Quantum Approximate Optimization Algorithm

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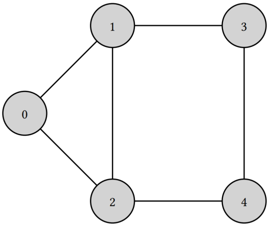
Presentation Overview

The Max-Cut problem

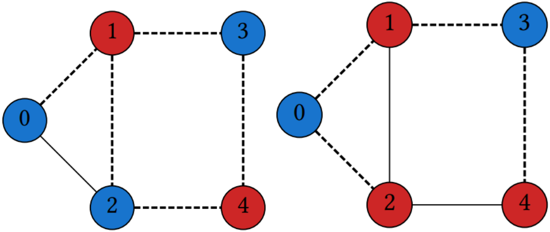
- The Max-Cut problem aims to find the **maximum cuts** in a **Graph**, a mathematical structure



- Divide the vertices into 2 sets in a way that the number of edges with extremes in different sets of the cut is maximum possible
- These edges are called **cut**, and their number define the **size of the cut**

The Max-Cut problem

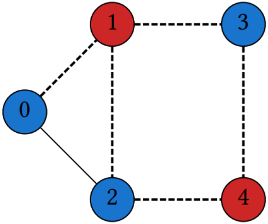
- Examples of cuts are shown below



- Left panel, cut size = 5, right panel, cut size = 4
- The maximum cut is **not** unique

The Max-Cut problem

- The Max-Cut problem can be formulated mathematically without graphical representation
- A combinatorial optimization problem
- Let z_i , $i = 0, 1, ..., n - 1$, be a vertex on the graph, with $z_i \in \{-1, 1\}$
- Each z assignment represent a cut, e.g., $z_0 = z_2 = z_3 = 1$, and $z_1 = z_4 = -1$ or vice versa



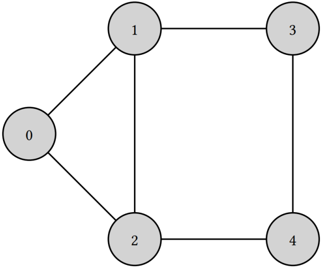
The Max-Cut problem as a combinatorial optimization

An edge between 2 vertices j and k is a cut if $z_j z_k = -1$

The Max-Cut problem

The Max-Cut problem can be formulated as

$$\begin{aligned} \text{minimize} \quad & \sum_{\{j,k\}\in E} z_j z_k \\ \text{subject to} \quad & z_j \in \{-1, 1\}, \end{aligned}$$

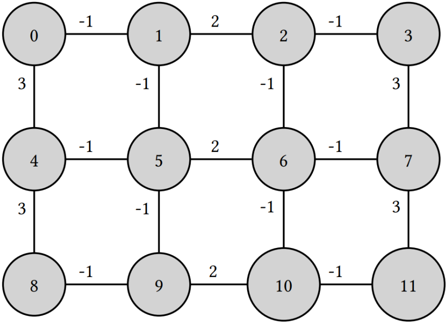


where E is the set of edges in the graph

$$\begin{aligned} \text{minimize} \quad & \sum_{\{j,k\}\in E} z_0 z_1 + z_0 z_2 + z_1 z_2 + z_1 z_3 + z_2 z_4 + z_3 z_4 \\ \text{subject to} \quad & z_j \in \{-1, 1\}, \end{aligned}$$

The Ising model

- The mathematical model describing the ferromagnetic interaction of particles with spin (lattice structure)
- The particle spin are represented by $z_j \in \{-1, 1\}$ (spin down and spin up) respectively

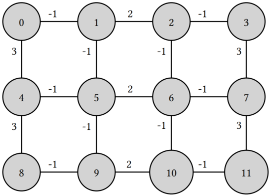


The Hamiltonian function of the Ising model read

$$-\sum_{j,k} J_{jk} z_j z_k - \sum_j h_j z_j \tag{1}$$

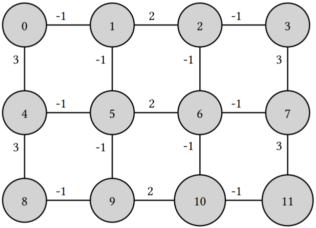
where

- The coefficients J_{jk} represent the interaction between particles j and k
- The coefficients h_j represent the influence of an external magnetic field on particles j



The Ising model

- Finding the minimum energy of the system $\iff$ obtaining a spin configuration that the Hamiltonian reaches its minimum
- The coefficients J_{jk} are the numbers on the edges
- Homogeneous magnetic field, i.e. the coefficients $h_j = 1$



$$\min \sum_{\{j,k\} \in E} z_0 z_1 - 2 z_1 z_2 - 3 z_0 z_4 + z_1 z_5 + 2 z_2 z_6 - 3 z_3 z_7 + \cdots$$
$$\text{s.t.} \quad z_j \in \{-1, 1\},$$

Thinking quantum: mapping problems into quantum computing

Z matrix

Rotation operator of the state vector around the z-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

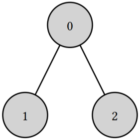
- $Z\,|0\rangle = |0\rangle$, $Z\,|1\rangle = -\,|1\rangle$, $\langle 0|\,Z\,|0\rangle = 1$, and $\langle 1|\,Z\,|1\rangle = -1$

Key to quantum formulation

The Z matrix can be used to evaluate terms in the function to be minimized

- Consider the following Max-Cut problem

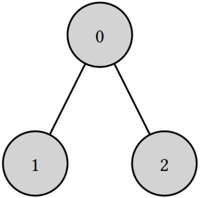
$$\begin{aligned} \text{minimize} \quad & z_0 z_1 + z_0 z_2 \\ \text{subject to} \quad & z_j \in \{-1, 1\},\ j = 0, 1, 2, \end{aligned}$$



Thinking quantum: mapping problems into quantum computing

- Working with 3 vertices, 3 qubits are used
- Apply the tensor product $Z \otimes Z \otimes I$ on the basis $|010\rangle$ leads to $\langle 010| Z \otimes Z \otimes I |010\rangle = -1$
- $|010\rangle$ can be interpreted as a cut where vertices 0 and 2 are assigned to one set and vertex 1 to the other
- $\langle 010| Z \otimes Z \otimes I |010\rangle = -1$: the edge (0,1) has extremes in different sets of the cut
- $Z \otimes Z \otimes I$, Z operators act on qubits 0 and 1
- $Z \otimes Z \otimes I$ is denoted as $Z_0 Z_1$

$$\begin{aligned} \text{minimize} \quad & z_0 z_1 + z_0 z_2 \\ \text{subject to} \quad & z_j \in \{-1, 1\}, j = 0, 1, 2, \end{aligned}$$



Thinking quantum: mapping problems into quantum computing

- Furthermore $\langle 010| Z_0 Z_2 |010\rangle = 1$
- Edge (0,2) is not a cut
- In analogy, the $Z_0 Z_1$ and $Z_0 Z_2$ can be applied on any basis $|x\rangle \in \{000, \cdots, 111\}$
- By linearity

$$\langle x| Z_0 Z_1 + Z_0 Z_2 |x\rangle$$

- Furthermore $Z_j Z_k |x\rangle = |x\rangle$ or $Z_j Z_k |x\rangle = -|x\rangle$
- $|x\rangle$ is an eigenvector of $Z_j Z_k$ with eigenvalues of 1 or -1, therefore for $x \neq y$

$$\langle y| Z_j Z_k |x\rangle = \pm \langle y|x\rangle = 0$$

Thinking quantum: mapping problems into quantum computing

- $|\Psi\rangle = \sum_x a_x |x\rangle$
- By linearity

$$\langle\Psi| Z_j Z_k |\Psi\rangle = \sum_x |a_x|^2 \langle x| Z_j Z_k |x\rangle$$

In particular for the Max-Cut case,

$$\langle\Psi| Z_0 Z_1 + Z_0 Z_2 |\Psi\rangle = \sum_x |a_x|^2 \langle x| Z_0 Z_1 + Z_0 Z_2 |x\rangle$$

Given that $\sum_x |a_x|^2 = 1$, then this sets an upper limit according to

$$\sum_x |a_x|^2 \langle x| Z_0 Z_1 + Z_0 Z_2 |x\rangle \geq \sum_x |a_x|^2 \langle x_{\text{min}}| Z_0 Z_1 + Z_0 Z_2 |x_{\text{min}}\rangle$$
$$= \langle x_{\text{min}}| Z_0 Z_1 + Z_0 Z_2 |x_{\text{min}}\rangle$$

Thinking quantum: mapping problems into quantum computing

- $|x_{\text{min}}\rangle$ is a basis state (or a set of them) for which $\langle x| Z_0Z_1 + Z_0Z_2 |x\rangle$ is minimum
- $|x_{\text{min}}\rangle$ represents a maximum cut
- The Maximum-Cut can now therefore be written as

$$\begin{aligned} \text{minimize} \quad & \langle\Psi| Z_0Z_1 + Z_0Z_2 |\Psi\rangle \\ \text{where} \quad & |\Psi\rangle \in \text{set of qauntum states of 3 qubits} \end{aligned}$$

- Minimization over all possible quantum states

Thinking quantum: mapping problems into quantum computing

The quantity

$$\langle\Psi| \sum_{j,k\in E} Z_j Z_k |\Psi\rangle = \sum_{j,k\in E} \langle\Psi| Z_j Z_k |\Psi\rangle$$

- Is the **expected value** of $\sum_{j,k\in E} Z_j Z_k$
- Reaches its minimum at *one* of the **eigenvectors**
- This eigenvector is called the **ground state**

For the Ising model, the quantum model reads

$$\begin{aligned} \text{minimize} \quad & \sum_{j,k\in E} J_{jk} \langle\Psi| Z_j Z_k |\Psi\rangle - \sum_{j} h_j \langle\Psi| Z_j |\Psi\rangle \\ \text{where} \quad & |\Psi\rangle \in \text{set of qauntum states of n qubits} \end{aligned}$$

Next

1. The Max-Cut problem
2. The Ising model
3. Thinking quantum: mapping problems into quantum computing
4. **Quantum Unconstrained Binary Optimization (QUBO) - Introduction**
5. **QUBO and the Ising model**
6. **Combinatorial optimization and the QUBO model**

The End

Questions? Comments?