Notes on Hawking radiation

Alessio Miscioscia

Deutsches Elektronen-Synchrotron DESY, Notkestr. 85, 22607 Hamburg, Germany

E-mail: alessio.miscioscia@desy.de

ABSTRACT: In these notes we explore the basics of Hawking radiation in black holes' physics. Notes prepared for the Desy theory Workshop (13.06.2023).

Contents

1 Recap of Black holes and Penrose diagrams

In this section we firstly recall the basics of black holes; in particular we proceed by using as explicit case of study the Schwarzschild metric, i.e. the metric obtained by considering a spherical (uncharged and non-rotating) black hole. We will describe the main feature of this metric in the Kruskal coordinates and we will describe its Penrose diagram. The Schwarzschild metric is derive in appendix B and it is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) \ , \tag{1.1}$$

where M is the black hole mass and G is the Newton constant. At $r = 0$ the metric is singular; this divergence can be understood in a coordinate independent way by considering the divergence of the fully contracted Riemann tensor $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$.
The radius $r_s = 2GM$ is called the Schwarzschild radius; the metric in equation (1.1) seems to be divergent also for $r = r_s$, however this last singularity is just due to a bad choice of coordinates and indeed the fully contracted Riemann tensor is not singular in that point. Nevertheless for $r = r_s$ one can easily observe that the sign of the coefficients in front of $\mathrm{d}t^2$ and $\mathrm{d}r^2$ switch. The coordinate r becomes timelike and any particle that falls in the region $r < r_s$ proceed necessarily toward the singularity. For this reason the surface defined by $r = r_s$ is called the black hole horizon.
In order to study the near horizon physics it is convenient to choose different coordinates; a null geodesic in Schawrzschild is given by $t = \pm r_\star + C$, where C is a constant and $r_\star$ is the tortoise coordiante

$$r_\star = r + \log(r - 1) \ , \tag{1.2}$$

where we choose from here on $r_s = 1$. Let's define the Kruskal coordinate

$$U = -e^{(r_\star - t)/2} , \qquad V = e^{(r_\star + t)/2} . \tag{1.3}$$

By construction lines defined by U or V constant are null geodesic and the singularity is when $UV = 1$, while the horizon is defined by either $U = 0$ or $V = 0$. The metric is now

$$\mathrm{d}s^2 = -\frac{4}{r} e^{-r} \mathrm{d}U \mathrm{d}V + r^2 \mathrm{d}\Omega^2 , \tag{1.4}$$

where r can be implicitly given by $UV = (1-r)e^r$. The off-diagonality of such a coordinates can be removed by defining

$$U = T - X , \qquad V = T + X . \tag{1.5}$$

Indeed in these coordinates

$$\mathrm{d}s^2 = \frac{4}{r} e^{-r} (-\mathrm{d}T^2 + \mathrm{d}X^2) + r^2 \mathrm{d}\Omega^2 . \tag{1.6}$$

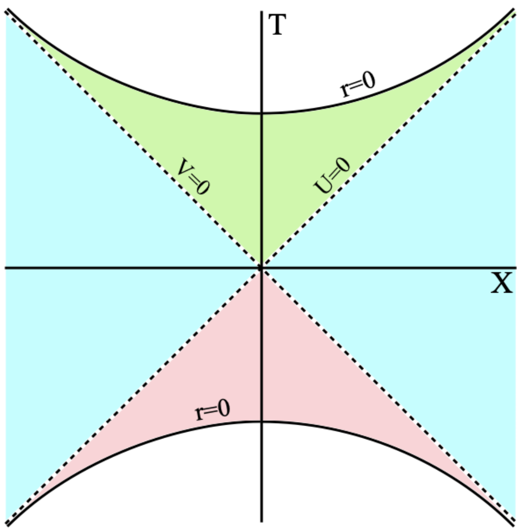

Figure 1. The Schawzschild solution in the (T, X); the light blue region is the region defined by $r > 1$, the $r < 1$ spacetime is the given by the green and the red $(X^2 - T^2 < 0)$; finally the singularity is given by $X^2 - T^2 = -1$. Figure taken by [1].

It is usually convenient to draw simplest diagrams to understand the causality proprieties of the black hole geometry. A way to do that is to observe that conformally equivalent metric, i.e. metrics related by

$$\tilde{g}_{\mu\nu}(x) = e^{2\omega(x)} g_{\mu\nu}(x) , \tag{1.7}$$

have the same null geodesics and timelike/spacelike curves in one metric will be time-like/spacelike curves in the other. A skatch of the proof is given in appendix A but, with this simple observation, it is possible to define the Penrose diagrams in the following way:

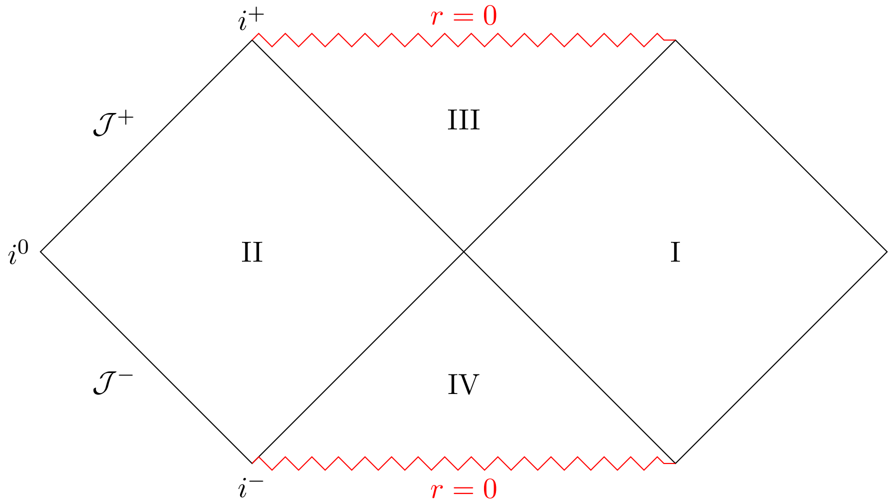

Figure 2. Penrose diagram of a Schwarschild spacetime. $i^{\mp}$ is the past/future timelike infinity, $\mathcal{J}^{\mp}$ is the past/future null infinity and i^0 is the spatial infinity. $i^{\pm}$ are where timelike geodesics come from. The red zigzag lines are where the singularity is located.

1. Choose a set of the coordinates of the sapcetime defined in a finite range such that
$$\mathrm{d}s^2 = \frac{1}{\omega^2(x)}\mathrm{d}\hat{s}^2 \ , \tag{1.8}$$
where $\mathrm{d}\hat{s}^2$ is regular on the boundary, i.e. at the infinity of the previous spacetime coordinates.

2. The spacetime defined by $\mathrm{d}\hat{s}^2$ has the same causality proprieties of the spacetime defined by $\mathrm{d}s^2$ and therefore we can study the causality proprieties of $\mathrm{d}s^2$ by studying $\mathrm{d}\hat{s}^2$. This spacetime is defined in a finite range.

Let's apply this procedure to the Schwarzschild case; starting from the first region T and X one can define
$$T' + X' = \arctan(T + X) \ , \qquad T' - X' = \arctan(T - X) \ , \tag{1.9}$$
and these parameters are defined in a finite domain $-\frac{\pi}{2} < \tilde{U}, \tilde{V} < \frac{\pi}{2}$; the metric can be written as
$$\mathrm{d}s^2 = \overbrace{\frac{e^{-r}}{4r\cos^2(T' + X')\cos^2(T' - R')}}^{\omega^{-2}} \overbrace{\left(-(\mathrm{d}T')^2 + (\mathrm{d}X')^2 + r^2\mathrm{d}\Omega^2\right)}^{\mathrm{d}\hat{s}^2} \tag{1.10}$$
The new coordinates are such that $|X' \pm T'| \leq \pi/2$ (and throw out the region $|T| > \pi/4$). Therefore we obtain the Penrose diagram in figure 2.

2 The Hawking radiation

The Hawking radiation is one of the most important phenomena in black hole physics [2]. It explains how a black hole can collapse even if it seems to violate the fact that the area of the black hole cannot decrease. This violation must be caused by a flux of negative energy across the event horizon which balances the positive energy flux emitted to infinity. Just outside the horizon there would be pairs of particles, one with positive and one with negative energy. The negative particle can tunnel inside the black hole where the Killing vector which represents time translations is spacelike. In this region the particle can exists as a real particle with a timelike momentum vector even if its energy relatie to infinity as measured by the time translation Killing vector is negative. The other particle can escape to infinity where it consitutes a part of the thermal emission we are going to describe. The probability of the negative energy particle tunnelling the horizon is governed by the surface gravity κ[1], which indeed measure how fast the Killing vector becomes spacelike.
In the following we will study a free massless scalar theory so that computations can be really be done explicitly, however let us just comment that a fully non-perturbative proof of the Unruh effect (and similarly Hawking radiation) is possible for every field theory satisfying the Wightmann axioms. For a case of a generic scalar theory (with arbitrary potential) see [3].
Let's be concrete and consider a free massless field satisfying the equation of motion [2]

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0 \ . \tag{2.1}$$

The field operator ϕ can be expanding as

$$\phi = \sum_i f_i \boldsymbol{a}_i + \overline{f}_i \boldsymbol{a}_i^\dagger \ , \tag{2.2}$$

where f_i satisfy the wave equations $g^{\mu\nu}\nabla_\mu\nabla_\nu f_i = 0$ and, by choosing the f_i so that they satisfy the orthonormality condition

$$\frac{i}{2}\int_{\mathcal{J}^-} \left(f_i \nabla_\nu \overline{f}_j - \overline{f}_j \nabla_\nu f_i\right) \mathrm{d}\Sigma^\nu = \delta_{ij} \ , \tag{2.3}$$

$\boldsymbol{a}_i/\boldsymbol{a}_i^\dagger$ are the annihilation and construction operators for particle at past null infinity ($\mathcal{J}^-$); those are ingoing particles. Observe that there is an ambiguity in the definition of the ingoing particles because of the presence of the white hole solution from which particles can be consider[4]. However this ambiguity is eliminated by the fact that we will consider a collapsing spherical body instead of the Schwarschild solution (see Penrose diagram in Figure 3): in this case the white hole region simply does not exists are we conclude that ingoing particles are only considering in $\mathcal{J}^-$. The field ϕ can be expressed everywhere as in (2.2), however in the region outside the event horizion one can express it in terms of

[1] Remember that, if k^a is the properly normalized Killing vector, the surface gravity is defined as $k^a\nabla_a k^b = \kappa k^b$. In the Schwarzschild case $\kappa = 1/(4GM) = 1/(2r_s)$.

[2] Same results can be obtained by using conformally invariant wave equation $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + R/6\phi = 0$.

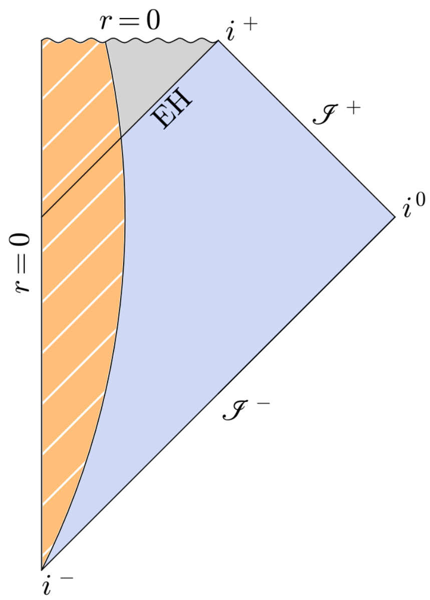

Figure 3. Penrose diagram of a collapsing spherical body. $i^{\mp}$ is the past/future timelike infinity, $\mathscr{J}^{\mp}$ is the past/future null infinity and i^0 is the spatial infinity. $i^{\pm}$ are where timelike geodesics come from. The red zigzag lines are where the singularity is located.

future null infinity ($\mathscr{J}^+$) data:

$$\phi = \sum_i p_i \boldsymbol{b}_i + \overline{p}_i \boldsymbol{b}_i^\dagger + q_i \boldsymbol{c}_i + \overline{q}_i \boldsymbol{c}_i^\dagger \ , \tag{2.4}$$

where $\{p_i\}$ are solutions purely outgoing, i.e. with zero Cauchy data on the event horizon, and $\{q_i\}$ are solution with zero Cauchy data on $\mathscr{J}^+$. This splitting is due to the fact that the outgoing Hilbert space is given by

$$\mathcal{H}_{\text{out}} = \mathcal{H}_{\text{out},\mathscr{J}^+} \oplus \mathcal{H}_{\text{out,BH}} \ . \tag{2.5}$$

The p_i and q_i satisfy

$$\frac{i}{2}\int_{\mathscr{J}^+} \left(p_i \nabla_\nu \overline{p}_j - \overline{p}_j \nabla_\nu p_i\right) \mathrm{d}\Sigma^\nu = \delta_{ij} \ , \tag{2.6}$$

and

$$\frac{i}{2}\int_{\text{Event Hor.}} \left(q_i \nabla_\nu \overline{q}_j - \overline{q}_j \nabla_\nu q_i\right) \mathrm{d}\Sigma^\nu = \delta_{ij} \ , \tag{2.7}$$

respectively. To pass from the decomposition in equation (2.2) to the decomposition in (2.4) we can apply the Bogolibov transformation

$$p_i = \sum_j \alpha_{ij} f_j + \beta_{ij} \overline{f}_j \ , \qquad q_i = \sum_j \gamma_{ij} f_j + \eta_{ij} \overline{f}_j \ , \tag{2.8}$$

$$\boldsymbol{b}_i = \sum_j \overline{\alpha}_{ij} \boldsymbol{a}_j - \overline{\beta}_{ij} \boldsymbol{a}_j^\dagger \ , \qquad \boldsymbol{c}_i = \sum_j \overline{\gamma}_{ij} \boldsymbol{a}_j - \overline{\eta}_{ij} \boldsymbol{a}_j^\dagger \ . \tag{2.9}$$

It is known that the Bogolibov's transformations "do not preserve the vacuum", in the sense that the vacuum of $\mathscr{J}^-$, i.e. the vacuum defined by $\boldsymbol{a}_i \left|0, in\right\rangle$, is not the vacuum in $\mathscr{J}^+$ since

$$\left\langle 0, in\right| \boldsymbol{b}_i^\dagger \boldsymbol{b}_i \left|0, in\right\rangle = \sum_j |\beta_{ij}|^2 \ . \tag{2.10}$$

The proof of the above equation is provided in appendix C. β_{ij} contain the information about the particles created by the gravitation field and emitted to infinity and we then want to estimate these quantities. Let us expand the solution in spherical harmonics

$$f_{\omega' lm} \sim r^{-1}(\omega')^{-1/2} F_{\omega'}(r) e^{i\omega' v} Y_{lm}(\theta, \varphi) \ , \tag{2.11}$$

$$p_{\omega lm} \sim r^{-1}\omega^{-1/2} P_{\omega}(r) e^{i\omega u} Y_{lm}(\theta, \varphi) \ , \tag{2.12}$$

where v and u are andvaned and retarted coordinates

$$v/u = t \pm r \pm 2M \log\left|\frac{r}{2M} - 1\right| \ . \tag{2.13}$$

In general

$$p_\omega = \int_0^\infty \left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} \overline{f}_{\omega'}\right) \mathrm{d}\omega' \ . \tag{2.14}$$

There are two-component of p_ω: one will be scattered outside the collapsing body and will end up on $\mathscr{J}^-$; the other will enter in the collapsing body, call it $\tilde{p}_\omega$. In order to estimate the last contribution near the latest time that a null geodesic could leave $\mathscr{J}^-$, v_0, we can define a null vector tangent to the horizon l^μ and n^μ the future directed null vector at x which is directly radially and such that $l^\mu n_\mu = -1$. A vector $-\epsilon n^\mu$ connects a point x on the event horizon with a nearby null surface of constant retarded time u. We have to transport (l^μ, n^μ) back along the point in which future and past event horizon intersected. Define the affine parameter λ [3] which is related to the retarded time u on the part horizon by

$$\lambda = -Ce^{-\kappa u} = (v_1 - v_0) \ , \tag{2.15}$$

where C is a constant and κ is the surface gravity. Therefore

$$u = -\frac{1}{\kappa} \log(v - v_0) \ , \tag{2.16}$$

which means that

$$e^{-i\omega u} \sim (v - v_0)^{i\omega/\kappa} \ , \tag{2.17}$$

therefore

$$\tilde{p}_\omega \sim \omega^{-1/2} r^{-1} P_\omega(2M)(v - v_0)^{i\omega/\kappa} \ . \tag{2.18}$$

The Frourier transform of $\tilde{p}_\omega$ will give us $\tilde{\alpha}_{\omega\omega'}$ and $\tilde{\beta}_{\omega\omega'}$; in particular for large ω'

$$\tilde{\alpha}^{(2)}_{\omega\omega'} \sim e^{i(\omega - \omega')v_0} \left(\frac{\omega'}{\omega}\right)^{1/2} \Gamma\left(1 - \frac{i\omega}{\kappa}\right) (-i\omega')^{-1+i\omega/\kappa} \ , \tag{2.19}$$

[3] The affine parameter is that it satisfies the geodesic equation. Another way is to say that if the parametrization is affine, parallel transport preserves the tangent vector.

$$\tilde{\beta}_{\omega\omega'} \sim -i\tilde{\alpha}_{\omega(-\omega')} \ . \tag{2.20}$$

The logarithmic singularity of $(-i\omega')^{-1+i\omega/\kappa}$ in $\omega' = 0$ can be cured analytically continue $\tilde{\alpha}$ anticlockwise round the singularity. Therefore

$$|\tilde{\alpha}_{\omega\omega'}| \sim e^{\pi\omega/\kappa}|\tilde{\beta}_{\omega\omega'}| \ . \tag{2.21}$$

The total number of created particle in $\mathcal{J}^+$ in the frequency range $\omega + \mathrm{d}\omega$ is given by $\mathrm{d}\omega \int_0^\infty |\beta_{\omega\omega'}|\mathrm{d}\omega'$ which is divergent. However this divergence is due to the fact that there is a finite steady rate of emission continuing for an infinite time. Construct then the wave-packets

$$p_{jn} = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} \mathrm{d}\omega \ e^{2\pi i n\omega/\epsilon} p_\omega \ , \tag{2.22}$$

where j and n are integers. For ϵ small these wavepackets have frequency $j\epsilon$ and are peacked around $u = 2\pi n/\epsilon$ (with width ϵ^{-1}). We can express p_{jn} in terms of the f_ω as

$$p_{jn} = \int_0^\infty (\alpha_{jn\omega'} f_{\omega'} + \beta_{jn\omega'} \overline{f}_{\omega'})\mathrm{d}\omega' \ , \tag{2.23}$$

where

$$|\alpha_{jn\omega'}| = \left| \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} \mathrm{d}\omega \ e^{2\pi i n\omega/\epsilon} \alpha_{\omega\omega'} \right| \sim \omega^{-1/2} \Gamma\left(1 - \frac{i\omega}{\kappa}\right) (\epsilon\kappa)^{-1} (\omega')^{-1/2} \sin(\epsilon/(2\kappa)) \ . \tag{2.24}$$

The expectation value of the number of particles created and emitted to infinity ($\mathcal{J}^+$) in the wavepacket mode p_{jn} is given by

$$\int_0^\infty \mathrm{d}\omega |\beta_{jn\omega}|^2 \ , \tag{2.25}$$

and this quantity can be computed by considering the fraction Γ_{jn} of wavepacket that will enter in the collapsing body

$$\Gamma_{jn} = \int_0^\infty \mathrm{d}\omega \ \left(|\tilde{\alpha}_{jn\omega}|^2 - |\tilde{\beta}_{jn\omega}|^2 \right) \ , \tag{2.26}$$

where the minus sign is because negative frequency contribution make negative contribution to the flux. From (2.21) it's clear that

$$|\tilde{\alpha}_{jn\omega'}| = e^{\pi\omega/\kappa}|\tilde{\beta}_{jn\omega'}| \ . \tag{2.27}$$

Therefore the total number of particle created in the mode p_{jn} is given by

$$\frac{\Gamma_{jn}}{e^{2\pi\omega/\kappa} - 1} \ . \tag{2.28}$$

Observe that this is the emission cross-section is exactly that for a body with a temperature of $T = \kappa/2\pi$. This temperature is called Hawking temperature and observe that it is related to the mass of the black hole, unintuitively, as $T \sim 1/M$[4]. Therefore the temperature of the

[4]Reintroducing also $\hbar$, c and G we have $T = \hbar c^3/(8\pi GM)$

black hole increase when the black hole is collapsing; or in another way the smaller is the black hole mass the more it emits particles. Further observe that all this calculation can be repeted for a massless free fermions (e.g. an approximation for neutrinos) and it gives the very same result up to equation (2.26) where the sign in front of $|\beta_{jn\omega}|$ is the opposite (i.e. a plus); this is because negative frequency components gives, due to the spin-statistics, a positive contribution to the flux. The number of particles emitted is therefore

$$\frac{\Gamma_{jn}}{e^{2\pi\omega/\kappa}+1} , \tag{2.29}$$

which is the Fermi-Dirac statistics. To better interpret this phenomena simply recall that the ingoing Hilbert space is just $\mathcal{H}_{\text{in},\mathcal{J}^-}$, whereas the outgoing Hilbert space is given in equation (2.5); the outgoing Fock-space is then given by

$$\mathcal{F}(\mathcal{H}_{\text{out}}) = \mathcal{F}(\mathcal{H}_{\text{out},\mathcal{J}^+}) \otimes \mathcal{F}(\mathcal{H}_{\text{out,BH}}) . \tag{2.30}$$

This means that the out-states are "joint" (entangled) states of particles that reach $\mathcal{J}^+$ and particles that will fall down in the black holes.

3 Back reaction on the metric

The back reaction on the metric due to the particle creation implies the slow decreasing of the black hole mass. Unfortunately it is not meaninful to talk about local energy-momentum of creating particles; this is similar to the problem of defining gravitational energy in classical general relativity. Nevertheless we can define an total flux, by integrating over a suitable surface. The stress energy tensor of a free scalar is given by[5]

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi . \tag{3.1}$$

let k^μ be a time parameter defined along the generators of the horizon in the final quasi-stationary state. We are interesting in the quantity

$$\frac{1}{u_1-u_2}\int_{u_1}^{u_2} d^dx \ \langle 0, in| : T_{\mu\nu} : |0, in\rangle \ k^\mu \mathrm{d}\Sigma^\nu , \tag{3.2}$$

where the integration is performed by fixing r, considering two retarded times u_1 and u_2 and $|0, in\rangle$ is the vacuum of $\mathcal{J}^-$. For convenience let's define the wave-packets $x_{jn} = p^{(2)}_{jn} + q^{(2)}_{jn}$ and $y_{jn} = p^{(1)}_{jn} + q^{(1)}_{jn}$; the first quantity represents the part of p_{jn} and q_{jn} passes through the collapsing body, the second quantities do not contain any negative frequencies and therefore do not contribute to the flux of the stress tensor. On the other hand

$$x_{jn} = \int_0^\infty \mathrm{d}\omega \left(\chi_{jn\omega} f_\omega + \xi_{jn\omega}\overline{f}_\omega\right) , \tag{3.3}$$

[5] In general we have to renormalize this quantity; however every tensor which is stationary, satisfies $\nabla_\mu T^{\mu\nu}$ and agree near $\mathcal{J}^+$ gives the same result for the flux.

but close enough to $\mathcal{J}^+$ we have that $x_{jn} \simeq \sqrt{\Gamma_{jn}} p_{jn}$. This implies that

$$\int_{u_1}^{u_2} d^d x \ \langle 0, in| : T_{\mu\nu} : |0, in\rangle \ k^\mu \mathrm{d}\Sigma^\nu =$$
$$= \mathrm{Re}\left[\sum_{jn}\sum_{pl}\int_0^\infty \mathrm{d}\omega'' \int_{u_1}^{u_2} \mathrm{d}u \ \omega\omega' \sqrt{\Gamma_{jn}} p_{jn} \bar{\xi}_{jn\omega'} \left(\sqrt{\bar{\Gamma}_{pl}} \bar{p}_{pl} \chi_{pl\omega'} - (\sqrt{\Gamma_{pl}} p_{pl} \xi_{pl\omega'}\right)\right] , \tag{3.4}$$

where ω and ω'' are the frequencies of the wave-packets p_{jn} and p_{pl} respectively. By considering $u_2 - u_1 \gg 1$ we only the first term in the integrand contribues; furthermore by repeating argument similar to the discussion in the above section we conclude that

$$\int_0^\infty |\xi_{jn\omega'}| \ \mathrm{d}\omega' = \frac{1}{e^{2\pi\omega/\kappa} - 1} . \tag{3.5}$$

Therefore

$$\frac{1}{u_1 - u_2} \int_{u_1}^{u_2} d^d x \ \langle 0, in| : T_{\mu\nu} : |0, in\rangle \ k^\mu \mathrm{d}\Sigma^\nu = \int_0^\infty \Gamma_\omega \frac{\omega}{e^{2\pi\omega/\kappa} - 1} , \tag{3.6}$$

where $\Gamma_\omega = \lim_{n\to\infty} \Gamma_{jn}$ is the fraction of wave-packet of frequency that would be absorbed by the black hole.

3.1 Thermodynamic interpretation

The energy flux computed in (3.6) exactly compensate the thermal emission computed above (see equation (2.29) for the number of emitted particles). This energy flux will cause the area of the event horizon to decrease and so the black hole will not, in fact, be in a stationary state. This can also be interpreted from the thermodynamic point of view considering the standard entropy definition

$$\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{1}{T} ; \tag{3.7}$$

in the case of the black hole we discussed above that

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi G M} . \tag{3.8}$$

Therefore by using this temperature in the above definition and by identifying M as the energy we have that

$$S = \frac{A}{4G} , \tag{3.9}$$

where A is the area of the black hole. This entropy seems to suggests that the area of a black hole can never decrease; nevertheless, due to the Hawking radiation discussed above, $\langle T_{\mu\nu}\rangle$ does not satisfy the energy condition assumed in the proof of the area law (i.e. the law $\delta A \geq 0$). Nevertheless it is clear that also the second law of thermodynamics for the matter outside the black hole is violated; indeed the entropy S_m of the matter outside the black hole can decrease by dropping matter inside the black hole. Notice however that the violation of the two laws compensate each other, in the sense that when $\delta S_m < 0$ by dropping matter in the black hole then $\delta A > 0$, whereas when $\delta A < 0$ the black hole emits particles that increase the entropy of the matter outside the black hole $\delta S_m > 0$. In order to solve this problem Bekenstein proposed the *generalized entropy* [5][6]

$$S' = S_m + \frac{A}{4G} \ , \tag{3.10}$$

and the second law, given by

$$\delta S' \geq 0 \ , \tag{3.11}$$

is then valid. As shown above the black hole is not in a stationary states, because of particle emission. This implies that the black hole is evaporating. The Penrose diagram is given in figure 3; the generalized second law makes this process possible since the fact that the area of the black hole is decreasing is compensate by the particle production.
The last comment concern the fact that the Hawking temperature is very small when the mass of the black hole is smaller then the mass Plank. This implies the number of particle produced by a massive (mass grater then the Planck mass) (per unit of time) black hole is small. Therefore, even if the black hole is not in a stationary state it is a reasonable assumption to consider the black hole in a sequence of stationary states (quasi-stationary states).

A Conformally equivalent metrics

We prove here that a null geodesic for for a metric is a null geodesic also for any conformally equivalent metric. The trajectory of a null geodesic is

$$g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = 0 \tag{A.1}$$

and this trajectory is the same for $\hat{g}_{\mu\nu}$ in fact

$$\hat{g}_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = e^{2\omega} g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = 0 \ . \tag{A.2}$$

The same is true if we consider a space/time-like curve, since $e^{2\omega} > 0$. However observe that the geodesic equations in one metric is different from the geodesic equation in the other one and therefore time/space-like geodesics in one metric are not necessarily time/space-like geodesics in the other one.

B Schwarzschild solution

Let's solve the Einstein equations for a particular case: let's assume that the system is static, which means that is invariant under spacetime translations and temporal inversion, and spherically symmetric. A particular solution for the Minkowski case is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)$$

[6] Originally Bekenstein proposed the generalized entropy before the Hawking paper on black hole radiation; the law was based on merely observations, but the Hawking radiation put solid ground on the theoretical meaning of such an entropy.

Let's use it as a starting point and let us deformed it; a general ansaz satisfying the above conditions is

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{2B(r)}\mathrm{d}r^2 + e^{2C(r)}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)$$

Assuming that $\partial_r C(r)$ is not vanishing, introducing the coordinate $\tilde{r} = e^{C(r)}$ and redefining $\tilde{r} = r$ one can obtain

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{2B(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)$$

We have only two function to be found: $A(r)$ and $B(r)$. Let's use the Einstein equation in vacuum when $T^{\mu\nu} = 0$. Let's use the Vielbein formalism [7] and consider

$$e^0 = e^{A(r)}\mathrm{d}t \qquad e^1 = e^{B(r)}\mathrm{d}r \qquad e^2 = r\mathrm{d}\theta \qquad e^3 = r\sin\theta\mathrm{d}\varphi$$

We want to find the spin connection. In order to do that we have to impose the null torsion condition and the condition $\nabla_\mu \eta_{ab} = 0$ (which defines the Levi-Civita connection); these two conditions can be written as

$$\mathrm{d}e^a + \omega^a_b \wedge e^b = 0 \qquad\qquad \omega_{ab} = -\omega_{ba}$$

The first equation is

$$\omega^0_b \wedge e^b = \omega^0_1 \wedge e^1 + \omega^0_2 \wedge e^2 + \omega^0_3 \wedge e^3 = -\mathrm{d}e^0 = -\dot{A}e^A\mathrm{d}r \wedge \mathrm{d}t$$

Where we read $\omega^0_2 \propto e^2$, $\omega^0_3 \propto e^3$ and $\omega^0_1 \wedge e^1 = \dot{A}e^{A-B}\mathrm{d}t \wedge e^1$. Then one can write

$$\omega^1_b \wedge e^b = \omega^1_1 \wedge e^1 + \omega^1_2 \wedge e^2 + \omega^1_3 \wedge e^3 = -\mathrm{d}e^1 = 0$$

from which $\omega^1_2 \propto e^2$, $\omega^1_3 \propto e^3$ and $\omega^1_0 \propto e^0$ and so the first solution is $\omega^0_1 = \dot{A}e^{A-B}\mathrm{d}t$. From the equation

$$\omega^2_b \wedge e^b = \omega^2_1 \wedge e^1 + \omega^2_2 \wedge e^2 + \omega^2_3 \wedge e^3 = -\mathrm{d}e^2 = -\mathrm{d}r \wedge \mathrm{d}\theta$$

one can conclude that $\omega^2_3 \propto e^3$ and so $\omega^0_2 = 0$ e $\omega^1_2 = -e^{-B}\mathrm{d}\theta$. The last equation is

$$\omega^3_b \wedge e^b = \omega^3_1 \wedge e^1 + \omega^3_2 \wedge e^2 + \omega^3_3 \wedge e^3 = -\mathrm{d}e^3 = -\sin\theta\mathrm{d}r \wedge \mathrm{d}\varphi - r\cos\theta\mathrm{d}\theta \wedge \mathrm{d}\varphi$$

form which $\omega^0_3 = 0$ and so $\omega^1_3 = -\sin\theta e^{-B}\mathrm{d}\varphi$ and $\omega^2_3 = -\cos\theta\mathrm{d}\varphi$.
Summarizing

$$\begin{array}{cc} \omega^0_1 = \dot{A}e^{A-B}\mathrm{d}t & \omega^0_2 = 0 \\ \omega^0_3 = 0 & \omega^1_3 = -\sin\theta e^{-B}\mathrm{d}\varphi \\ \omega^2_3 = -\cos\theta\mathrm{d}\varphi & \omega^1_2 = -e^{-B}\mathrm{d}\theta \end{array}$$

Recalling that the curvature tensor is

$$R^a_b = \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b$$

[7] The same calculation without using the Vielbein formalism is provided in [6].

with simple steps one can find

$$R^0_{\ 1} = -\left(\ddot{A} + \dot{A}^2 - \dot{A}\dot{B}\right) e^{-2B} e^0 \wedge e^1 \qquad R^0_{\ 2} = -\frac{\dot{A}}{r} e^{-2B} e^0 \wedge e^2$$
$$R^0_{\ 3} = -\frac{\dot{A}}{r} e^{-2B} e^0 \wedge e^2 \qquad R^1_{\ 2} = \frac{\dot{B}e^{-2B}}{r} e^1 \wedge e^2$$
$$R^1_{\ 3} = \frac{\dot{B}e^{-2B}}{r} e^1 \wedge e^3 \qquad R^2_{\ 3} = \frac{1-e^{2B}}{r} e^2 \wedge e^3$$

From which the Ricci tensor is[8]

$$R_{00} = \left(\ddot{A} + \dot{A}^2 - \dot{A}\dot{B} + \frac{2\dot{A}}{r}\right) e^{-2B} \qquad R_{11} = \left(-\left(\ddot{A} + \dot{A}^2 - \dot{A}\dot{B}\right) + \frac{2\dot{B}}{r}\right) e^{-2B}$$

$$R_{22} = R_{33} = \left(-\dot{A} + \dot{B} + \frac{e^{2B} - 1}{r}\right) \frac{e^{-2B}}{r}$$

The fact that $G_{ab} = 0$ means that $R_{ab} = 0$ and so $R_{00} + R_{11} = 0$ which means $\dot{A} + \dot{B} = 0$. From this one can conclude $B = -A + c$ but rescaling properly the time and setting $c = 0$ the result is $A = -B$. Moreover $R_{22} = 0$ means that

$$e^{2A} = 1 + \frac{c}{r}$$

and the equation $R_{00} - R_{11}$ is automatically solved.
In order to find the value of c one can recall that for $r \to \infty$ the metric has to be flat. So using the weak field approximation

$$h_{00} = -\frac{c}{r} = -2\Phi = 2\frac{GM}{r} \Rightarrow c = -2GM$$

In conclusion for an spherically symmetric object [9] the metric is

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right)$$

This is the *Schwarzschild solution*.

C Proof of equation (2.10)

In order to prove equation (2.10) simply substistute $\boldsymbol{b}_i$ and $\boldsymbol{b}_i^\dagger$ with their decomposition in $\boldsymbol{a}_i$ and $\boldsymbol{a}_i^\dagger$. In particular

$$\begin{aligned} \langle 0, in| \, \boldsymbol{b}_i^\dagger \boldsymbol{b}_i \, |0, in\rangle &= \sum_j \sum_l \langle 0, in| \, (\alpha_{il} \boldsymbol{a}_l^\dagger - \beta_{il} \boldsymbol{a}_l)(\overline{\alpha}_{ij} \boldsymbol{a}_j - \overline{\beta}_{ij} \boldsymbol{a}_j^\dagger) \, |0, in\rangle = \\ &= \sum_j \sum_l \langle 0, in| \, \beta_{il} \overline{\beta}_{ij} \boldsymbol{a}_l \boldsymbol{a}_j^\dagger \, |0, in\rangle = \\ &= \sum_j \sum_l \delta_{jl} \beta_{il} \overline{\beta}_{ij} = \sum_j |\beta_{ij}|^2 \ . \end{aligned} \tag{C.1}$$

[8]Observe that $R_{22} = R_{33}$, as we expected from the symmetry of the problem.
[9]For instance a planet or a black hole.

References

[1] D. Harlow, *Jerusalem Lectures on Black Holes and Quantum Information*, Rev. Mod. Phys. **88** (2016) 015002 [1409.1231].

[2] S.W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. **43** (1975) 199.

[3] J.J. Bisognano and E.H. Wichmann, *On the Duality Condition for a Hermitian Scalar Field*, J. Math. Phys. **16** (1975) 985.

[4] R.M. Wald, *General Relativity*, Chicago Univ. Pr., Chicago, USA (1984), 10.7208/chicago/9780226870373.001.0001.

[5] J.D. Bekenstein, *Black holes and entropy*, Phys. Rev. D **7** (1973) 2333.

[6] S.M. Carroll, *Spacetime and Geometry*, Cambridge University Press (7, 2019).