

Analysis of $(n + 1)$ and n -parton contributions for computing QCD jet cross sections in the LASS scheme

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Universität Hamburg

DER FORSCHUNG | DER LEHRE | DER BILDUNG

- ▶ The general structure of the NNLO correction:

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n VV \delta_n(X) + \int d\Phi_{n+1} RV \delta_{n+1}(X) + \int d\Phi_{n+2} RR \delta_{n+2}(X)$$

- ▶ In the Local Analytic Sector Subtraction scheme this is rewritten as:

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n VV_{\text{sub}}(X) + \int d\Phi_{n+1} RV_{\text{sub}}(X) + \int d\Phi_{n+2} RR_{\text{sub}}(X),$$

where

$$RR_{\text{sub}}(X) \equiv RR \delta_{n+2}(X) - K^{(1)} \delta_{n+1}(X) - \left(K^{(2)} - K^{(12)} \right) \delta_n(X),$$

$$RV_{\text{sub}}(X) \equiv \left(RV + I^{(1)} \right) \delta_{n+1}(X) - \left(K^{(\text{RV})} + I^{(12)} \right) \delta_n(X),$$

$$VV_{\text{sub}}(X) \equiv \left(VV + I^{(2)} + I^{(\text{RV})} \right) \delta_n(X).$$

- ▶ See Adam's talk for the discussion regarding RR_{sub}
- ▶ Focus of this talk: RV_{sub} and VV_{sub}

$$RR_{\text{sub}}(X) \equiv RR \delta_{n+2}(X) - K^{(1)} \delta_{n+1}(X) - \left(K^{(2)} - K^{(12)} \right) \delta_n(X),$$

$$RV_{\text{sub}}(X) \equiv \left(RV + I^{(1)} \right) \delta_{n+1}(X) - \left(K^{(\text{RV})} + I^{(12)} \right) \delta_n(X),$$

$$VV_{\text{sub}}(X) \equiv \left(VV + I^{(2)} + I^{(\text{RV})} \right) \delta_n(X).$$

- ▶ $K^{(1)}$ - captures the single unresolved singularities
- ▶ $K^{(2)}$ - captures the double unresolved singularities
- ▶ $K^{(12)}$ - accounts for the overlap of $K^{(1)}$ and $K^{(12)}$
- ▶ $K^{(12)}$ - accounts for the overlap of $K^{(1)}$ and $K^{(12)}$
- ▶ $K^{(\text{RV})}$ - captures the phase-space singularities of RV
- ▶ I -terms - the integrated versions of the above terms

The beauty of the LASS scheme: *The clever design of K -terms*

- ▶ Every contribution is free of **explicit ϵ -poles** and **phase-space singularities**
- ▶ All the subtraction terms are analytical expressions, requiring only numerical **evaluation** for each phase-space point during the integration process

- ▶ All the building blocks of the scheme are given in the references [[arXiv:1806.09570](#)], [[arXiv:2010.14493](#)], [[arXiv:2212.11190](#)] (Magnea et al. 2019, 2020, 2022)

Implementation strategy:

1. Rederive every analytical expression given in these papers
 - ▶ Protects against accidental typos in lengthy equations
 - ▶ Confirmation that one has a correct understanding of the procedure
2. Check the explicit ϵ -pole cancellation
3. Check the singular behavior during the phase-space integration
4. Design an automatic generator of the subtraction terms for any process
5. Implement an efficient MC generator
6. Study the physical processes $e^+e^- \rightarrow 3\text{jets}, 4\text{jets}, \dots$

- ▶ In some cases the analytical integration of the subtraction terms can be quite involved
- ▶ Following the strategy of [\[arXiv:2010.14493\]](https://arxiv.org/abs/2010.14493) every integral can be reduced to the following form

$$I_{a,A,B,C,D,E,F,G} = \int_0^1 dy' \int_0^1 dz \int_0^1 dz' \int_0^1 dw' \times$$

$$\left(w' (1 - w') \right)^{-1/2+\epsilon} (1 - y')^A (y')^B (1 - z)^C z^D (1 - z')^E (z')^F$$

$$\times \frac{1}{\left[1 - (1 - y') z' \right]^{G-1} \left[z (1 - z') + y'(1 - z)z' + 2 (1 - 2w') \sqrt{y'} \sqrt{(1 - z)z} \sqrt{(1 - z') z'} \right]^a}.$$

- ▶ For the specific values of (a, A, B, C, D, E, F, G) which appear in the problem, this was possible to integrate analytically.
- ▶ Some of the master integrals were double-checked using pySecDec (Heinrich et al.).
- ▶ The procedure was automatized in FORM and Mathematica packages.
- ▶ FORM package can rederive all the integrated subtraction terms from the LASS papers within ~ 5 seconds.

where we only integrate in the first line the full-particle chains... the first spin-up electron pair can have three... the full spin-up electron pair can have three...

The next iterate obtained for three-line parts when given an (n-2) loop photon space...

Series of equations involving sums over indices and integrals, representing higher-order corrections.

We emphasize that the three-line parts in this section apply to one hadron vertex and three vertices...

4.4. Assembling the complete integrated counterterms

After summing all contributions that were differentially stopped, exhibiting symmetry, and testing on the four-line basis...

Equation defining the counterterm contribution K^3,3 in terms of various integrals.

Furthermore, we can reduce the double correlation functions K^2,2 and K^2,4 to the same components...

Equations showing the reduction of K^2,2 and K^2,4 to K^3,3 and K^3,4 terms.

Since the pole parts of both K^3,3 and K^3,4 are explicitly known, the necessary counterterms can be deduced...

With these counterterms in hand, we can assemble the counterterms for the diagrams of the three-line diagrams...

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Thanks to the fact that the full and the integral counterterms coincide with the integrated ones, the latter is still available in a straightforward way...

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the integrals J^1,1, J^2,1, J^2,2, J^2,3 and J^2,4 are defined in appendix A, and W_{2,2,1} is given by eq. (C.2).

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Here we list the full-particle counterterms obtained for single-line, double-line, eq. (3.11) and K^3,4 in the double-correlation counterterm, defined as eq. (A.7).

Integrating double-line, with three-line vertices and double-line vertices counterterms, the last of which may involve three or five three-line particles...

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Here the counterterm integrals are given by eq. (B.4). The left three-line vertices counterterm functions are:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the rules +, - as defined in eq. (B.3), govern a line being loop or jet. In eq. (B.4) we have introduced the following, with three-line vertices integrals:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

These counterterm integrals can be found in eq. (B.13). Next, we give the counterterm contributions for the diagrams of the three-line diagrams...

Again the result eq. (3.16) is to be given as representative. It is to be understood that the counterterm or integrals (with + or -) are to be summed over all lines...

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

With the counterterms in hand, we can assemble the counterterms for the diagrams of the three-line diagrams...

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Since the counterterms themselves, at this stage, of the right parts have already been specified, the latter counterterms, K^3,3^c, is given by eq. (3.18). We can only do diagram eq. (3.18) by eq. (3.19).

By using the counterterms in hand, we can assemble the counterterms for the diagrams of the three-line diagrams...

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Starting from eq. (3.20), it is then straightforward to obtain:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the relevant counterterm integrals are given by eq. (B.11). Finally, we come to the double-line vertices counterterm involving two three-line particles, which reads:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

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Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the counterterm integrals are given by eq. (B.11). Similarly to (3.17), for the integral of the strongly-ordered counterterms, K^3,4, we give the counterterm with both integrated and operational vertex functions, as we do in eq. (3.17).

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the three-line part of vertex functions, K^3,3, are given by eq. (B.11). We can use eq. (B.11) as input for the counterterm with both integrated and operational vertex functions...

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the counterterm integrals are given by eq. (B.11) and in eq. (B.13). We notice that the counterterm integrals, K^3,3, are given by eq. (B.11). We notice that the counterterm integrals, K^3,3, are given by eq. (B.11).

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the left lines of the propagator vertex functions, A_{2,1}, is defined in eq. (3.20). The three-line vertices counterterm K^3,3^c is obtained by combining eqs. (A.7) with eqs. (3.11) and (3.16), dropping the double-pole parts. The result is entirely symmetric and can be written for the gauge-invariant part of the strongly-ordered counterterms...

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where we introduced the double-line vertices:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Notice that also in eq. (3.21) the kinematical dependence is expressed only in terms of single logarithms. One may stop to use to integrate the real and virtual counterterms, and add back the result to eq. (3.20).

Integration of the real-virtual counterterms

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

and these counterterm contributions of the same order and same integrals as in eq. (3.21). The first (with + or -) are:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the double-line part of vertex functions, W_{2,2,1}, is given by eq. (3.18), and again two three-line vertices have to be introduced. Finally, the left-to-right vertex has a similar counterterm and reads (with + or -):

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

As already stated, a more compact notation can be obtained using conventional vertex functions. We use now:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where the full counterterms are given by eq. (3.27) and the hadronic counterterm K^3,3^c is defined as:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where we introduced the double-line vertices:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

In eq. (3.18, 3.21) we have defined the counterterms K^3,3^c, that resulted so far to build the real-virtual and virtual counterterms of K^3,3, integrals to which in (3.1) both photon space, and line of poles as. The K^3,3^c counterterms need to be integrated in the (n-2) loop dimension in the radiation phase space and then the result can be added back, line-by-line to the real-virtual counterterms in eq. (3.20) (3.18). In order to compute the integrated counterterms, K^3,3^c, we defined in (3.21), we proceed by inserting all vertex W_{2,2,1}, that is vertex functions drop out of the calculation, using in the same order they satisfy (that have three lines in (3.18)). We thus perform the integration over the radiation phase space with the measure d^3k_{1,2,3}, naturally reduced by the missing lines, according to:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

where W_{2,2,1} is defined in eq. (3.18). The integration of K^3,3^c is carried out following the method described in ref. [36] and using that the real-virtual counterterms are proportional to the branch cuts, Q_{2,1}^+ and Q_{2,2}^+, which can be integrated, and defined in appendix B. The final expression for the integrated counterterm of K^3,3^c can be written as:

Equation defining the counterterm contribution K^3,3^c in terms of various integrals.

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- ▶ The cancellation of the explicit poles of RV by $I^{(1)}$ is manifest
- ▶ In case of VV , upon expansion of $I^{(2)} + I^{(RV)}$ we could obtain the general pole structure of VV [Catani 1998] with the opposite sign (as advertised by the LASS papers):

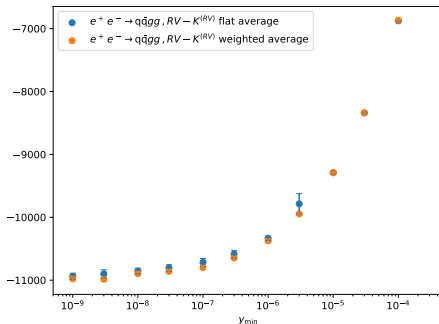
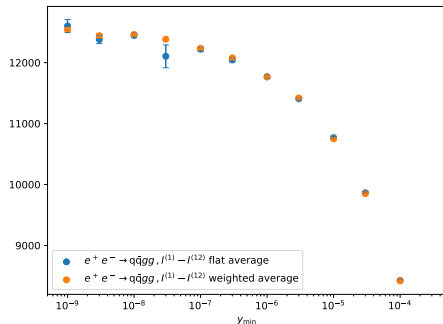
$$\begin{aligned}
 VV_{\text{poles}} = 2 \operatorname{Re} & \left[-\frac{1}{2} \left\langle \mathcal{M}^{(0)} \left| I^{(1)}(\epsilon) I^{(1)}(\epsilon) \right| \mathcal{M}^{(0)} \right\rangle - \frac{\beta_0}{\epsilon} \left\langle \mathcal{M}^{(0)} \left| I^{(1)}(\epsilon) \right| \mathcal{M}^{(0)} \right\rangle \right. \\
 & + \left\langle \mathcal{M}^{(0)} \left| I^{(1)}(\epsilon) \right| \mathcal{M}^{(1)} \right\rangle + e^{-\epsilon\gamma} \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{\beta_0}{\epsilon} + K \right) \left\langle \mathcal{M}^{(0)} \left| I^{(1)}(2\epsilon) \right| \mathcal{M}^{(0)} \right\rangle \\
 & \left. + \left\langle \mathcal{M}^{(0)} \left| H^{(2)}(\epsilon) \right| \mathcal{M}^{(0)} \right\rangle \right] \\
 & + \operatorname{Re} \left[2 \left\langle \mathcal{M}^{(1)} \left| I^{(1)}(\epsilon) \right| \mathcal{M}^{(0)} \right\rangle - \left\langle \mathcal{M}^{(0)} \left| I^{(1)\dagger}(\epsilon) I^{(1)}(\epsilon) \right| \mathcal{M}^{(0)} \right\rangle \right]
 \end{aligned}$$

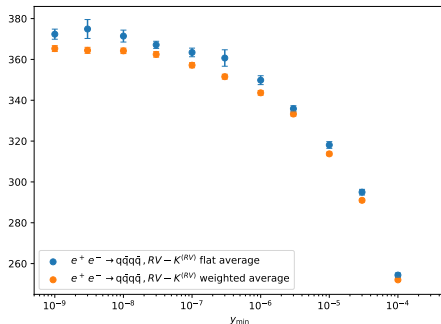
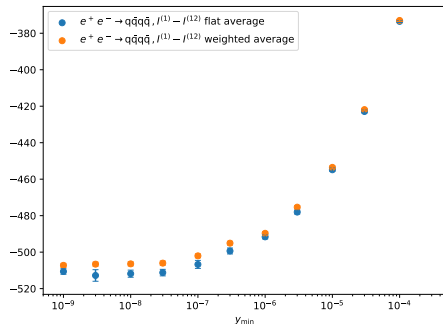
$$\begin{aligned}
 (I^{(2)} + I^{(RV)})_{\text{poles}} = & B \left(\frac{\alpha_s}{2\pi} \right)^2 \left[\frac{1}{\epsilon^4} \left(\frac{CA^2}{2} + \left(2CF - \frac{17}{3} \right) CA + 2CF^2 - \frac{34CF}{3} \right) \right. \\
 & + \frac{1}{\epsilon^3} \left(\left(\log(\mu^2) + \log(s_{12}) - \log(s_{13}) - \log(s_{23}) + \frac{11}{24} \right) CA^2 \right. \\
 & + \left(-\frac{nfTr}{6} - \frac{34 \log(\mu^2)}{3} - 6 \log(s_{12}) + \frac{26 \log(s_{13})}{3} \right. \\
 & + CF \left(4 \log(\mu^2) - 2 \log(s_{13}) - 2 \log(s_{23}) + \frac{47}{12} \right) + \frac{26 \log(s_{23})}{3} - \frac{164}{9} \left. \right) CA \\
 & + \frac{34nfTr}{9} + CF^2 \left(4 \log(\mu^2) - 4 \log(s_{12}) + 6 \right) \\
 & + CF \left(-\frac{nfTr}{3} - \frac{68 \log(\mu^2)}{3} + \frac{32 \log(s_{12})}{3} + 6 \log(s_{13}) + 6 \log(s_{23}) - \frac{98}{3} \right) \left. \right) \\
 & + \frac{1}{\epsilon^2} (\dots) + \frac{1}{\epsilon} (\dots) \left. \right]
 \end{aligned}$$

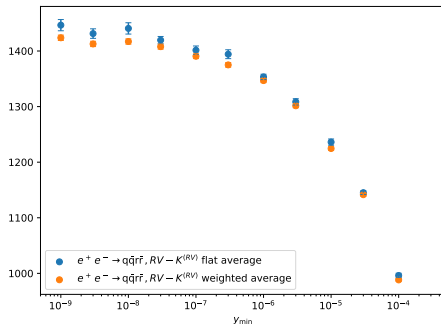
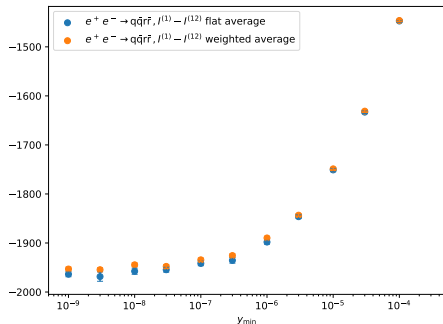
- ▶ With the opposite sign this exactly agrees with the standard $e^+e^- \rightarrow 3$ jet references [Garland, Gehrmann et al. 2001], [Gehrmann-De Ridder, Gehrmann et al. 2007].

$$RV_{\text{sub}}(X) \equiv \left(RV + I^{(1)} \right) \delta_{n+1}(X) - \left(K^{(RV)} + I^{(12)} \right) \delta_n(X) \quad (1)$$

- ▶ Automatic generation of $I^{(1)}$, $K^{(RV)}$ and $I^{(12)}$ using FORM.
- ▶ The RV matrix element is extracted from [Bern, Dixon, Kosower 1997].
- ▶ Everything is fed to the MC integrator (see Adam's talk).
- ▶ Time to perform y_{\min} study (see Adam's talk for the definition).







- ▶ After reassuring in the stability of the integration procedure, one can fix the y_{\min} value and produce the distributions of some event shape observables.
- ▶ In the following we will examine following event shape observables:

1. τ -parameter (thrust):

$$T = \max_{\vec{n}} \left(\frac{\sum_i |\vec{n} \cdot \vec{p}_i|}{\sum_i |\vec{p}_i|} \right), \quad \tau \equiv 1 - T.$$

2. C-parameter:

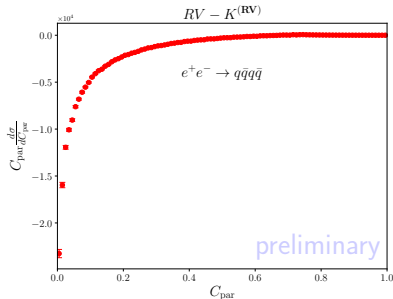
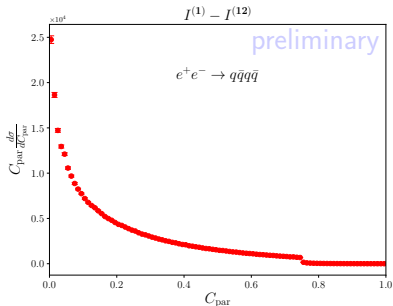
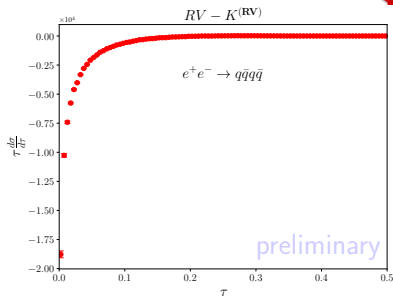
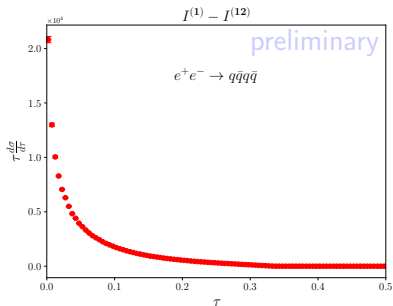
$$C_{\text{par}} = \frac{3}{2} \frac{\sum_{i,j} |\vec{p}_i| |\vec{p}_j| \sin^2 \theta_{ij}}{(\sum_i |\vec{p}_i|)^2}.$$

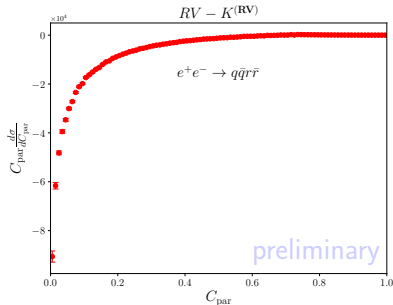
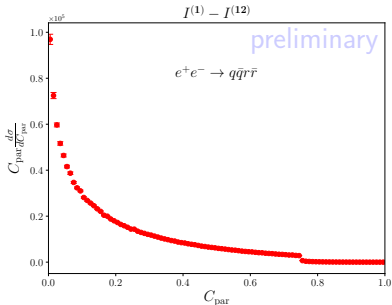
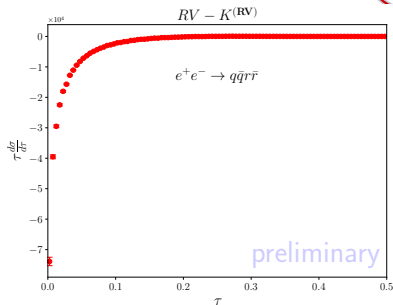
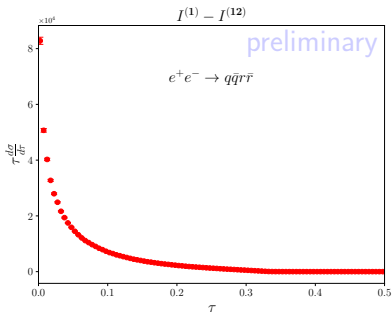
3. Energy-energy correlation:

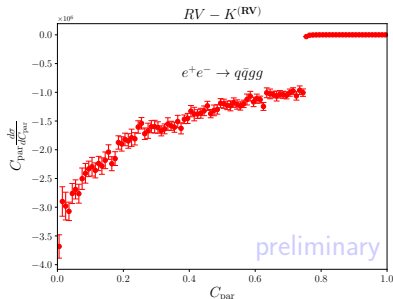
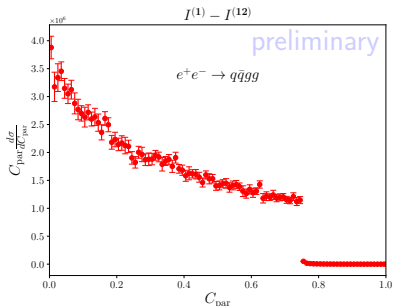
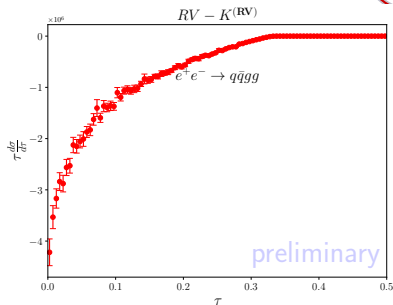
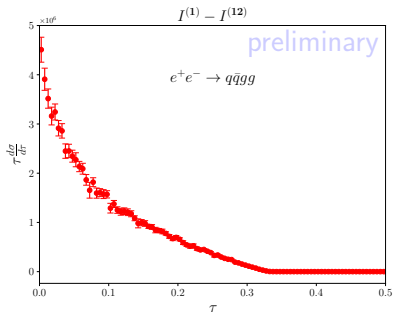
$$\text{EEC}(\chi) = \frac{1}{\sigma_{\text{had}}} \sum_{i,j} \int \frac{E_i E_j}{Q^2} d\sigma_{e^+e^- \rightarrow ij+\chi} \delta(\cos \chi + \cos \theta_{ij}).$$

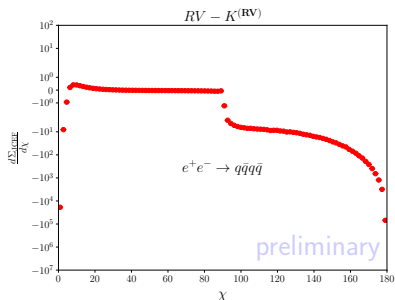
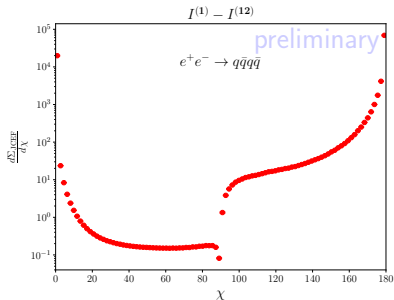
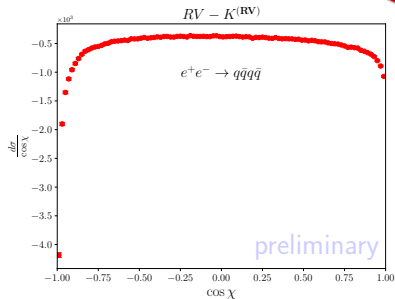
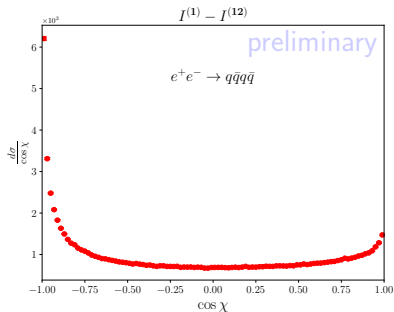
4. Jet-cone energy fraction:

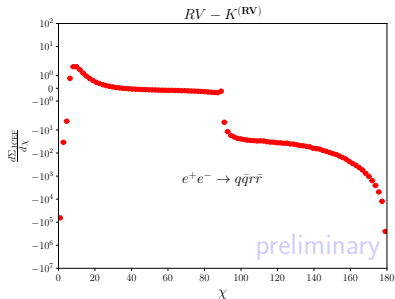
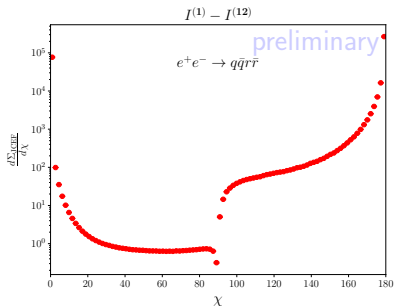
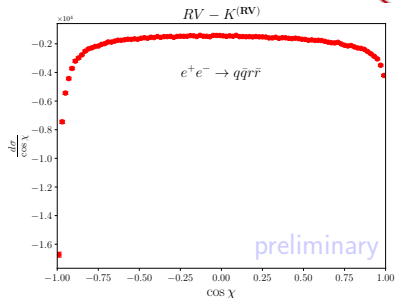
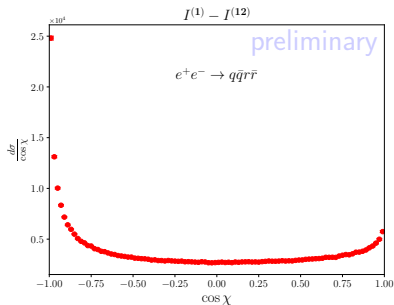
$$\frac{d\Sigma_{\text{JCEF}}}{d \cos \chi} = \sum_i \int \frac{E_i}{Q} d\sigma_{e^+e^- \rightarrow i+\chi} \delta \left(\cos \chi - \frac{\vec{p}_i \cdot \vec{n}_T}{|\vec{p}_i|} \right).$$

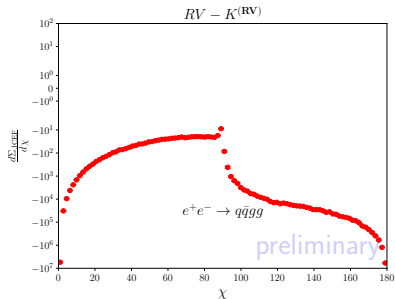
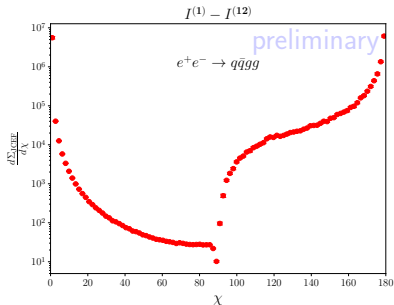
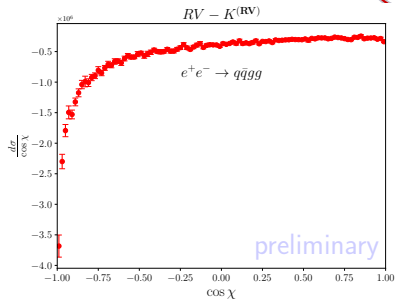
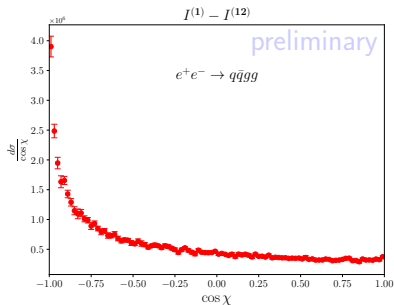












- ▶ We demonstrated a proof-of-concept implementation of the LASS scheme.
- ▶ Numerous expressions from the LASS papers were successfully cross-checked.
- ▶ We have full analytic and numeric control of RR , RV , and VV contributions.
- ▶ Efficient automatic generation and integration of the subtraction terms were achieved.
- ▶ First distributions of the event shape observables using the LASS scheme were obtained.
- ▶ Significant progress was made towards the full automation of the current LASS scheme.

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Thanks for your attention!



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