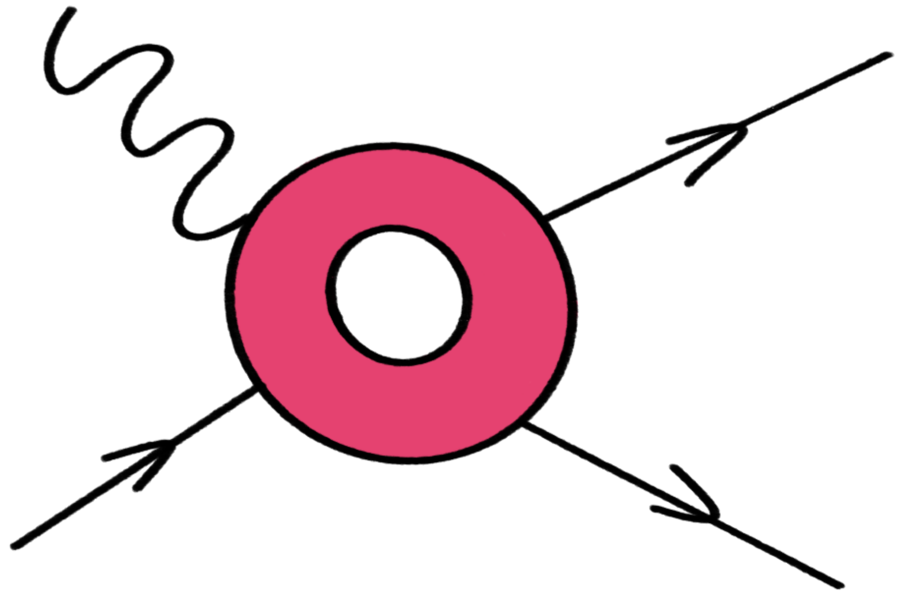
$2 \to 3$



$$I_{RR} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{\not{D}_1} \frac{1}{\not{D}_2} \frac{1}{\not{D}_3} \prod_j \frac{1}{D_j^{\alpha_j}}$$

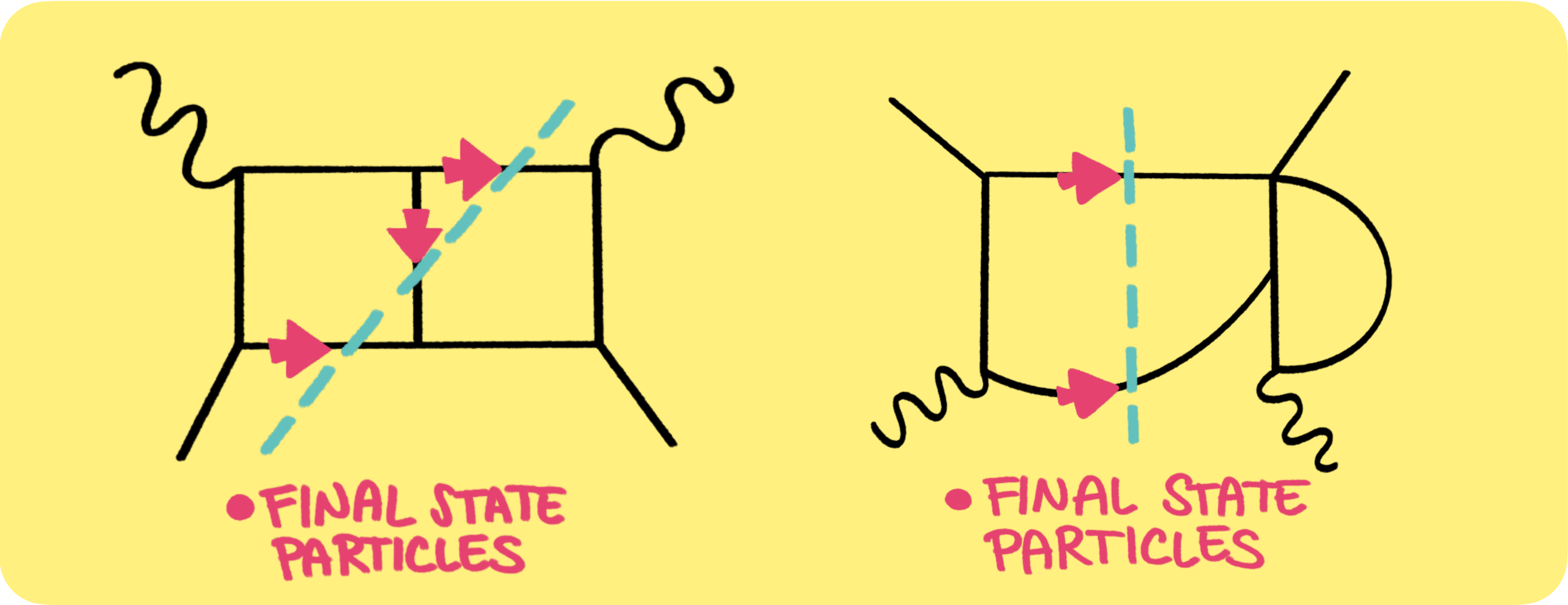
RV

$2 \to 2$



$$I_{RV} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{\not{D}_1} \frac{1}{\not{D}_2} \prod_j \frac{1}{D_j^{\alpha_j}}$$

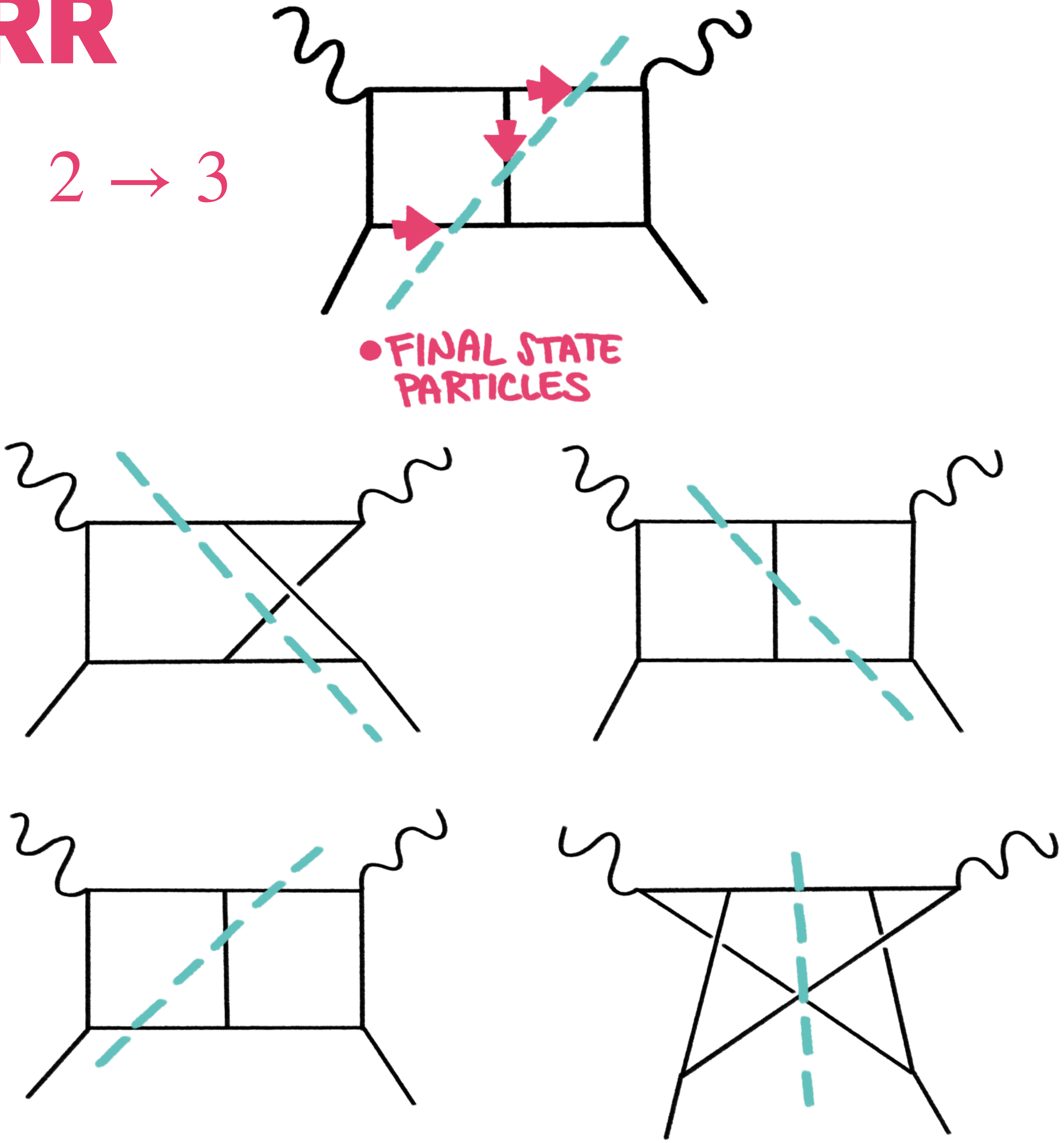
- Write down the forward DIS scattering amplitude at NNLO
- Find physical cuts
    - 2 cuts → phase space $2 \rightarrow 2$ @ 1loop
    - 3 cuts → phase space $2 \rightarrow 3$ @ tree level



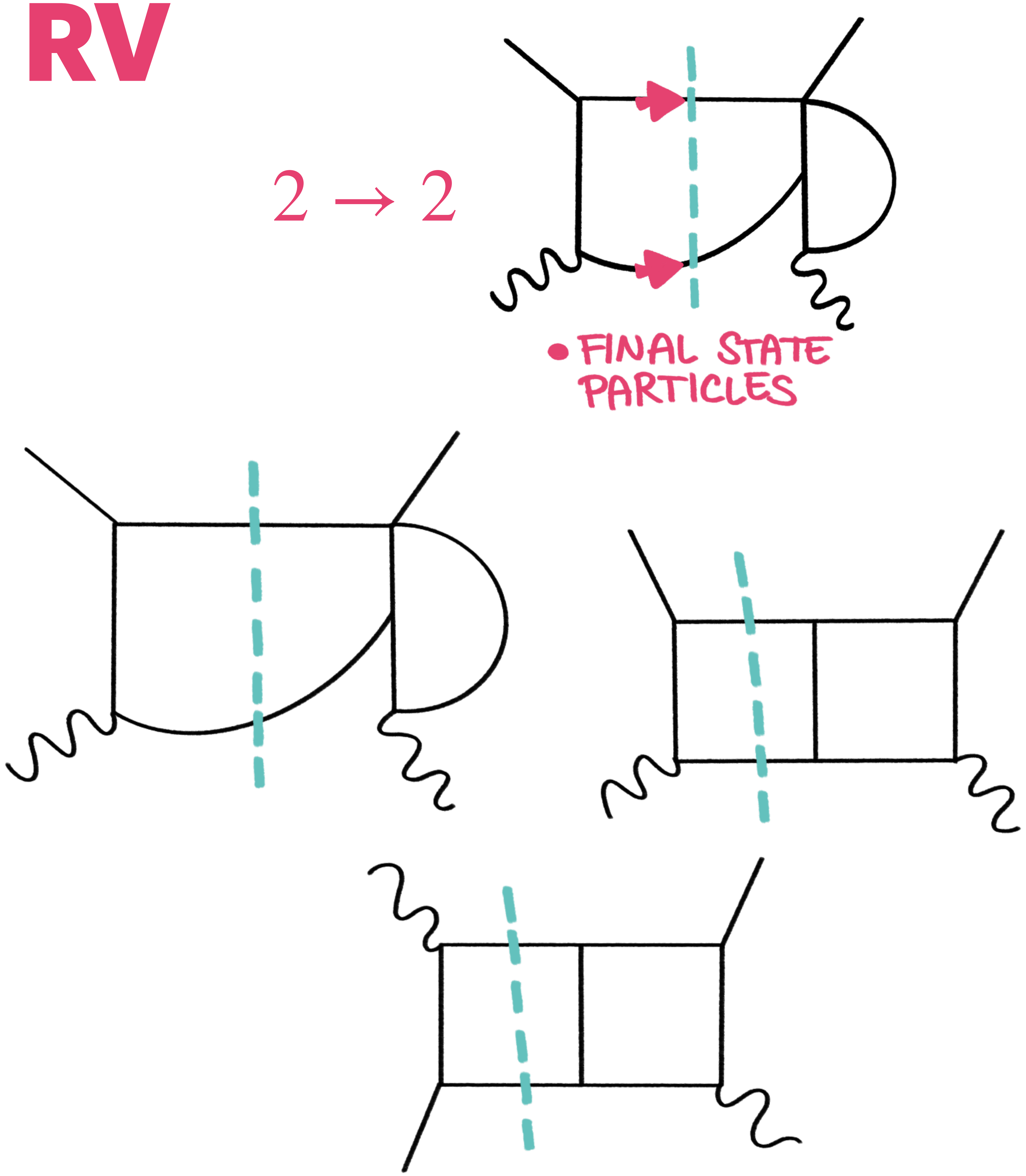
- Write all integrals as a function of a minimal, linearly independent set of master integrals using IBP identities

master integral families

RR



RV



DEs for master integrals

How to solve a differential equation:

- Generic solution
- Boundary condition

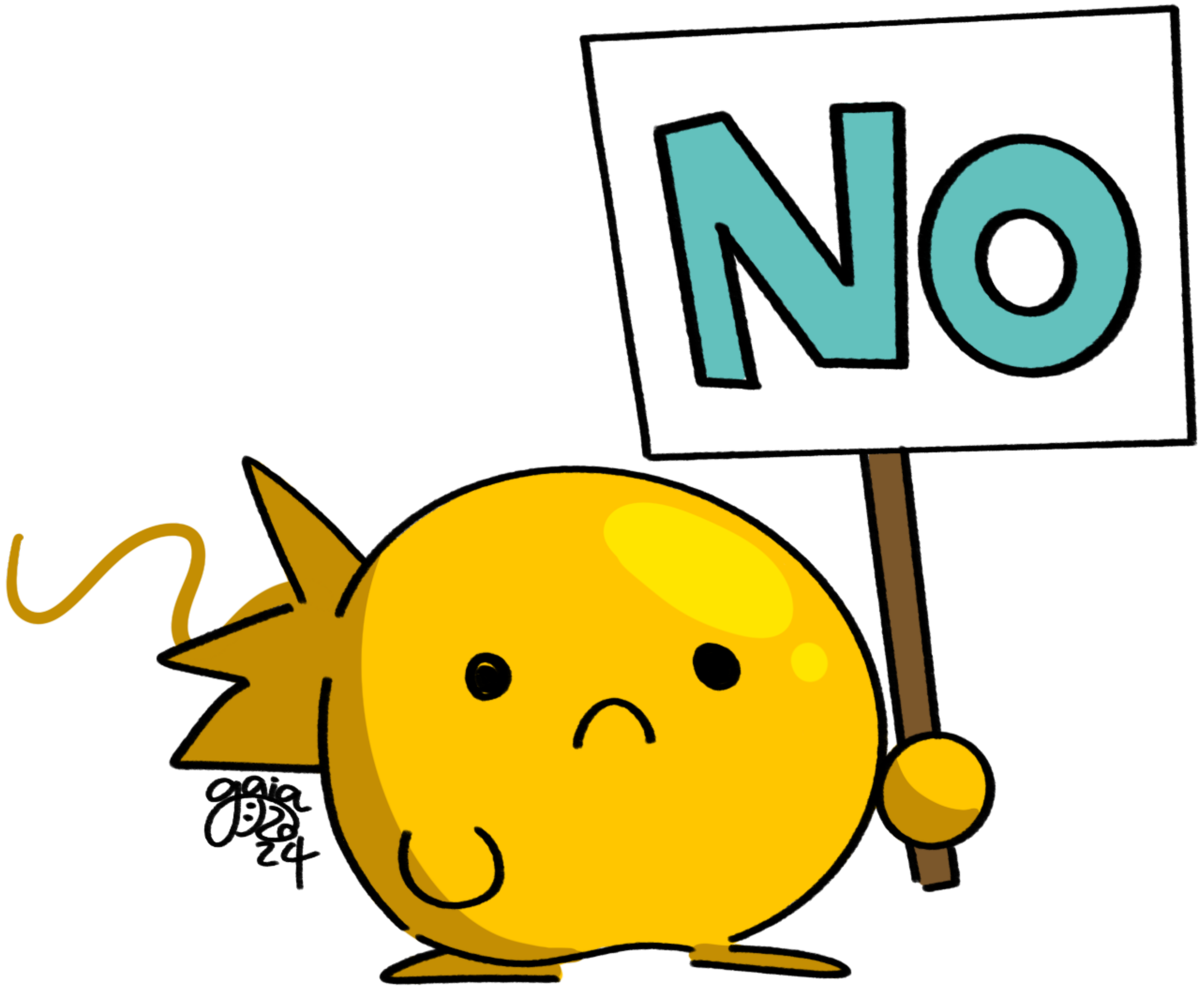
Generic solution

- Canonical form $\partial_z \vec{g} = \epsilon A \cdot \vec{g}$ **Libra** Lee (2020)
- Generic solution in terms of iterated integrals

Alphabet:

$$\mathscr{A}_C = \left\{ \omega_0 = \frac{1}{x}, \omega_{-1} = \frac{1}{1+x}, \omega_1 = \frac{1}{-1+x} \right\}$$

Are we finished?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NO
T YET

Boundary conditions

- Consistency conditions
  - Finding relations between boundaries
- Evaluation in some kinematic limit
  - Fix the remaining ones

Consistency conditions

We look at the kinematic limit $z \to 1 \Rightarrow s \to 0$ **(soft limit)**

RR

$$I_i^{RR} \sim (1-z)^{n_i - 2\epsilon} \sum_j c_j(\epsilon)(1-z)^j, \quad n_i \in \mathbb{Z}$$

- Extract the **leading behavior** of the MIs
- Rescaling the integrals w.r.t. their leading behavior → regularity
- Imposing that in this limit the terms $\log(1-z)$ and poles in $(1-z)$ vanish
- **Relations between boundaries** of different MIs

RV

$$I_i^{RV} \sim (1-z)^{m_i-2\epsilon} \sum_j d_j(\epsilon)(1-z)^j + (1-z)^{l_i-\epsilon} \sum_j e_j(\epsilon)(1-z)^j, \quad m_i, l_i \in \mathbb{Z}$$

- Extract the **leading behavior** of the MIs
- Rescaling the integrals w.r.t. their leading behavior → regularity
- Imposing that in this limit the terms $\log(1-z)$ vanish
- **Relations between boundaries** of different MIs

Now we need to fix the remaining boundaries!

$$I_i^{RR} \sim (1-z)^{n_i - 2\epsilon} \sum_j c_j(\epsilon)(1-z)^j, \quad n_i \in \mathbb{Z}$$

We need $c_0(\epsilon)$

$$I_i^{RV} \sim (1-z)^{m_i - 2\epsilon} \sum_j d_j(\epsilon)(1-z)^j + (1-z)^{l_i - \epsilon} \sum_j e_j(\epsilon)(1-z)^j, \quad m_i, l_i \in \mathbb{Z}$$

We need $d_0(\epsilon), e_0(\epsilon)$

Wishlist:

- Analytic boundaries
- General algorithm to obtain them

AMFlow framework

Liu, Ma (2022)

- Fully **numerical**
- Evaluate FI at any loop order in a **non-singular** point

Outline:

- Add aux mass η^2 to some propagators → auxiliary family

$$I^{phys}(\epsilon, \vec{z}) \to I^{aux}(\epsilon, \vec{z}, \eta^2)$$

- Derive DE with respect to the mass
  - $\partial_{\eta^2} \vec{I}^{aux} = A_\eta \cdot \vec{I}^{aux}$
  - Boundaries @ $\eta^2 \to \infty$ (easy!)
- "Flow" $\eta^2 \to 0$ for physical solution: $\lim_{\eta^2 \to 0} I^{aux} = I^{phys}$
- All implemented in a MATHEMATICA package

Analytic Auxiliary Mass Flow

AAMFlow

GF, Gehrmann, Schönwald (2024)

- Fully **analytical** → can be used near **singular** points

Outline:

- Add aux mass η^2 to **chosen** propagators:
  - limits in kinematical variable and η^2 need to **commute**
- Derive DE with respect to η^2 & solve it
- Fix constants of integration in $\eta^2 \to \infty$ limit (easy!)
- "Flow" to $\eta^2 \to 0$ for physical solution:
  - **method of regions** to extract the physical solution

We look at the boundaries in $z \to 1$: **kinematical endpoint singularity**

- ★ **"Conservation of complexity":**
  - ★ We add an auxiliary mass to one/some propagators
  - ★ This complicates the DE system
  - ★ But makes the boundaries trivial
  - ★ We look at the integrals in a certain limit to simplify the DE

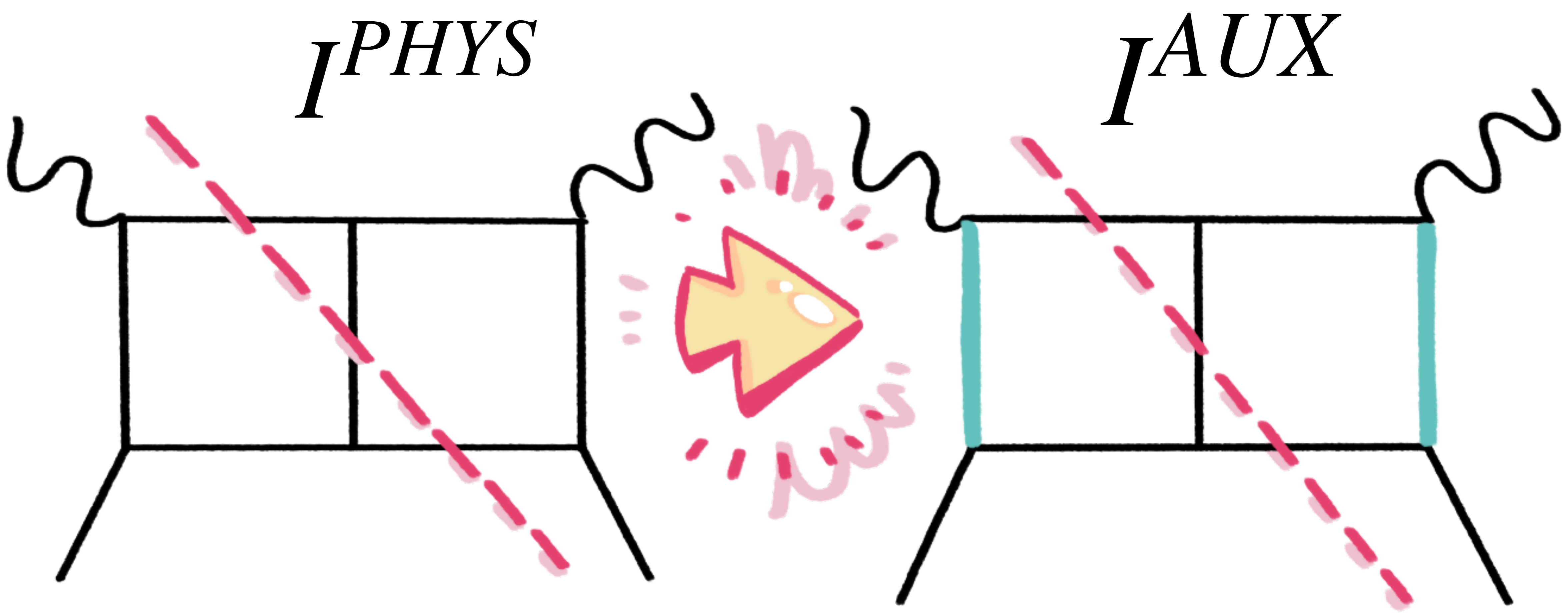
RECIPE:

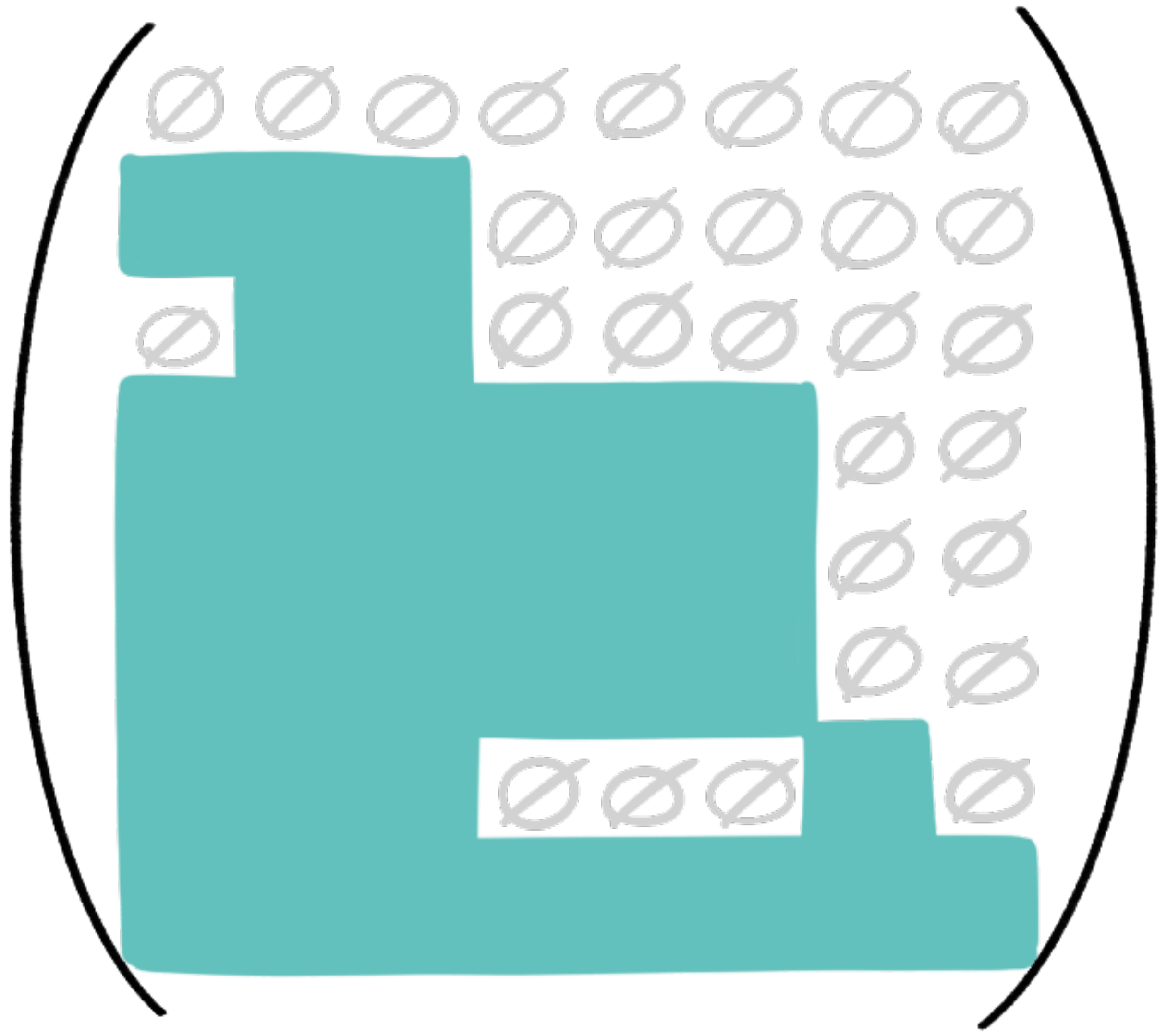
- ★ Choose a family for which to calculate the boundaries
- ★ Choose propagators to which add an auxiliary mass
- ★ Derive DE with respect to $u = 1/\eta^2$
- ★ Fix constants of integration in $\lim u \to 0$
- ★ Limit $\eta^2 \to 0$ & disentangle regions
- ★ Extract physical region

Proof of concept

$$I_i^{RR} \sim (1-z)^{n_i - 2\epsilon} \sum_j c_j(\epsilon)(1-z)^j, \quad n_i \in \mathbb{Z}$$

- We need $c_0(\epsilon)$ of this top sector:





8 master integrals

- Add auxiliary mass → auxiliary topology
- Differential equation wrt $u = 1/\eta^2$ for the $c_0(\epsilon)$

Intermezzo: large mass limit

Beneke, Smirnov (1997)

- Depends on scaling of loop moms

soft $k \sim \mathcal{O}(1)$ or large $k \sim \mathcal{O}(\eta)$

- **SOFT propagators:**

$$\frac{1}{(k+p)^2 - \eta^2} \sim -\frac{1}{\eta^2}$$

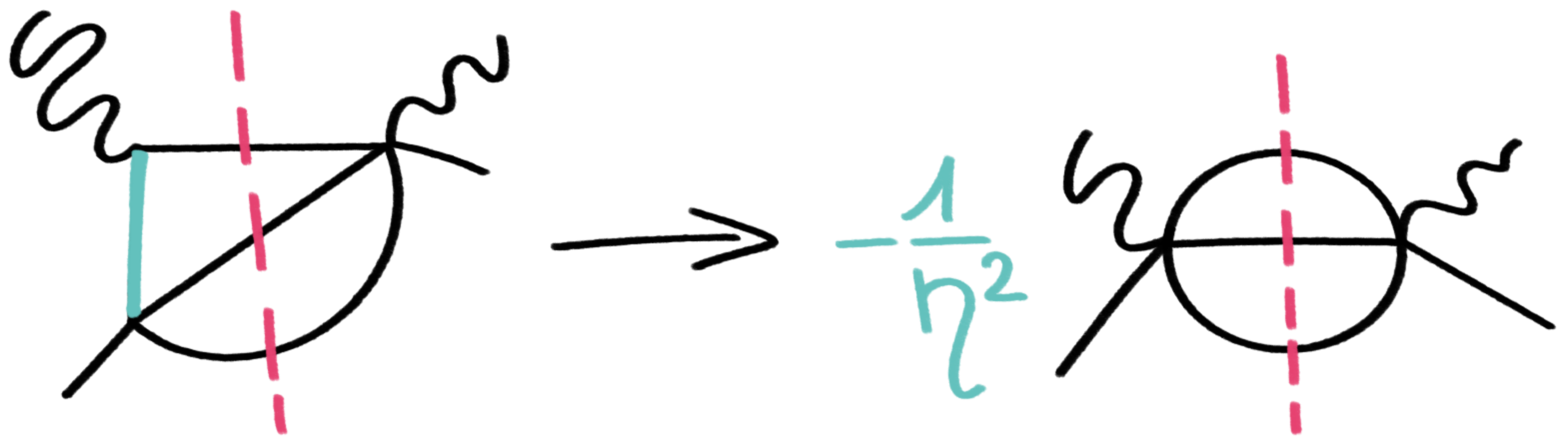
- **LARGE propagators:**

$$\frac{1}{(k+p)^2 - \kappa\eta^2} \sim -\frac{1}{k^2 - \kappa\eta^2}, \quad \kappa \in \{0,1\}$$

Large mass limit: RR ints

- Loop momentum scales **only soft**
  - **“Pinch” propagators with auxiliary mass**
- Example:

$$\frac{1}{(k+p)^2-\eta^2} \sim -\frac{1}{\eta^2}$$

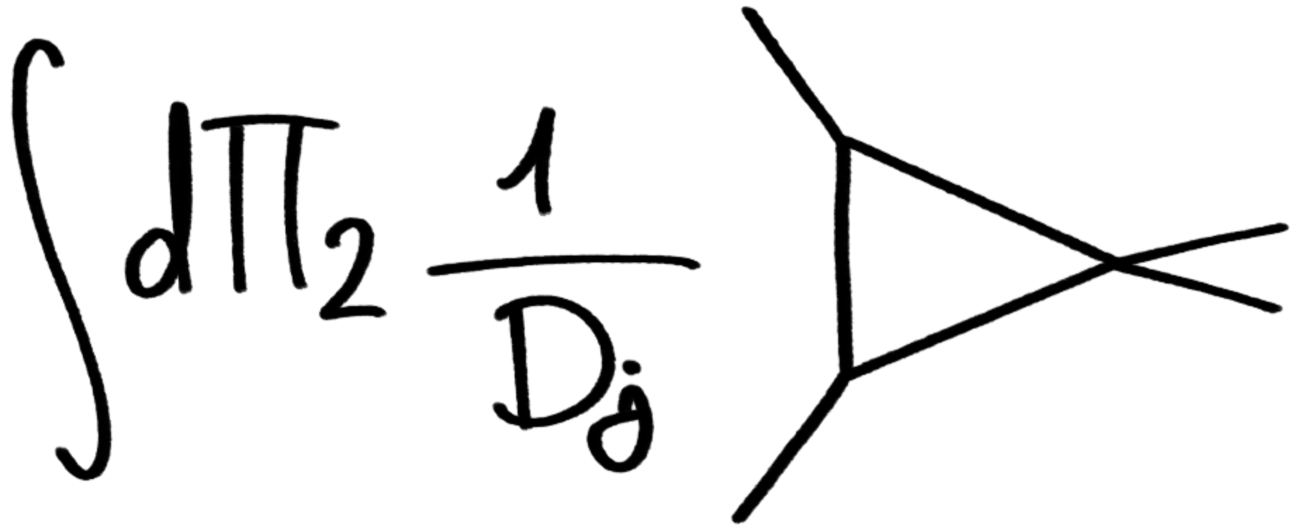


Integrated $2 \rightarrow 3$ phase space

Large mass limit: RV ints

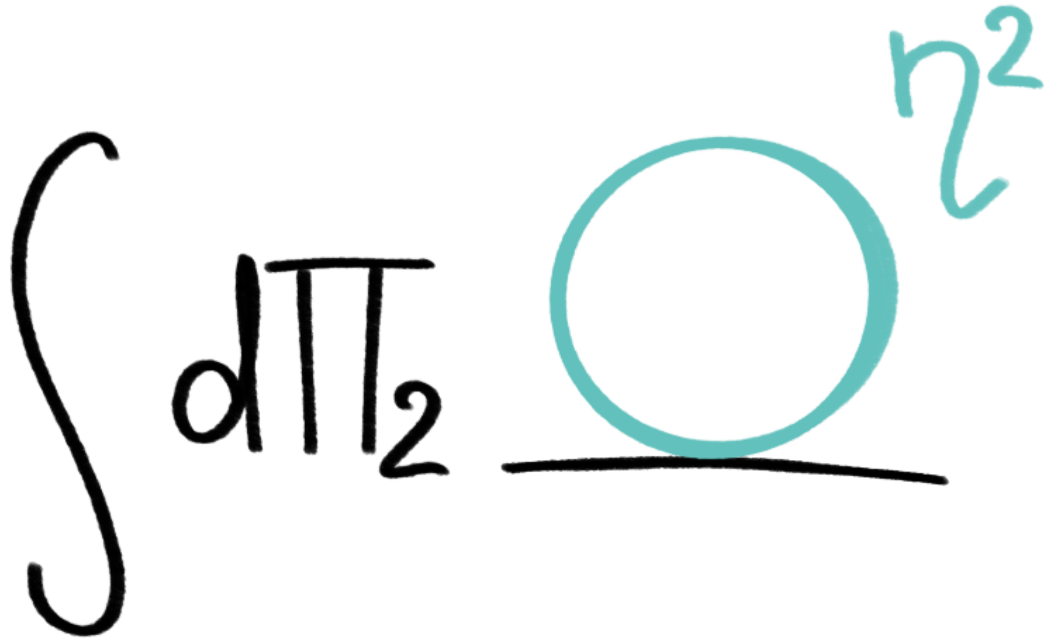
- Loop momentum scales **soft** or **large**
- We have two regions
- $$\lim_{\eta^2\to\infty} \vec{I}^{aux}_{RV} = \lim_{\eta^2\to\infty, k\sim SOFT} \vec{I}^{aux}_{RV} + \lim_{\eta^2\to\infty, k\sim LARGE} \vec{I}^{aux}_{RV}$$

$k \sim SOFT$



- Most complicated soft region
- D_j depends only on kinematics

$k \sim LARGE$



- All large regions are massive tadpoles

Solution of $I^{AUX}(u)$

$$I^{AUX} = S_\Gamma \pi\, y^{-1+2\epsilon} \frac{2u}{2+u} \Bigg\{ \frac{1}{\epsilon^2}\Big(2H_{-1}(u)\Big) + \frac{1}{\epsilon}\Big(4H_{0,-1}(u) + 4H_{-1,-1}(u) - 4H_{-2,-1}(u)\Big)$$

$$+8H_{0,0,-1}(u) + 8H_{0,-1,-1}(u) - 8H_{0,-2,-1}(u) + 16H_{-1,0,-1}(u)$$

$$-16H_{-1,-1,-1}(u) + 8H_{-1,-2,-1}(u) - 8H_{-2,0,-1}(u) - 8H_{-2,-1,-1}(u)$$

$$+8H_{-2,-2,-1}(u) - 10\zeta_2 H_{-1}(u) + \mathcal{O}(y^1, \epsilon) \Bigg\}$$

$$u = 1/\eta^2$$

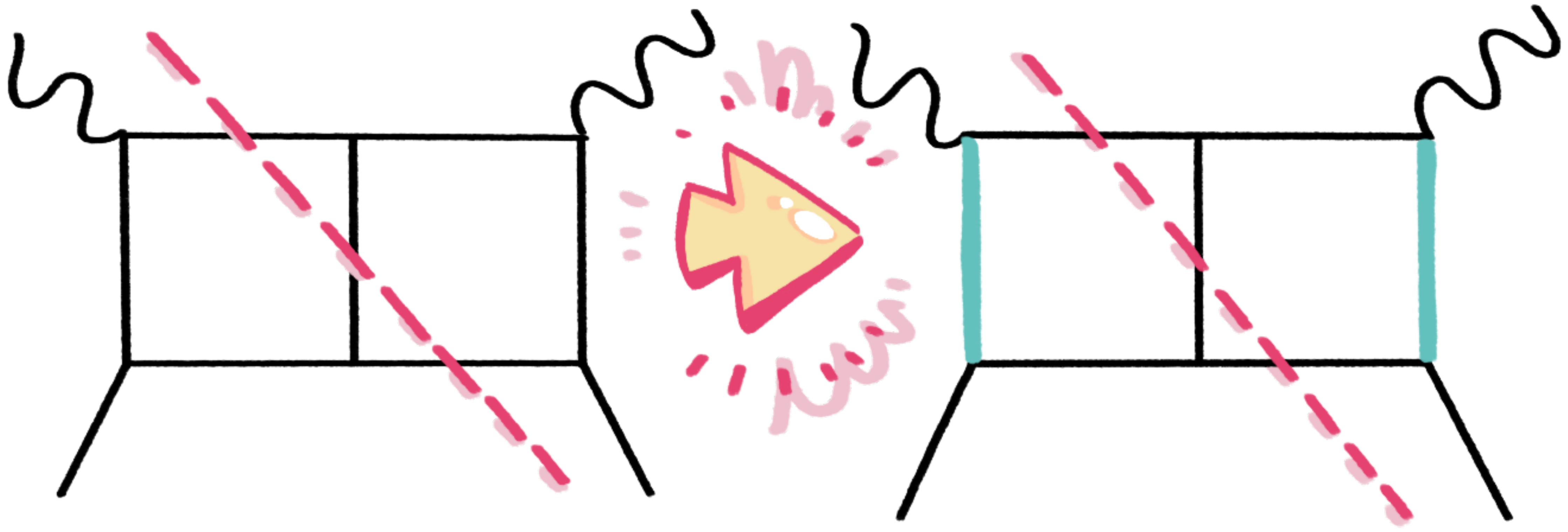
Alphabet:

$$\mathcal{A}_u = \left\{ \omega_0 = \frac{1}{x}, \omega_{-1} = \frac{1}{1+x}, \omega_{-2} = \frac{1}{2+x} \right\}$$

What we want

$$I^{PHYS} \sim (1-z)^{n-2\epsilon} \sum_j c_j(\epsilon)^{PHYS} (1-z)^j, \quad n \in \mathbb{Z}$$

- $c_0(\epsilon)^{PHYS} = c_0(\epsilon)$



What we have

$$I^{AUX} \sim (1-z)^{n-2\epsilon} \sum_j c_j(\epsilon,\eta)^{AUX} (1-z)^j, \quad n \in \mathbb{Z}$$

- $c_0(\epsilon,\eta)^{AUX} = c_0(\epsilon,\eta)$

$$\lim_{\eta^2\to 0, z\to 1} I^{AUX} = I^{PHYS}?$$

aka

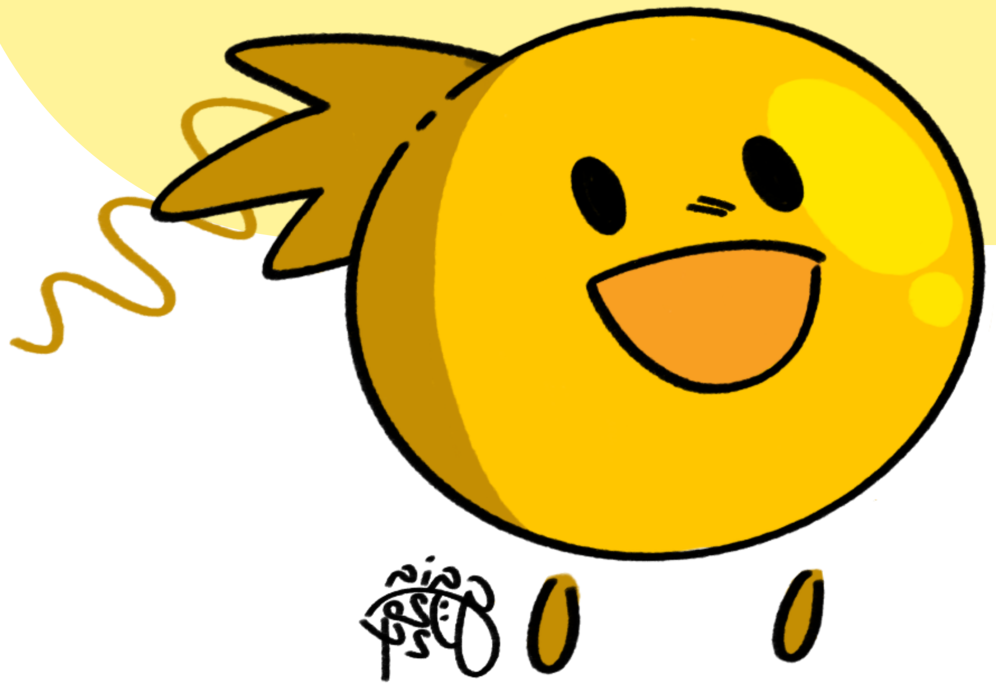
$$\lim_{\eta\to 0} c_0(\epsilon, \eta^2) = c_0(\epsilon)?$$

Not really...

$$\lim_{\eta\to 0} c_0(\epsilon,\eta^2) =$$

$$c_0(\epsilon)+\eta^{-\epsilon} d_1(\epsilon) + \eta^{-2\epsilon} d_2(\epsilon)$$

Extract the hard region from the vanishing mass limit!



Flow to vanishing auxiliary mass

We take naively the limit $\eta^2 \to 0$ in our solution and obtain this expansion:

$$c_0(\eta, \epsilon) = \sum_{k=min}^{\infty} \epsilon^k \left[r_{k,0} + \sum_{m=1}^{k} r_{k,m} \log^m(\eta) \right]$$

$r_{k,m}$ known!

Since we also know the analytic structure of the limit

$$c_0(\eta, \epsilon) = c_0(\epsilon) + \eta^{-\epsilon} d_1(\epsilon) + \eta^{-2\epsilon} d_2(\epsilon) + \mathcal{O}(\eta)$$

Hard region = physical region

We can obtain e.g. $c_0^{(0)}$ by comparing the two $\epsilon-$expansions

$$c_0(\eta,\epsilon) = r_{0,0} + \ldots + \epsilon\, r_{1,1} \log(\eta) + \ldots + \epsilon^2\, r_{2,2} \log^2(\eta)$$

$r_{0,0}, r_{1,1}, r_{2,2}$ known!

$$\begin{aligned} c_0(\eta,\epsilon) = {} & c_0^{(0)} + d_1^{(0)} + d_2^{(0)} \\ & + \epsilon \left(c_0^{(1)} + \left(-d_1^{(0)} - 2d_2^{(0)} \right) \log(\eta) + d_1^{(1)} + d_2^{(1)} \right) \\ & + \epsilon^2 \left(c_0^{(2)} + d_1^{(2)} + d_2^{(2)} + \frac{1}{2} \left(d_1^{(0)} + 4d_2^{(0)} \right) \log^2(\eta) + \left(-d_1^{(1)} - 2d_2^{(1)} \right) \log(\eta) \right) \\ & + \mathcal{O}(\epsilon^3) \end{aligned}$$

$$c_0(\eta,\epsilon) = r_{0,0} + \ldots + \epsilon\, r_{1,1} \log(\eta) + \ldots + \epsilon^2 r_{2,2} \log^2(\eta)$$

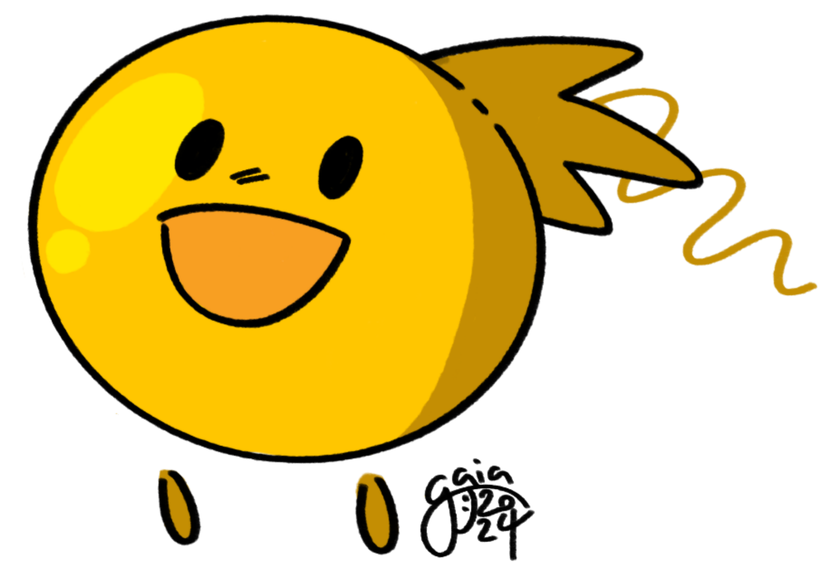
$$\begin{aligned} c_0(\eta,\epsilon) = {} & c_0^{(0)} + d_1^{(0)} + d_2^{(0)} \\ & + \epsilon \left(c_0^{(1)} + \left(-d_1^{(0)} - 2d_2^{(0)}\right) \log(\eta) + d_1^{(1)} + d_2^{(1)} \right) \\ & + \epsilon^2 \left(c_0^{(2)} + d_1^{(2)} + d_2^{(2)} + \frac{1}{2}\left(d_1^{(0)} + 4d_2^{(0)}\right) \log^2(\eta) + \left(-d_1^{(1)} - 2d_2^{(1)}\right) \log(\eta) \right) \\ & + \mathcal{O}(\epsilon^3) \end{aligned}$$

$r_{0,0}, r_{1,1}, r_{2,2}$ known!

Set up this system of eq.s to obtain $c_0^{(0)}$

$$\begin{cases} -d_1^{(0)} - 2d_2^{(0)} = r_{1,1}, \\ d_1^{(0)}/2 + 2d_2^{(0)} = r_{2,2}, \\ c_0^{(0)} + d_1^{(0)} + d_2^{(0)} = r_{0,0} \end{cases}$$

Repeat for all ϵ orders!



- Analogous system for all $c_0^{(i)}$
- Fixed all $\epsilon-$expansion of hard region: ✓

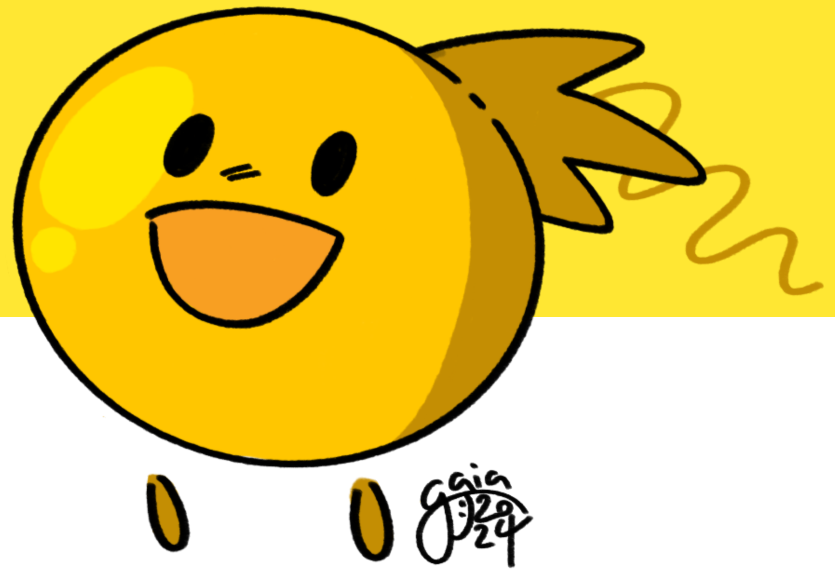
$$\lim_{z\to 1} I = (1-z)^{-1+2\epsilon}\Bigg\{ -\frac{1}{\epsilon^3} + \frac{5\pi^2}{6\epsilon} + \frac{38\zeta_3}{3} + \frac{7\pi^4}{72}\epsilon + \left(\frac{562\zeta_5}{5} - \frac{74\pi^2\zeta_3}{9}\right)\epsilon^2 + \left(\frac{155\pi^6}{1008} - \frac{191\zeta_3^2}{9}\right)\epsilon^3 + \mathcal{O}(\epsilon^4)\Bigg\} + \mathcal{O}\Big((1-z)^0\Big)$$

- Same procedure for RV-integrals

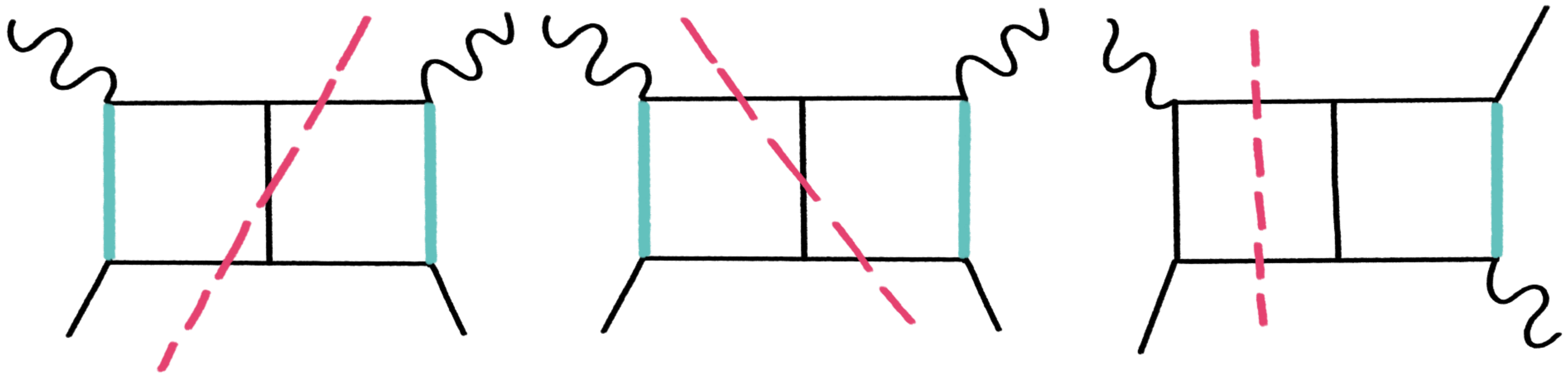
Reduction of constants

- $\partial_u I^{AUX} = A_u \cdot I^{AUX}$, solution in terms of $H(u)$
    - Easier to take the limit $u \to 0$ ($\eta \to \infty$)
- Analytic continuation to perform the $\eta \to 0$ limit
    - $H(u) \to H(1/\eta)$
- This generates constants evaluated at 1 over the alphabet
    - $\mathscr{A}_\eta = \left\{ \omega_0 = \frac{1}{x}, \omega_{-1} = \frac{1}{1+x}, \omega_{-2} = \frac{1}{2+x}, \omega_{-1/2} = \frac{1}{1/2+x} \right\}$
    - Replacing $2 \to t$
    - Fibration of HPLs of argument t, evaluation of $t = 2$ HPLs in terms of known constants

Results



- Procedure applied to fix all nontrivial RR and RV boundaries
- Required the following auxiliary topologies:



- Results used to derive IF antenna functions at higher epsilon order

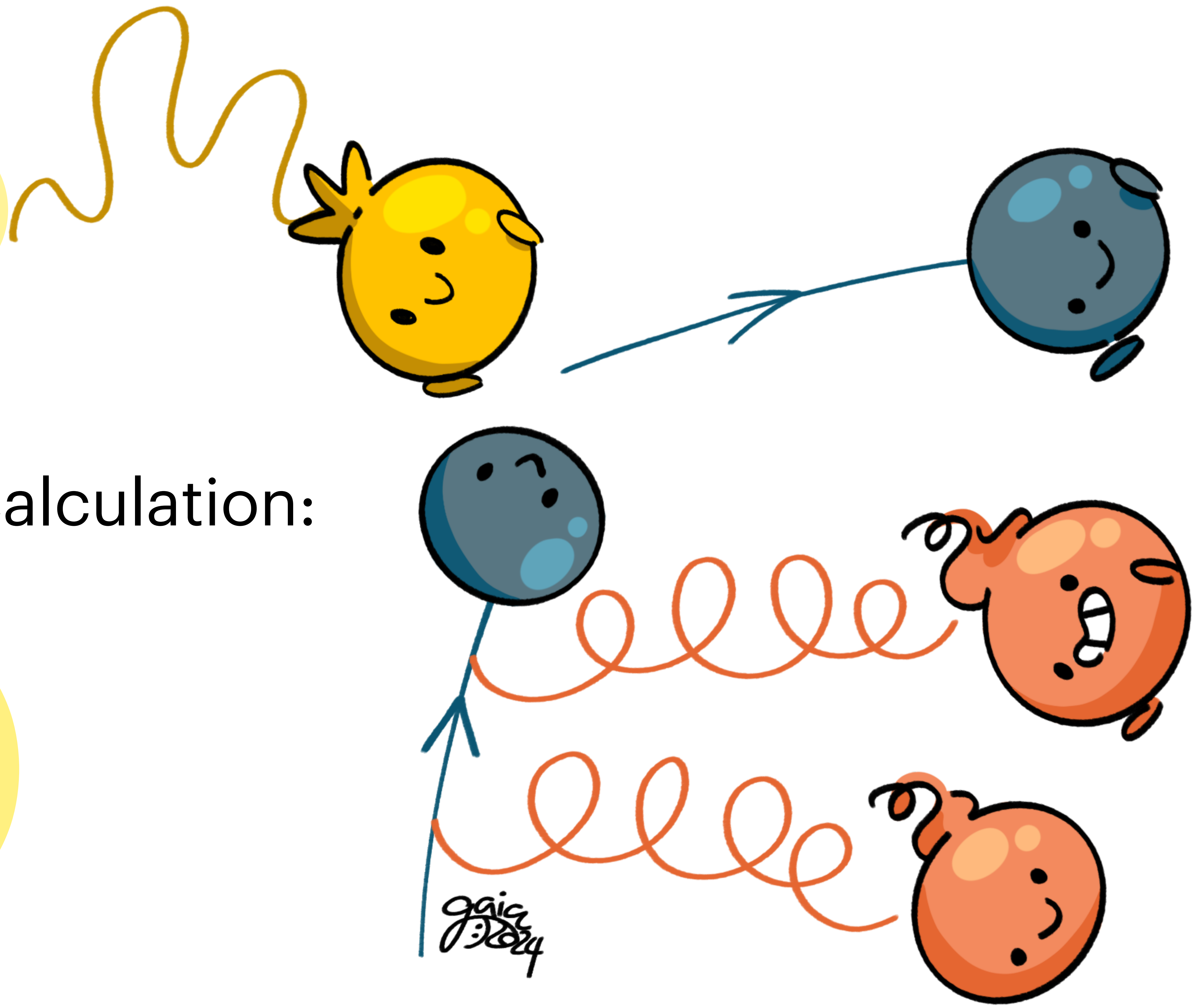
Conclusion

- Analytical extension of auxiliary-mass-flow method
- Feasible to study integrals near singular kinematical points
- Algorithmic procedure

Outlook

- Derivation of IF antennae at 3 loops
- Based on DIS kinematics. Layers of the calculation:
  - RRR
  - RRV
  - RVV
  - VVV

Current focus



Looking for UT integrals

Conjecture: Dlog integrand → UT

Arkani-Hamed, Bourjaily, Cachazo , Trnka (2010)

- Dlog candidates → find as many as possible that are linearly independent
  - Use of Baikov representation & package "DLogBasis" Wasser (2020)
  - Current application on the first RRR family (19 MIs)

$$I = \int \left(\prod_{i=1}^{\ell} \mathrm{d}^d k_i \right) \frac{1}{z_1^{\nu_1} \dots z_n^{\nu_n}} = K \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \, B(\mathbf{z})^\gamma \frac{1}{z_1^{\alpha_1} \dots z_n^{\alpha_n}}$$

- Challenges:
  - Techniques for candidates that are UT but not Dlog
  - Baikov representation of a forward kinematic integral
  - Automatisation

Thank you for your attention!