A new method for the reconstruction of rational functions

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Outline

I. **Introduction**

II. **The method**

III. **Examples**

IV. **Summary and outlook**

➢ Multiloop scattering amplitudes

- Construct the amplitude

$$\mathcal{A} = \sum c_i I_i$$

  - I_i: scalar Feynman integrals in dimensional regularization

$$I(\vec{\nu}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0)^{\nu_K}}$$

- Compute the scalar integrals: reduction + computation
  - reduction: express all the scalar integrals in terms of a smaller set of independent integrals (master integrals)

$$I_i = \sum_j b_{ij} M_j$$

  - computation: compute the master integrals as expansions in the dimensional regulator $\epsilon = (4 - D)/2$

$$M_j = \sum_{l=-2L} d_{jk} \epsilon^k$$

Feynman integrals

- Integration-by-parts reduction [Chetyrkin and Tkachov, Nucl. Phys. B, 1981] [Laporta, Int. J. Mod. Phys. A, 2000]
  - AIR [Anastasiou and Lazopoulos, JHEP, 2004]
  - FIRE [Smirnov, JHEP, 2008] [Smirnov, Smirnov, Comput. Phys. Commun., 2013] [Smirnov, Comput. Phys. Commun., 2015] [Smirnov and Chuharev, Comput. Phys. Commun., 2020]
  - Reduze [Studerys, Comput. Phys. Commun., 2010] [Manteuffel and Studerus, e-Print: 1201.4330]
  - Kira [Maierhofer, Usovitsch and Uwer, Comput. Phys. Commun., 2018] [Klappert, Lange, Maierhofer and Usovitsch, Comput. Phys. Commun., 2021]
  - LiteRed [Lee, 2012] [Lee, 2014]
  - NeatIBP [Wu, Boehm, Ma, et al, Comput. Phys. Commun., 2024]
- Finite-field reconstruction [Manteuffel and Schabinger, Phys. Lett. B, 2015] [Peraro, JHEP, 2016]
  - FiniteFlow [Peraro, JHEP, 2019]
  - FireFly [Klappert and Lange, Comput. Phys. Commun., 2020]

Summary

$$\text{time} = \frac{\text{time for a single sample} \times \text{number of samples}}{\text{number of CPUs}}$$

Refined IBP systems:

syzygy equations [Gluza, Kajda and Kosower, Phys. Rev. D, 2011] [Larsen and Zhang, Phys. Rev. D, 2016]

block-triangular systems [Guan, **XL**, Ma, Chin.Phys.C, 2020]

new version of Kira → talks by Matteo Fael, Fabian Lange

More powerful linear solver:

RATRACER [Magerya, e-Print: 2211.03572]

better interpolation methods [Klappert and Lange, Comput.Phys.Commun. 2020] [Belitsky, Smirnov, Yakovlev, 2023.02511]

more compact ansatz[Badger, Hansen, Chicherin, et al, JHEP 2021][Laurentis, Page, JHEP 2022][Abreu, Laurentis, Ita, et al, 2305.17056]

Q-linear relations → talk by Giuseppe De Laurentis

P-adic reconstruction → talk by Herschel Chawdhry

the method in this talk

Outline

Motivation

➢ A simple observation

- Traditional strategy: reconstructing functions individually & neglecting common structures
- Example

$$f_i(x) = \left(\frac{1+x}{1-x}\right)^{i-1}, \quad i \in [1, 100]$$

  - approximately 200 samples using Thiele's interpolation formula
  - a system of relations

$$(1-x) f_{i+1}(x) - (1+x) f_i(x) = 0, \quad i \in [1, 99]$$

  - ansatz + linear fit → 4 samples

$$(a_i + b_i x) f_{i+1}(x) + (c_i + d_i x) f_i(x) = 0$$

- Linear relations → common structures utilized → number of samples reduced

The method

➢ General description

- Relations among nonzero functions $f_1(\vec{x}), \dots, f_n(\vec{x})$

$$Q_1(\vec{x}) f_1(\vec{x}) + \cdots + Q_n(\vec{x}) f_n(\vec{x}) = 0$$

- Comments:
  - Number of independent relations is $n - 1$
    - not n, otherwise $f_i(\vec{x}) = 0$
    - no less than $n - 2$, otherwise the basis contains at least two functions f_i and f_j. However, $f_i = \frac{f_i}{f_j} \times f_j$
  - Polynomial-coefficients $Q(\vec{x})$ exist
- Goal: find $n - 1$ independent relations with polynomial-coefficients

The method

- Algorithm from [Guan, **XL**, Ma, Chin.Phys.C, 2020]

  1. start with $n = 0$

  2. make a degree-n ansatz for the relations

  3. generate numerical samples of functions

  4. determine the unknown parameters in the ansatz through a linear fit

  5. count the number of independent relations: if sufficient, end; otherwise increase n by 1 and go to step 2

The method

$$f_1(x) = 1, \qquad f_2(x) = \frac{1+x}{1-x}$$

- degree-0 ansatz: $Q_1 f_1(x) + Q_2 f_2(x) = 0$
  - $x = 100 \quad \to \quad f_1 = 1, f_2 = 101/99 \quad \to \quad Q_1 + 101/99 Q_2 = 0$
  - $x = 101 \quad \to \quad f_1 = 1, f_2 = 102/100 \quad \to \quad Q_1 + 102/100 Q_2 = 0$
  - no solution $\to$ go to degree-1 ansatz
- degree-1 ansatz: $(Q_{10} + Q_{11}x) f_1(x) + (Q_{20} + Q_{21}x) f_2(x) = 0$
  - $x = 100 \quad \to \quad f_1 = 1, f_2 = \frac{101}{99} \quad \to \quad (Q_{10} + 100 Q_{11}) + \frac{(Q_{20} + Q_{21} 100) 101}{99} = 0$
  - $x = 101 \quad \to \quad f_1 = 1, f_2 = \frac{102}{100} \quad \to \quad (Q_{10} + 101 Q_{11}) + \frac{(Q_{20} + Q_{21} 101) 102}{100} = 0$
  - $x = 102 \quad \to \quad f_1 = 1, f_2 = \frac{103}{101} \quad \to \quad (Q_{10} + 102 Q_{11}) + \frac{(Q_{20} + Q_{21} 102) 103}{101} = 0$
  - $x = 103 \quad \to \quad f_1 = 1, f_2 = \frac{104}{102} \quad \to \quad (Q_{10} + 103 Q_{11}) + \frac{(Q_{20} + Q_{21} 103) 104}{102} = 0$
  - One solution: $Q_{10} = Q_{11} = Q_{20} = -1, Q_{21} = 1$

$$(1+x) f_1(x) + (1-x) f_2(x) = 0$$

The method

- Summary
  - generator of samples
    - e.g., IBP system + linear solver over finite fields
  - linear relations
    - make various ansatz for $Q_1(\vec{x})f_1(\vec{x}) + \cdots + Q_n(\vec{x})f_n(\vec{x}) = 0$
    - generate samples
    - linear fit (dense linear system over finite fields)
- About the obtained relations
  - can be easily solved by e.g.
    - traditional rational functions reconstruction strategy
    - additional finite fields + rational numbers reconstruction (Chinese Remainder Theorem + Wang's algorithm [Wang, 1981])

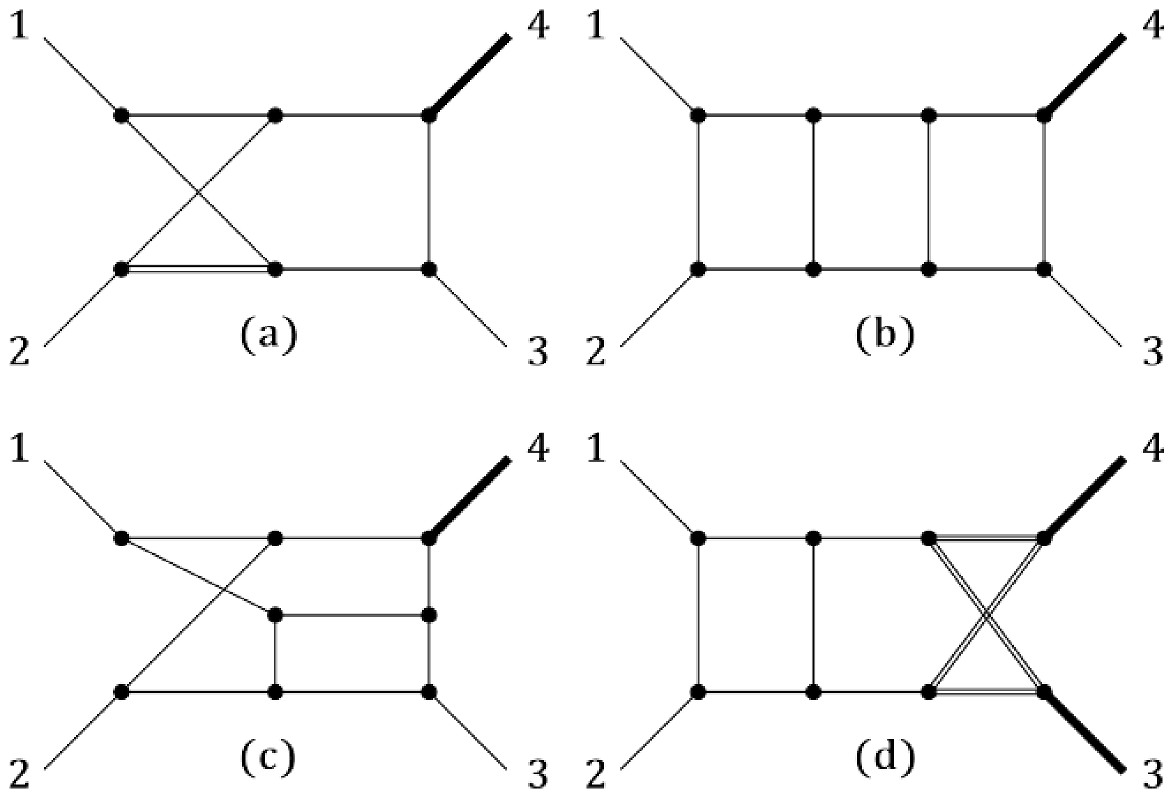
Outline

Examples

- Reduction coefficients of Feynman integrals or amplitudes

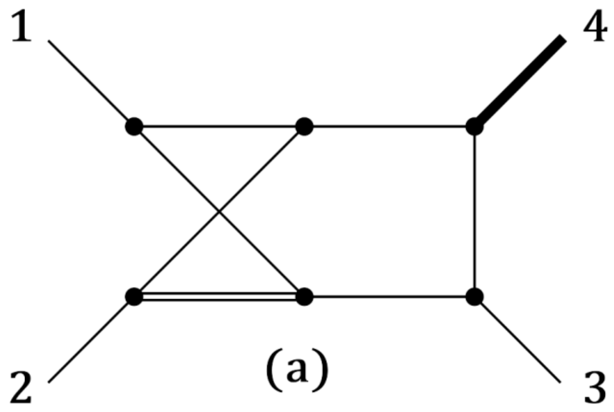
$$\mathcal{A} = f_1 \mathcal{M}_1 + \cdots + f_n \mathcal{M}_n$$

  - a common set of denominators reflecting the singularities



Examples

- Topology (a): two-loop amplitude of the mixed QCD-electroweak correction to $pp \to Z + j$ [Bargiela, Caola, Chawdhry, **XL**, 2312.14145]
- Setup
  - $m_Z^2 = 1, m_W^2 = 7/9$
  - remaining: $\{\epsilon, s_{12}, s_{13}\}$
  - 56 master integrals $\Rightarrow$ 56 rational functions
  - auxiliary function $f_{57} = 1$
  - LiteRed + FiniteFlow



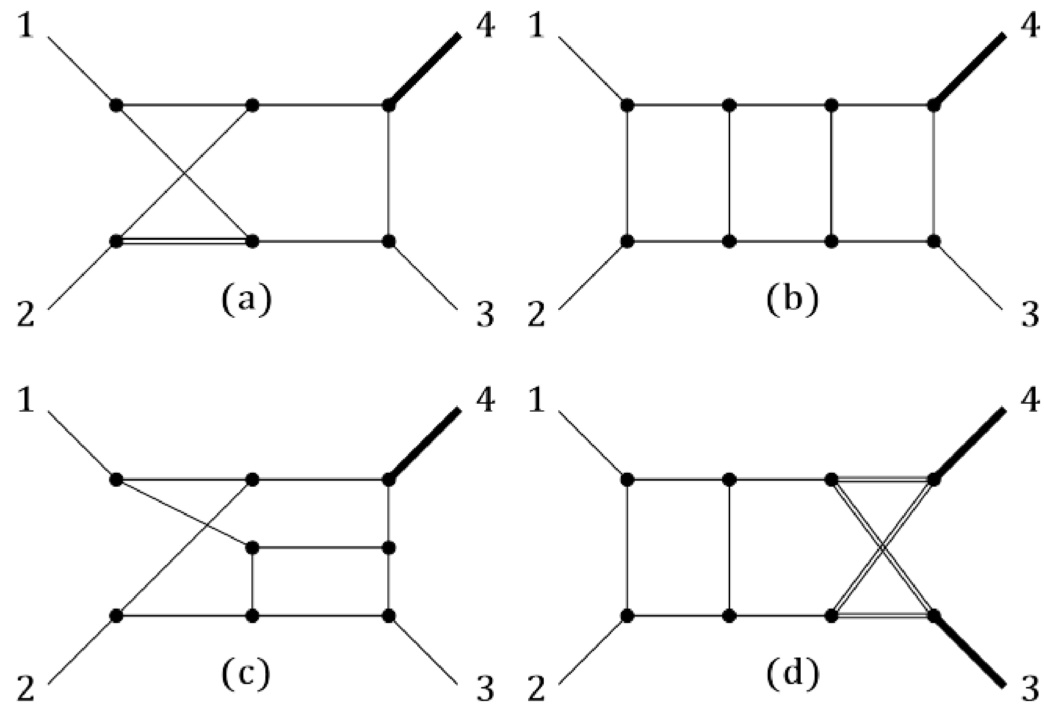
(a)

- Details
  - Completed with degree-6 ansatz
  - Number of samples: from 54978 (FiniteFlow) to 5321 $\to$ a factor of 10.3
  - Computational cost: from 4.6h to 0.47h $\Rightarrow$ a factor of 9.8
  - The computational cost is dominated by samples generation.

Examples

topology	variables	generator	degree	samples	samples (old)	R_{samples}	cost	cost (old)	R_{cost}
(a), 56	ϵ, s_{12}, s_{13}	LR+FF	6	5321	54978	10.3	0.47h	4.6h	9.8
(b), 83	ϵ, s_{12}, s_{13}	NI+FF	8	15208	145722	9.6	8.2h	78.5h	9.6
(c), 280	ϵ, s_{12}, s_{13}	LR+FF	8	43662	2351622	54	8549h	450728h	53
(d), 336	ϵ, η	LR+FF	5	41350	473946	11.5	340h	2320h	9.5

- (a): amplitude
- (b): rank -6 integral
- (c): differential equations with respect to s_{12}, s_{13}
- (d): differential equations with respect to η (AMFlow)

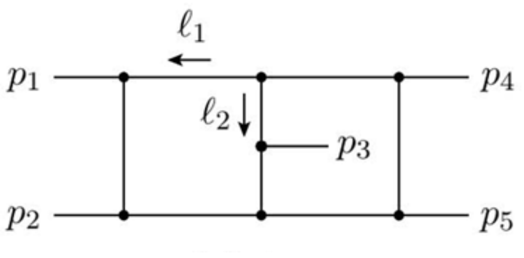


Examples

- Discussion
  - samples generation + linear fit + explicit solutions
  - linear fit
    - Undominated if the ansatz is not too big (less than 20000 parameters)
    - For most problems with less than 3 variables, this holds true.
      - For topology (c): the biggest ansatz contains 6810 parameters
    - For problems with five-point kinematics, this can become a problem.
  - explicit solutions
    - Undominated in most cases
    - For topology (b): IBP system costs 1.3s per point; our linear relations cost 0.0024s per point → 540 times faster

Examples

- A step towards multivariate problems **[in preparation]**
  - five variables: $\epsilon, s_{23}, s_{34}, s_{45}, s_{51}$ ($s_{12} = 1$)
  - full ansatz may contain O(100000) parameters
  - work in four-dimensional slices ($s_{23}, s_{34}, s_{45}, s_{51}$)
    - $\epsilon \to \epsilon_0$
    - For fixed ϵ, deal with the four-variate problem (ansatz size: 14000)



$$\frac{a_0 + a_1 s_{23} + a_2 s_{34} + a_3 s_{45} + \cdots}{1 + b_1 s_{23} + a_2 s_{34} + b_3 s_{45} + \cdots}$$

- - - a's and b's are rational functions of ϵ to be reconstructed
    - 400000 samples in total to reconstruct the fully analytic expressions over a finite field
    - FiniteFlow requires approximately $O(10^7)$ points

Outline

Summary and Outlook

- A new method for the reconstruction of rational functions is proposed, which works by exploiting all the independent linear relations among the target functions.
- The method effectively reduces the number of sample points required in the framework of finite-field sampling.
- The method works well for problems with no more than three variables, i.e., problems with three-point or four-point kinematics.
- For problems with five-point kinematics, some preliminary results shows that the method can still work, but further studies are required.

Thank you!