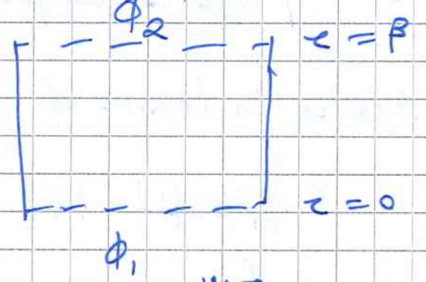
Unruh Effect from Euclidean Path Integrals (1)

1. Path integrals Euclidean / Lorentzian
2. Properties of Euclidean Correlators — Smooth, no branch cuts — Operators commute
3. ~~Boost operator~~, Half Spaces, Density matrix $\boxed{B \mid A}$, ρ_A $\rho_{pure} \to \rho_{mixed}$
4. $Q[\partial_\theta]$, Boost operator $\rho_A = e^{2\pi K_A}$ $T = \frac{1}{2\pi}$
5. Connection with Unruh effect $ds^2 = R^2 d\theta^2 + dR^2$ $\to = -R^2 d\eta^2 + dR^2 + dy_\perp^2$ $\boxed{R = a^{-1}, \eta = a\tau}$
6. Purification of a state, $\psi \leftrightarrow \rho$ correspondence. $\psi_\rho = $ $\beta/2$
7. Thermofield double, Θ_{CRT}
8. Concluding remarks.

(1) Path Integrals

$$\langle \phi_2 | e^{-\beta H} | \phi_1 \rangle = \int_{\phi(\tau=0)=\phi_1}^{\phi(\tau=\beta)=\phi_2} D\phi \, e^{-S_E[\phi]}$$



$$\langle \phi_2(\vec{x}) | e^{-iHT} | \phi_1(\vec{x}) \rangle$$

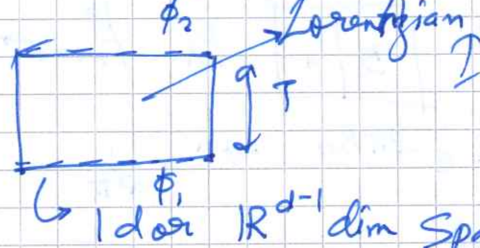
Reference: http://www.hartmanhep.net/topics2021/

→ Course on QGBH by Tom Hartman

1 Consider fields and operators specified on a spatial surface

$\phi(\vec{x}) \quad \mathcal{O}(\vec{x}, 0)$

Transition amplitudes $\langle \phi_2(\vec{x}) | e^{-iHT} | \phi_1(\vec{x}) \rangle = \int_{\phi(t=0)=\phi_1(\vec{x})}^{\phi(t=T)=\phi_2(x)} D\phi \, e^{-iS[\phi]}$

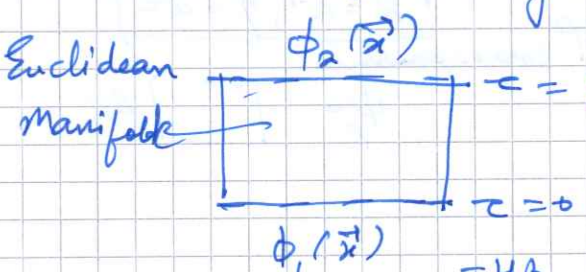


Interpret this as a wave function

$\langle \phi_2(\vec{x}) | \psi \rangle = \Psi[\phi_2(\vec{x})]$

↳ 1 dor $\mathbb{R}^{d-1}$ dim spatial surface $\quad |\psi\rangle = e^{-iHT}|\phi_1(\vec{x})\rangle$

We can equivalently consider $\langle \phi_2(\vec{x}) | e^{-HB} | \phi_1(\vec{x}) \rangle$



$= \int_{\phi(\tau=0)=\phi_1(\vec{x})}^{\phi(\tau=\beta)=\phi_2(\vec{x})} D\phi \, e^{-S_E[\phi]}$

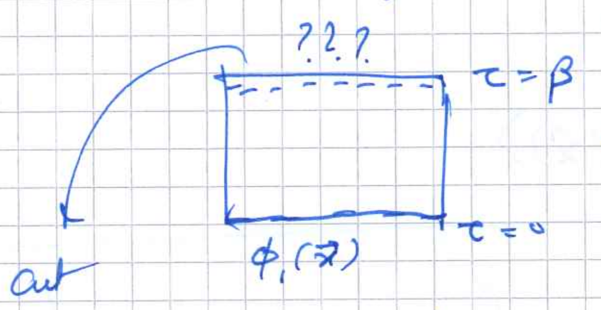
$\tau = +it$

~~Represent~~ $e^{-H\beta}$ is just some non unitary operator

Euclidean evolution prepares a new state $|\psi\rangle = e^{-\beta H}|\phi\rangle$

Lorentzian evolution = Physics happening

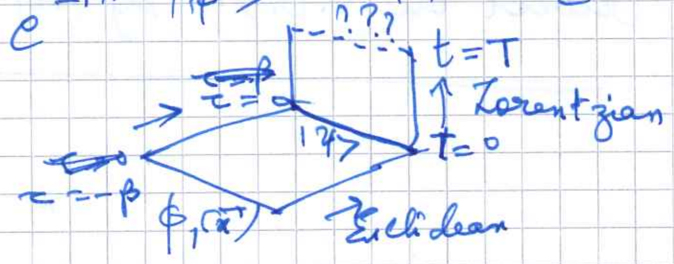
Represent ~~path integrals as~~ states in terms of path integrals



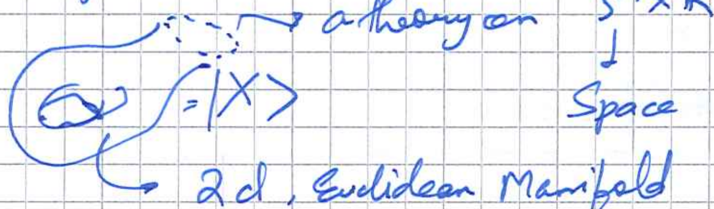
$= \int_{\phi(\tau=0)=\phi_1(\vec{x})}^{\phi(\tau=\beta)=???} D\phi \, e^{-S_E[\phi]}$

We can combine two evolutions by writing down a path integral on a manifold of mixed signature

$e^{-iHT}|\psi\rangle \qquad |\psi\rangle = e^{-\beta H}|\phi_1\rangle$

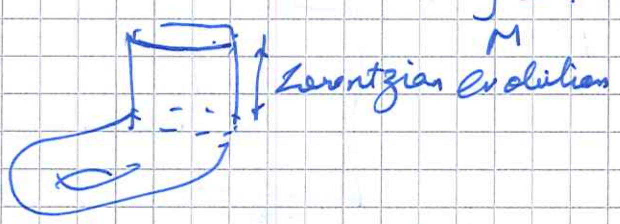


In this way any state can be specified by a Euclidean path integral



Has no clear time dir but that is OK

$$|X\rangle = \int_M D\phi \, e^{-S_E[\phi]}$$



Key point about Euclidean path integrals: τ is not special

→ Can choose any other set of coordinates to describe this surface.

→ Will be important later

~~Another crucial point:~~

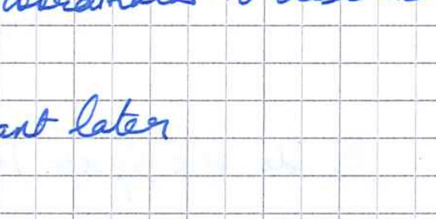
Vacuum State

$$|\psi\rangle = \sum_n \psi_n |n\rangle$$

$$e^{-\beta H} |\psi\rangle = \sum_n e^{-\beta E_n} \psi_n |n\rangle$$

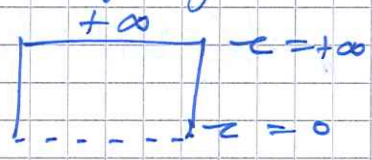
$\beta \to +\infty \Rightarrow |\psi\rangle \to |0\rangle$ [need some overlap with $|0\rangle$]

So vacuum state (unnormalized) $|0\rangle =$

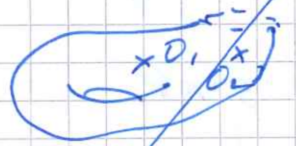


Any regular boundary cond. is OK

Likewise $\langle 0| =$



~~Can general Can include operators on Euclidean manifold~~

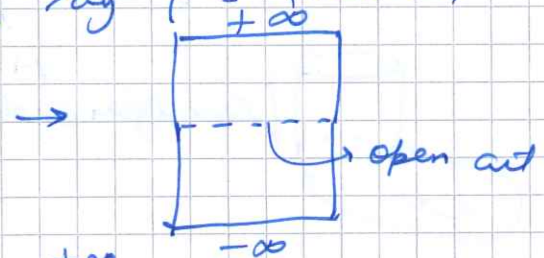


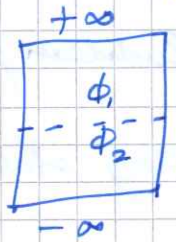
$= |x'\rangle$

Vacuum correlators:

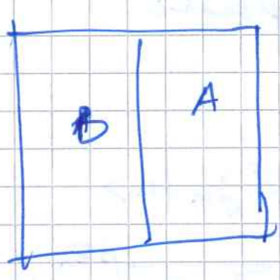
Next we can write down a density matrix by leaving both ends open

$\rho =$  Say $\rho = |0\rangle\langle 0|$



$\langle \phi_1 | \rho | \phi_2 \rangle =$  $= \langle \phi_1 | 0 \rangle \langle 0 | \phi_2 \rangle$

Now divide the space into two half spaces



A : $x^1 > 0$

B : $x^2 < 0$

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

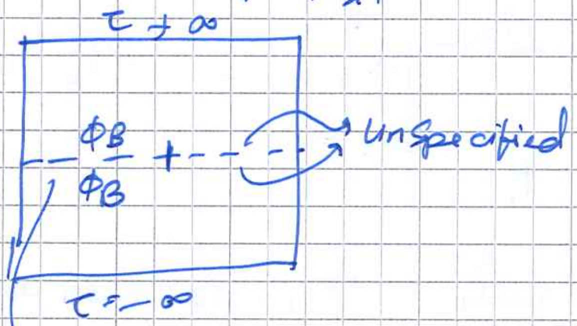
What is ρ_A? $\rho_A = \sum_{\phi_B \in B} \langle \phi_B | \rho | \phi_B \rangle$

$$\langle \phi_1 | \rho_A | \phi_2 \rangle = \sum_{\phi_B \in B} \langle \phi_B \phi_1 | \rho | \phi_B \phi_2 \rangle$$

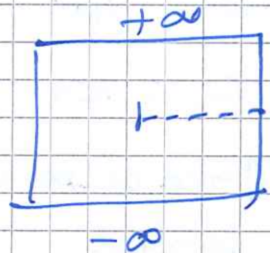
reduced density matrix for $O_A \in A$ [operators that only act on states in A]

$$\langle O_A \rangle = \mathrm{tr}(\rho\, O_A) = \mathrm{tr}_A(\rho_A O_A)$$

Our $\rho = |0\rangle\langle 0|$ so $\xrightarrow{\tau} x_1$

$\rho_A = \sum_{\phi_B}$ 

$\sum_{\phi_B} \longrightarrow$ this glues the two ~~sheets~~ sheets together

$\rho_A =$ 

$\Rightarrow$

Now claim

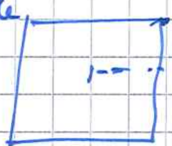
$$\rho_A = e^{-2\pi K_A}$$

K_A = ~~Analytic cont of~~ x_1-boost generator ~~to Euclidean space~~ on $\underline{A}$

Proof

Recall: This is a path integral on the Euclidean manifold with field data unspecified on top and bottom of the half cut

We have been using τ as Euclidean coordinate

e^{-H} 

~~W~~ Can choose to do this in polar coordinates

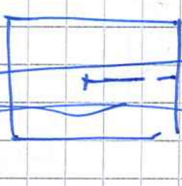
Choose Euclidean coordinates R, θ

$$ds^2 = dR^2 + R^2 d\theta^2$$

and do the integral as

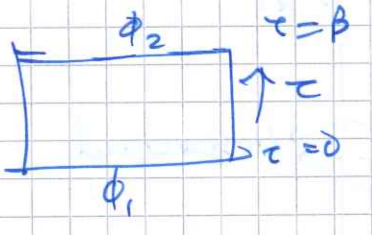
later .

Now boundary data specified on $\theta = 0$ and $\theta = 2\pi$

~~Previously we had~~  ~~$\tau = \int D\phi\, e^{-S_E[\phi]}$~~

Generator of τ-translations is Hamiltonian

(6)



$$= \langle \phi_2 | e^{-\beta H} | \phi_1 \rangle$$

More generally H is the conserved charge

$$= \int d^{d-1}x \; T_{tt}$$

Covariantly $= \int_\Sigma d\Sigma^\mu \, T_{\mu\nu} \, \xi^\nu$

$\xi^\nu \to$ Killing vector

$d\Sigma^\mu \to$ Area element of a spatial surface $\longrightarrow u^\mu$

Choice of Σ is irrelevant.

For boosts in x_1 direction

$$\xi^\nu = x_1 \hat{t}^\nu + t \hat{x}_1^\nu$$

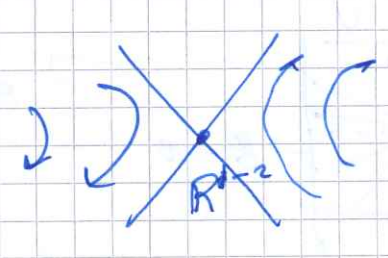
$$\xi^a = x_1 \partial_t^a + t \partial_{x_1}^a$$

Choose Σ to be surface at $t=0$

$$d\Sigma^\mu = \hat{t}^\mu dx_1 d^{d-2}x_\perp$$

$$K = \int dx_1 d^{d-2}x_\perp \; x_1 T_{tt}$$

ξ^ν :

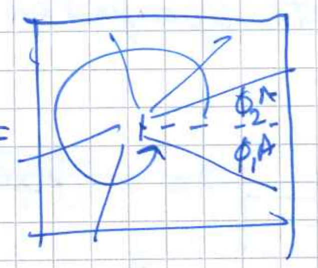


so $K_A = \int_0^\infty dx_1 d^{d-2}x_\perp \; x_1 T_{tt}$

$\theta \sim$ Euclidean "time"

Now

$\langle \varphi_2^A | \rho | \varphi_1^A \rangle =$



$$= \langle \phi_2^A | e^{-2\pi Q[\partial_\theta]} | \phi_1^A \rangle$$

Q is the charge associated with rotational symmetry in $x_1 - \tau$ plane

Killing vector: $\partial_\theta = \tau \partial_{x_1} - x_1 \partial_\tau$

$\tau = it$ ↳ Minkowskian

$\partial_\theta \to i(t\partial_{x_1} + x_1 \partial_t)$

Area element $\hat{\tau} d^D x^{-1} = -i\partial_t = \partial_{\tau_E}$

$Q[\partial_\theta] = K_A$

So $\rho_A = e^{-2\pi K_A}$

thermal with dimensionless temp "$T = \frac{1}{2\pi}$"

dimless because K_A is dimless

Connect With Unruh Effect

$$ds^2 = R^2 d\theta^2 + dR^2 \qquad \theta = i\eta$$

$$\to = -R^2 d\eta^2 + dR^2$$

η = Rindler time

For uniformly accelerating observer

$$\eta = a\tau \quad \text{↳ proper time } \cancel{\text{of}}$$

$$R = a^{-1}$$

Last time we wrote down → not the acceleration, just a parameter

$$ds^2 = e^{2a'\xi}(-d\eta^2 + d\xi^2)$$

~~Here~~ $\cancel{e^{a\xi}} =$ $R = e^{\xi}$ set $a' = 1$

$$x(\eta, R) = R\cosh\eta$$

$$t(\eta, R) = R\sinh\eta$$

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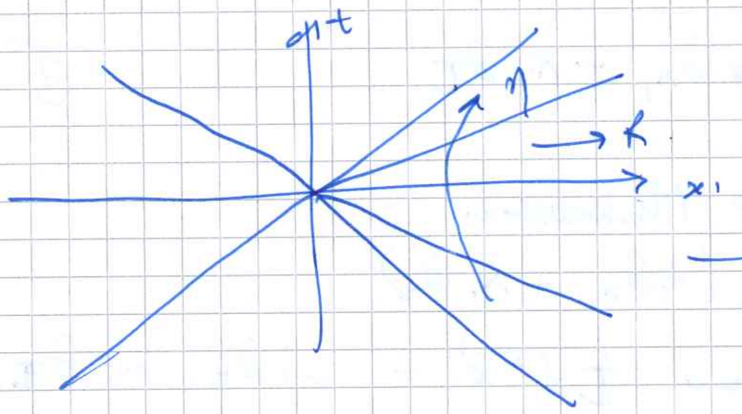
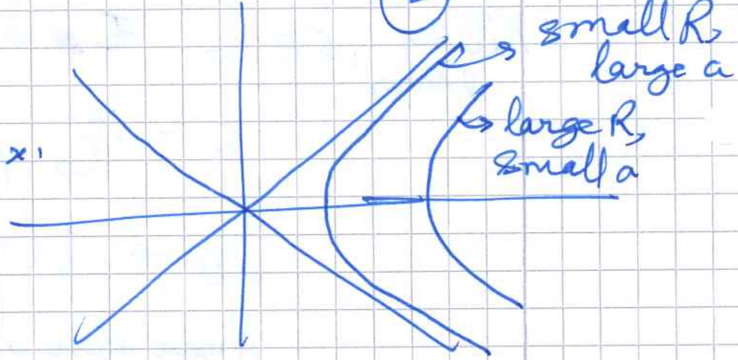




In Rindler time $\hat{K}_A$ is the "energy"

η is the "time"

ρ_A is "thermal"

Thermofield double

Any mixed state ρ can be purified by enlarging $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_{aux}$ such that $\exists\, |\psi\rangle \in \mathcal{H} \otimes \mathcal{H}_{aux}$

~~Diagonalize: $\rho = \sum p_i |i\rangle\langle i|$~~

$$\rho_\psi = |\psi\rangle\langle\psi|$$

$$\rho = \mathrm{tr}_{aux}\, \rho_\psi$$

$$\langle O_1 O_2 \dots\rangle_\rho = \langle\psi| O_1 O_2 \dots |\psi\rangle$$

Can take $\mathcal{H}_{aux} \simeq \mathcal{H}$.

$$\rho = \sum p_i |i\rangle\langle i|$$

take $\psi = \sum_i \sqrt{p_i}\, |i\rangle|\tilde{i}\rangle \qquad \tilde{i} \in \mathcal{H}_{aux}$

$$\rho_\psi = |\psi\rangle\langle\psi|$$

Clearly $\sum_{\tilde{i}} \langle\tilde{i}|\psi\rangle\langle\psi|\tilde{i}\rangle = \rho$

Thermo field double:

$$|\beta\rangle = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n/2} |n\rangle_1 |\tilde{n}\rangle_2$$

$$|\tilde{n}\rangle = \Theta^\dagger |n\rangle_1$$

C.P.T.

But here really CRT $\quad (t, x, y_\perp) \to (-t, -x, y_\perp)$

Claim: $|\beta\rangle$ is the purification of $\rho = e^{-\beta H}$

isomorphism

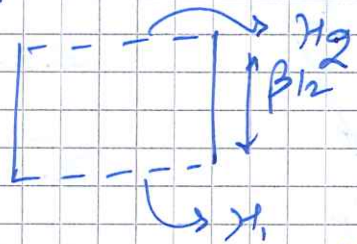
Operators $\rho : \mathcal{H} \to \mathcal{H}$

States $\psi \in \mathcal{H} \otimes \tilde{\mathcal{H}}$

$$\underbrace{|m\rangle_1 |n\rangle_2 \longrightarrow |m\rangle \langle n|}_{\text{Antilinear}}$$

$$\alpha |n\rangle_2 + \beta |n'\rangle_2 \longrightarrow \alpha^* \langle n|_1 + \beta^* \langle n|_2$$

$$|\beta\rangle_{\mathcal{H}\otimes\mathcal{H}} \longleftrightarrow \sqrt{\rho_\beta} \quad \mathcal{H} \to \mathcal{H}$$

$|\beta\rangle =$ 

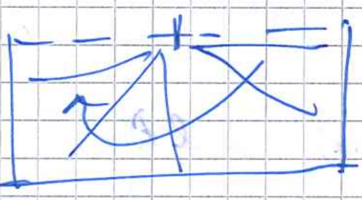
$$\langle \phi_1 \tilde{\phi}_2 | \beta \rangle = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n/2} \langle \phi_1 | n_1 \rangle \langle \tilde{\phi}_2 | \tilde{n} \rangle_2$$

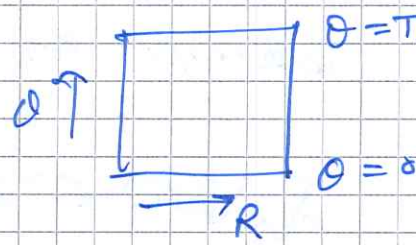
$$= \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n/2} \langle \phi_1 | n_1 \rangle \underbrace{\langle n | \phi_2 \rangle}$$

used the antilinear isomorphism

$$= \langle \phi_1 | \rho | \phi_2 \rangle$$

Claim: $|0\rangle_{\text{Mink}} = |\beta\rangle_{\text{Rind} \otimes \text{Rind}}$

$|0\rangle_M =$  $=$ Strip of size π in θ dir



$= e^{-\pi K_A}$

$= |\beta\rangle$

|

$A_+ = A\cdot n \quad A_- = A\cdot\bar{n}$

$a_\mu = a\bar{n}_\mu$

$$x'^\mu = \frac{x^\mu + a^\mu x^2}{1+2a\cdot x + a^2x^2} = \frac{x^\mu + \bar{n}^\mu a x^2}{1+2ax_- + a^2x^2}$$

light ray $\Rightarrow x^2 = 0$

$$x'^\mu = \frac{x^\mu \cancel{+\bar{n}^\mu}}{1+2ax_-}$$

$$\boxed{x'_- = \frac{x_-}{1+2ax_-}}$$

$$\cancel{x'_+ = \frac{n\cdot\bar{n}}{1+2ax_-}}$$

$x^\mu = x_- n^\mu + x_+ \bar{n}^\mu + x_\perp^\mu$

αn

$\hookrightarrow$ − comp

$a = a\bar{n}_\mu$

$\hookrightarrow$ + comp.

$\sum_{+-} \Phi(\alpha) = s\Phi(\alpha)$

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$\alpha \to \alpha' = \frac{a\alpha+b}{c\alpha+d}$

$\alpha \in \mathbb{R} \qquad ad-bc = 1$

$\Phi(\alpha) = \Phi'(\alpha)$

$\delta_D \Phi(x) = (x\cdot\partial + \ell)\Phi(x)$

$\delta_P^\mu \Phi(x) = \partial^\mu \Phi(x)$

$\delta_K^\mu \Phi(x) = (2x^\mu x\cdot\partial - x^2\partial^\mu + 2\ell x^\mu - 2x_\nu \Sigma^{\mu\nu})\Phi(x)$

Now $\Phi(x) = \Phi(n\alpha)$

$\partial^\mu \Phi(n\alpha) \Rightarrow$ only − comp. rel.

$\delta_P^- \Phi(x) = \partial^- \Phi(x)$

$\delta_D \Phi(x) = x_+\partial^-\Phi(x) + \ell\Phi(x)$

$\to \ell\Phi(x)$

$\delta_K^- \Phi(x) = 2\ell$