

# Four-loop two-mass tadpoles and the $\rho$ parameter

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based on work in progress in collaboration with:  
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## NEUTRAL CURRENTS AND THE HIGGS MECHANISM

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*Institute for Theoretical Physics, University of Utrecht \**

### 5. Experimental consequences

For experimental purposes we will assume here that the physical Higgs particles are too heavy to be of interest with present day energies. Then the upshot of the preceding discussion is the introduction of a new parameter  $\beta$ , defined by

$$\beta = M_{\pm}^2/e^2M_0^2 \quad (36)$$

- Important precision observable in the Standard Model

[Ross, Veltmann '75]

- In “modern” notation:

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = 1 + \delta\rho$$

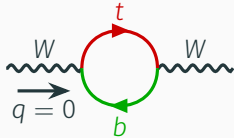
- Tests the structure of the electroweak sector (SM at tree-level:  $\delta\rho = 0$ )

# What to calculate

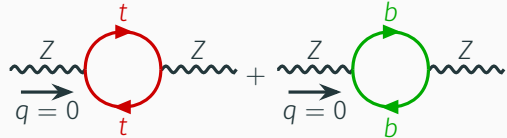
Corrections to the  $\rho$  parameter can be expressed in terms of  $W$  and  $Z$  boson self energies

$$\delta\rho = \frac{\Sigma_Z(0)}{m_Z^2} - \frac{\Sigma_W(0)}{m_W^2}$$

$W$  self energy:



$Z$  self energy:



Long history of higher-order corrections to the  $\rho$  parameter in the Standard Model

$O(\alpha)$

[Veltman '77]

$O(\alpha\alpha_s)$

[Djouadi, Vezeznassi '87]  
[Djouadi '87]  
[Kniehl et al. '88]  
[Kniehl '90]

$O(\alpha\alpha_s^2)$

[Anselm et al. '93]  
[Avdeev et al. '94]  
[Chetyrkin et al. '95]  
[Grigo et al. '12]  
[Blümlein et al. '18]  
[Abreu et al. '19]

$O(\alpha\alpha_s^3)$

[Steinhauser, Schröder '05]  
[Chetyrkin et al. '06]  
[Czakon, Boguezal '06]

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# State of the art

Long history of higher-order corrections to the  $\rho$  parameter in the Standard Model

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$O(\alpha^3)$ [van der Bij et al. '01] [Faist et al. '03]			

Our goal:  $O(\alpha\alpha_s^3)$  with  $m_b \neq 0$

# Evaluation of the $\rho$ parameter

- Self-energies with vanishing external momentum  $q^2 = 0$

$$\delta\rho = \frac{\Sigma_Z(0)}{m_Z^2} - \frac{\Sigma_W(0)}{m_W^2}$$

- Map to tadpole integrals (with different “mass colourings”:  $m_b$ ,  $m_t$  and  $m = 0$ )



- Use standard tools:
  - integration-by-parts reduction to master integrals
  - differential equations in mass ratio

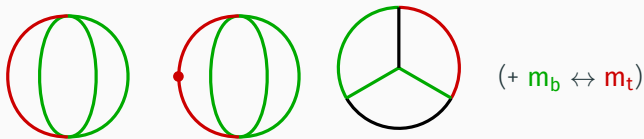
$$x = \frac{m_b}{m_t}$$

- numerical evaluation of master integrals via AMFlow  
[Liu, Ma, Wang '17] [Liu, Ma '21] [Liu, Ma '22a] [Liu, Ma '22b]
- Numerical impact of four-loop two-mass corrections is expected to be very small

# Analytic structure of the two-mass QCD contributions to the $\rho$ parameter

- Up to two-loop order: only harmonic polylogarithms
- Three-loop order: elliptic integrals appeared (two curves, do not mix)

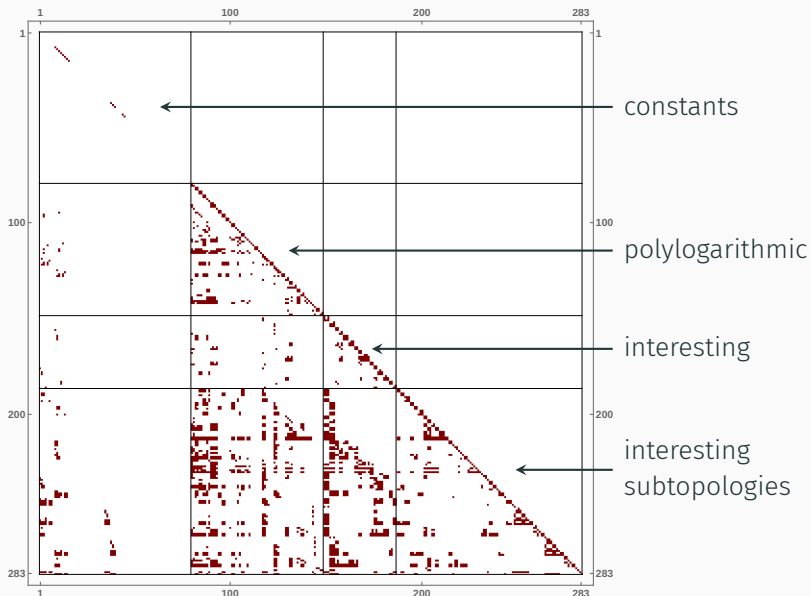
[Grigo et al. '12] [Blümlein et al. '18] [Abreu et al. '19]



- Four-loop: what comes next?
  - More elliptic curves?
  - Higher-genus curves, higher-dimensional geometries?



# Which integrals are interesting?



# Identifying interesting sectors

## Problem

- At four-loop order: 283 master integrals, organised into 197 sectors
- Many integrals are still just polylogarithmic
- How to find “interesting” integrals?

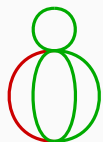
## Solution

- Analyse differential equations
- New structures arise from homogeneous part ( $\epsilon \rightarrow 0$ , no sub-sectors)
- Derive Picard-Fuchs operator  $L$  [Müller-Stach et al. '12] by uncoupling first-order systems into higher-order scalar differential equations [Zürcher '94] [Gerhold '02]
- Factor homogeneous differential operator  $L$  (e.g., using Maple's **DFactor** [van Hoeij '97])

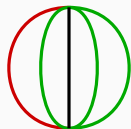
$$L = [\partial_x^2 + f_1(x)\partial_x + f_2(x)] [\partial_x + f_3(x)] \dots [\partial_x + f_r(x)]$$

→ higher-order factors can generate elliptic integrals and beyond

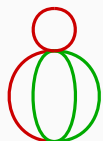
## Potentially elliptic sectors



$$2 \times 2 \text{ system} \\ L = \partial_x^2 + \# \partial_x + \#$$



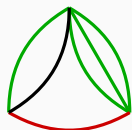
$$3 \times 3 \text{ system} \\ L = [\partial_x^2 + \# \partial_x + \#] [\partial_x + \#]$$



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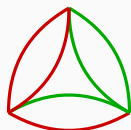
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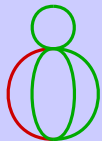


$$2 \times 2 \text{ system} \\ L = \partial_x^2 + \# \partial_x + \#$$

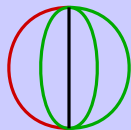


$$4 \times 4 \text{ system} \\ L = [\partial_x^2 + \# \partial_x + \#] [\partial_x + \#] [\partial_x + \#]$$

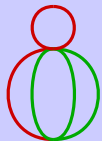
# Potentially elliptic sectors



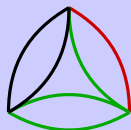
2 × 2 system  
 $L = \partial_x^2 + \# \partial_x + \#$



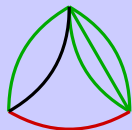
3 × 3 system  
 $L = [\partial_x^2 + \# \partial_x + \#] [\partial_x + \#]$



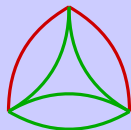
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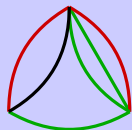
3 × 3 system  
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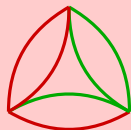
2 × 2 system  
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3 × 3 system  
 $L = [\partial_x^2 + \# \partial_x + \#] [\partial_x + \#]$



2 × 2 system  
 $L = \partial_x^2 + \# \partial_x + \#$



4 × 4 system  
 $L = [\partial_x^2 + \# \partial_x + \#] [\partial_x + \#] [\partial_x + \#]$

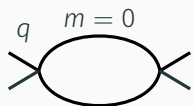
Have “twin” with  $m_b \leftrightarrow m_t$

Symmetric under  $m_b \leftrightarrow m_t$

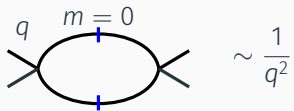
## Idea

- Maximal cut fulfils the homogeneous differential equation (maximal cut eliminates subtopologies)
- Analyse maximal cuts (in  $d = 2$ ) to find homogeneous solutions [Primo, Tancredi '16] [Primo, Tancredi '17]
- Work in loop-by-loop Baikov representation [Baikov '96] [Frellesvig, Papadopoulos '17]  
→ reuse building blocks to construct maximal cuts
- Identify geometric structure underlying the integrals that go beyond polylogarithms. E.g., elliptic curves show up as  $\int \frac{dz}{y}$  with  $y^2 = P(z)$  and  $P(z)$  third- or forth-order polynomial.

## Maximal cut analysis: bubble and sunrise examples


$$\sim \int dz_1 dz_2 \frac{(q^2)^{(2-d)/2}}{z_1 z_2} [(q^2)^2 - 2q^2(z_1 + z_2) + (z_1 - z_2)^2]^{(d-3)/2}$$

# Maximal cut analysis: bubble and sunrise examples

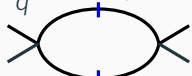


$q$   $m = 0$

$$\sim \frac{1}{q^2}$$

# Maximal cut analysis: bubble and sunrise examples

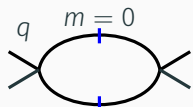
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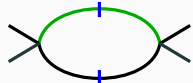

$$\sim \frac{1}{q^2 - m_b^2}$$



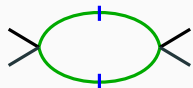
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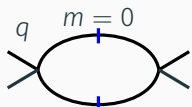


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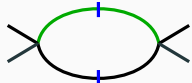


$$\sim \frac{1}{\sqrt{q^2[q^2 - 4m_b^2]}}$$

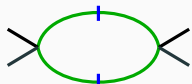
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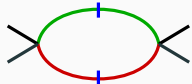
$$\sim \frac{1}{q^2}$$



$$\sim \frac{1}{q^2 - m_b^2}$$

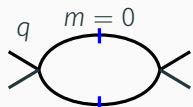


$$\sim \frac{1}{\sqrt{q^2[q^2 - 4m_b^2]}}$$

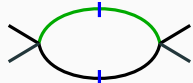


$$\sim \frac{1}{\sqrt{[q^2 - (m_b - m_t)^2][q^2 - (m_b + m_t)^2]}}$$

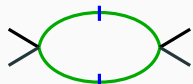
# Maximal cut analysis: bubble and sunrise examples



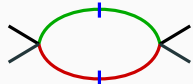
$$\sim \frac{1}{q^2}$$



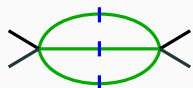
$$\sim \frac{1}{q^2 - m_b^2}$$



$$\sim \frac{1}{\sqrt{q^2[q^2 - 4m_b^2]}}$$

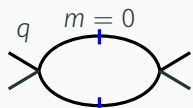


$$\sim \frac{1}{\sqrt{[q^2 - (m_b - m_t)^2][q^2 - (m_b + m_t)^2]}}$$

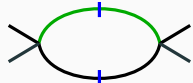


$$\sim \int \frac{dz}{\sqrt{[z - (\sqrt{q^2} - m_b)^2][z - (\sqrt{q^2} + m_b)^2]}\sqrt{z[z - 4m_b^2]}}$$

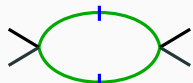
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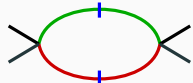
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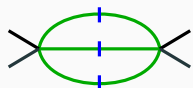
$$\sim \frac{1}{q^2 - m_b^2}$$



$$\sim \frac{1}{\sqrt{q^2[q^2 - 4m_b^2]}}$$



$$\sim \frac{1}{\sqrt{[q^2 - (m_b - m_t)^2][q^2 - (m_b + m_t)^2]}}$$



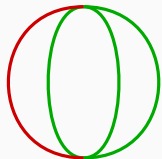
$$\sim \int \frac{dz}{\sqrt{[z - (\sqrt{q^2} - m_b)^2][z - (\sqrt{q^2} + m_b)^2]}\sqrt{z[z - 4m_b^2]}}$$

→ Maximal cut of sunrise is an elliptic integral [Primo, Tancredi '16] [Primo, Tancredi '17]

→ Is annihilated by sunrise differential operator

# Maximal cut analysis: three-loop example

Example: Analyse one of the elliptic tadpole integral from the three-loop  $\rho$  parameter

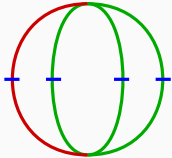


The diagram shows a tadpole integral with two internal loops. The left loop is a red circle, and the right loop is a green circle. They are connected at their top and bottom vertices by a single vertical line. The diagram is positioned to the left of the integral expression.

$$\sim \int d^d k B_{bt}(k^2) B_{bb}(k^2)$$

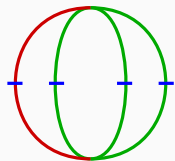

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$$\sim \int \frac{dk^2}{\sqrt{[k^2 - (m_t - m_b)^2][k^2 - (m_t + m_b)^2]} \sqrt{k^2[k^2 - 4m_b^2]}}$$

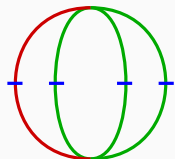
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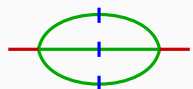

$$\sim \int \frac{dk^2}{\sqrt{[k^2 - (m_t - m_b)^2][k^2 - (m_t + m_b)^2]} \sqrt{k^2[k^2 - 4m_b^2]}}$$


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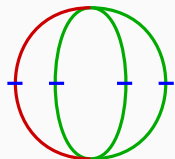
This is exactly the maximal cut of the on-shell ( $q^2 = m_t^2$ ) two-loop sunrise diagram:


$$\sim \int \frac{dz}{\sqrt{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]} \sqrt{z[z - 4m_b^2]}}$$



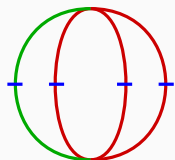
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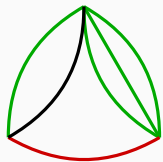
→ “sunrise curve”

The  $m_b \leftrightarrow m_t$  “twin” has a similar, but non-isomorphic curve:


$$\sim \int \frac{dk^2}{\sqrt{[k^2 - (m_t - m_b)^2][k^2 - (m_t + m_b)^2]} \sqrt{k^2[k^2 - 4m_t^2]}}$$

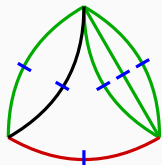
→ “sunrise twin curve”

# Maximal cut analysis: four-loop examples



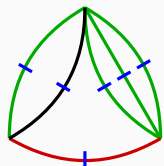
$$\sim \int d^d k B_{b0}(k^2) \frac{1}{k^2 - m_t^2} S_{bbb}(k^2)$$

# Maximal cut analysis: four-loop examples



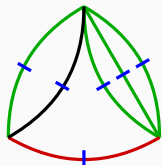
$$\sim \int dk^2 \frac{\delta(k^2 - m_t^2)}{k^2 - m_b^2} \int \frac{dz}{\sqrt{[z - (\sqrt{k^2} - m_b)^2][z - (\sqrt{k^2} + m_b)^2]}\sqrt{z[z - 4m_b^2]}}$$

# Maximal cut analysis: four-loop examples



$$\sim \int dk^2 \frac{\delta(k^2 - m_t^2)}{k^2 - m_b^2} \int \frac{dz}{\sqrt{[z - (\sqrt{k^2} - m_b)^2][z - (\sqrt{k^2} + m_b)^2]}\sqrt{z[z - 4m_b^2]}}$$

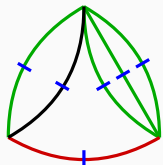
## Maximal cut analysis: four-loop examples



$$\sim \frac{1}{m_t^2 - m_b^2} \int \frac{dz}{\sqrt{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]}\sqrt{z[z - 4m_b^2]}}$$

→ “sunrise curve”

# Maximal cut analysis: four-loop examples



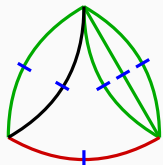
$$\sim \frac{1}{m_t^2 - m_b^2} \int \frac{dz}{\sqrt{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]}\sqrt{z[z - 4m_b^2]}}$$

→ “sunrise curve”



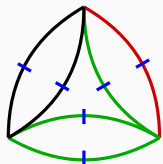
$$\sim \int d^d k B_{00}(k^2) B_{bb}(k^2) B_{bt}(k^2)$$

# Maximal cut analysis: four-loop examples



$$\sim \frac{1}{m_t^2 - m_b^2} \int \frac{dz}{\sqrt{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]}\sqrt{z[z - 4m_b^2]}}$$

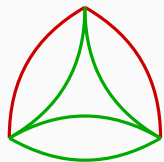
→ “sunrise curve”



$$\sim \int \frac{dz}{z\sqrt{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]}\sqrt{z[z - 4m_b^2]}}$$

→ “sunrise curve”

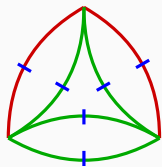
## Maximal cut analysis: four-loop examples (cont.)



$$\sim \int d^d k B_{bt}^2(k^2) B_{bb}(k^2)$$



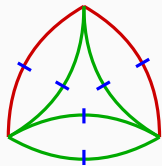
## Maximal cut analysis: four-loop examples (cont.)



$$\sim \int \frac{dz}{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]\sqrt{z[z - 4m_b^2]}}$$

→ algebraic solution, not elliptic!

## Maximal cut analysis: four-loop examples (cont.)



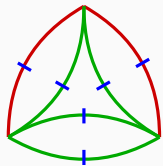
$$\sim \int \frac{dz}{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]\sqrt{z[z - 4m_b^2]}}$$

→ algebraic solution, not elliptic!



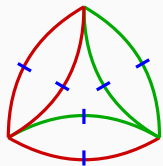
$$\sim \int d^d k B_{tt}(k^2) B_{bb}(k^2) B_{bt}(k^2)$$

## Maximal cut analysis: four-loop examples (cont.)



$$\sim \int \frac{dz}{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]\sqrt{z[z - 4m_b^2]}}$$

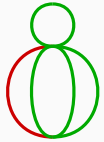
→ algebraic solution, not elliptic!



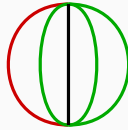
$$\sim \int \frac{dz}{z \sqrt{[z - 4m_t^2]}\sqrt{[z - 4m_b^2]}\sqrt{[z - (m_t - m_b)^2][z - (m_t + m_b)^2]}}$$

→ “new curve”

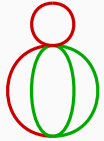
# Maximal cut analysis: results



→ Sunrise curve



→ Sunrise curve



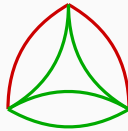
→ Sunrise curve



→ Sunrise curve



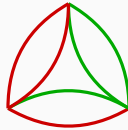
→ Sunrise curve



→ Algebraic, not elliptic

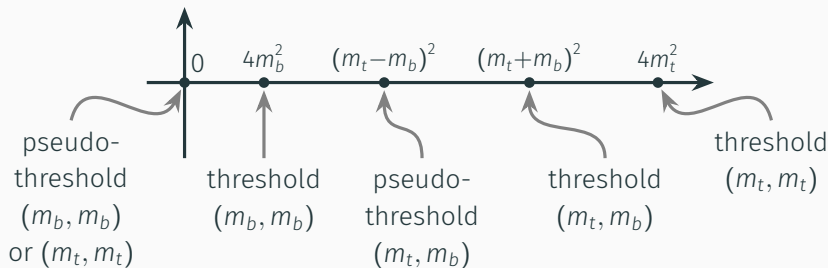


→ Sunrise curve



→ New curve

# Interpretation of the branch points



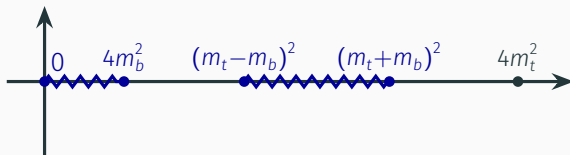
- Maximal cut picture lends itself to physical interpretation of the branch points
- Each curve uses four of the five possible points

Sunrise curve:  $y^2 = z [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2]$

Sunrise twin curve:  $y^2 = z [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] [z - 4m_t^2]$

New curve:  $y^2 = [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] [z - 4m_t^2]$

# Interpretation of the branch points



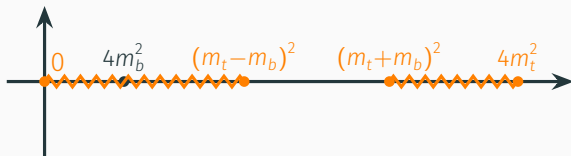
- Maximal cut picture lends itself to physical interpretation of the branch points
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Sunrise curve:  $y^2 = z [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2]$

Sunrise twin curve:  $y^2 = z [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] [z - 4m_t^2]$

New curve:  $y^2 = [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] [z - 4m_t^2]$

# Interpretation of the branch points



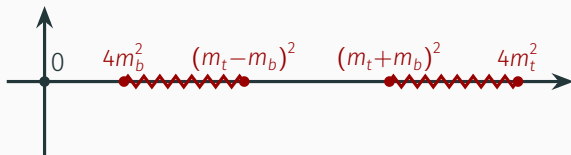
- Maximal cut picture lends itself to physical interpretation of the branch points
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Sunrise curve:  $y^2 = z [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2]$

Sunrise twin curve:  $y^2 = z [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] [z - 4m_t^2]$

New curve:  $y^2 = [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] [z - 4m_t^2]$

# Interpretation of the branch points



- Maximal cut picture lends itself to physical interpretation of the branch points
- Each curve uses four of the five possible points

Sunrise curve:  $y^2 = z [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2]$

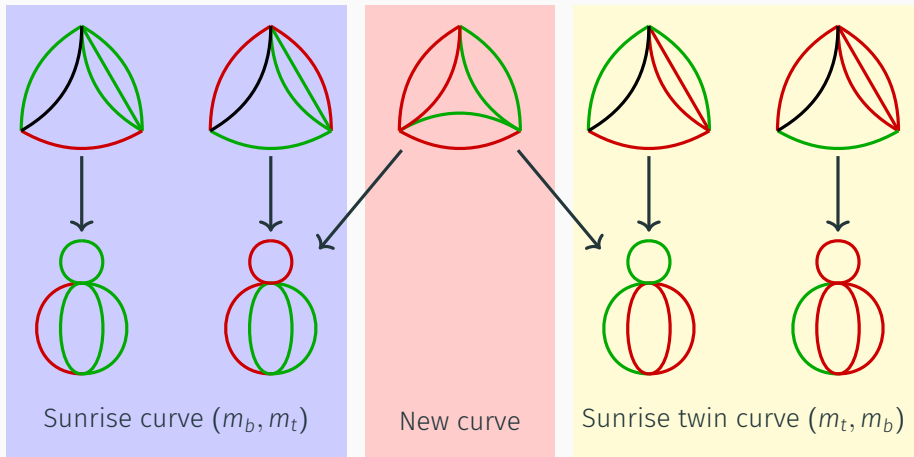
Sunrise twin curve:  $y^2 = z [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] [z - 4m_t^2]$

New curve:  $y^2 = [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] [z - 4m_t^2]$

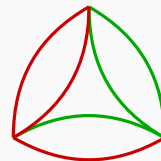
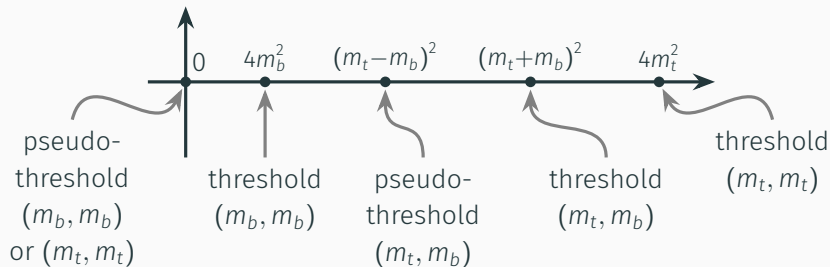


# Interaction of the elliptic curves

- Homogeneous differential equations only generate one elliptic curve per sector
- Inhomogeneous terms can mix different curves  
→ also for sectors that are themselves elliptic

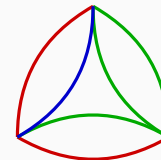


# Genus-2 curves from necklace integrals



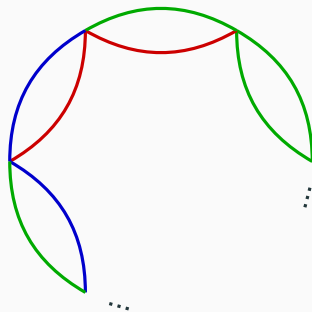
- With two masses we can only have 5 different (pseudo-)thresholds
- If we allow for three masses, we can construct an integral with a genus-2 curve (degree 6 polynomial) in its maximal cut

$$\begin{aligned}
 y^2 = & z [z - 4m_b^2] [z - (m_t - m_b)^2] [z - (m_t + m_b)^2] \\
 & \times [z - (m_t - m_3)^2] [z - (m_t + m_3)^2]
 \end{aligned}$$



# An integral for any genus

With enough distinct masses we can construct  $L$ -loop integrals with a genus- $(L - 1)$  curve in its maximal cut:



- Each bubble contributes two distinct (pseudo-)thresholds
- Interesting class of examples / toy models for further study!

## Conclusions

- We calculate the two-mass QCD contributions to the four-loop  $\rho$  parameter
- Interesting testing ground for special functions beyond MPLs
- We find three elliptic curves contributing to the integrals
- The three elliptic curves mix in the differential equation

## Outlook

- Finish calculation of numerical value for four-loop corrections
- Calculate series representations for the tadpole master integrals
- Find good description for expressing iterated integrals over these curves
- Study necklace integrals as examples for higher-genus curves